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REDUCTION OF FEYNMAN GRAPH AMPLITUDES TO A MINIMAL SET OF BASIC INTEGRALS*

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An algorithm for the reduction of Feynman integrals with any number of loops and external momenta to a minimal set of basic integrals is proposed. The method is based on the new algorithms for evaluating tensor integrals, representation of generalized recurrence relations for a given kind of integrals as a linear system of PDEs and the reduction of this system to a standard form. Basic integrals reveal as parametric derivatives of the system in the standard form and the number of basic integrals in the minimal set is determined by the dimension of the solution space of the system of PDEs.

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1. Introduction

Mass effects play an important role in confronting experimental data obtained at the high-energy colliders, like LEP and SLC, with theoretical predictions. Precise determinations of many physical parameters in the Standard Model (SM) require the evaluation of mass dependent radiative corrections.

In the SM, due to the complicated structure of integrals and the large number of diagrams (typically thousands), no complete calculation even two-loop self-energy has been carried out. Many different species of particles with different masses have to be taken into account and this makes the evaluation of multi-loop integrals a rather difficult problem. Existing numerical methods for evaluating Feynman diagrams cannot guarantee required accuracy for the sum of thousands of diagrams and therefore of the development of analytical or semi-analytical methods are of great importance.

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In calculating Feynman diagrams mainly three difficulties arise: tensor decomposition of integrals, reduction of scalar integrals to several basic integrals and the evaluation of scalar basic integrals.

For the reduction of scalar integrals to a minimal set of integrals recurrence relations are the most suitable tool. Up to now all attempts to extend the standard method of integration by parts [4] to multi-loop diagrams with arbitrary external momenta in the case when all the masses are different were unsuccessful. The first successful algorithm for the reduction of two-loop massive propagator integrals to a minimal set of integrals [5] was based on generalized recurrence relations proposed in Ref. [1].

Using the method described in [1] it is easy to write an enormous number of generalized recurrence relations for any kind of integrals. Not all recurrence relations are independent and therefore a prescription for determining an optimal set of recurrence relations is highly desirable. Another closely related and important problem is the determination of the minimal set of basic integrals.

In the present paper we propose an approach which in principle solves the problems mentioned before completely. We formulate an algorithm for transforming tensor integrals into a combination of scalar ones. We show how to reduce systems of generalized recurrence relations to a linear system of PDEs. The algebraic analysis of linear systems of PDEs is an intensely developing branch of contempory mathematics and many existing algorithms in this field can be used for our purposes. The reduction of linear systems of PDEs to an involutive or standard form solves the problem of finding the optimal set of recurrence relations and a minimal set of basic integrals.

The paper is organized as follows. In Section 2 a method for the representation of tensor integrals and integrals with irreducible numerators in terms of scalar ones with shifted space-time dimension d is shortly described. In Section 3 we describe the method of generalized recurrence relations. In Section 4 we show how to transform the system of generalized recurrence relations into a linear system of PDEs. In Section 5 we describe main features of the Standard Form algorithm by G. Reid and explain the correspondence between the standard form of the linear system of PDEs and our recurrence relations.

2. Evaluation of multi-loop tensor integrals

For the evaluation of L loop Feynman graph amplitudes one needs to calculate the integrals

$$G^{(d)}(\{q_i q_k\},\{m_s^2\}) = \prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P_{\overline{k}_j,m_j}^{\nu_j} \prod_{r=1}^{n_1} k_{1\mu_r} \prod_{s=1}^{n_L} \ldots k_{L\lambda_s}, \qquad (1)$$

where

$$P^j_{k,m} = \frac{1}{(k^2 - m^2 + i\epsilon)^j}, \qquad \overline{k}^\mu_j = \sum_{n=1}^{L} \omega_{jn} k^\mu_n + \sum_{m=1}^{E} \eta_{jm} q^\mu_m, \qquad (2)$$

q_m are external momenta, N is the number of lines, E is the number of external legs, ω and η are matrices of incidences of the graph with the matrix elements being ± 1 or 0.

The traditional way to evaluate tensor integral consists of the following steps:

- write for the integral the most general tensor as a polynomial made of monomials in external momenta and metric tensor $g_{\mu\nu}$
- multiply this tensor by appropriate tensor monomials and obtain a linear system of equations
- solve the system of algebraic equations

One-loop tensor integrals can always be reduced to a combination of scalar integrals without scalar products in the numerator. Evaluating multi-loop integrals one encounters so-called irreducible numerators, *i.e.* scalar products which cannot be expressed in terms of scalar factors in the denominator. The solution of this problem for multi-loop integrals as well as evaluation of multi-loop tensor integrals was proposed in [1]. The main idea of this method consists in the representation of tensor integrals in terms of scalar ones with shifted space-time dimension d. For tensor integral with any number of loops, internal lines and external momenta the following formula was proposed:

$$\prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} \prod_{r=1}^{n_1} k_{1\mu_r} \cdots \prod_{s=1}^{n_L} k_{L\mu_s}$$
$$= T_{\mu_1 \ldots \lambda_L}(\{q_i\}, \{\partial_j\}, \mathbf{d}^+) \prod_{i=1}^{L} \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j}, \qquad (3)$$

where $T_{\mu_1 \ldots \lambda_L}$ is a polynomial type tensor operator and

$$\partial_j \equiv \frac{\partial}{\partial m_j^2}, \qquad \boldsymbol{d}^+ G^{(d)} = G^{(d+2)}. \qquad (4)$$

The main ingredients of the derivation of the operator T are independent auxiliary vectors $a_i (i = 1, \ldots, L)$ and the use of the α- parametric representation. The tensor structure of the integrand on the left-hand side of (3)

can be written as

$$k_{1\mu_1}\dots k_{L\lambda_s} = \frac{1}{i^{n_1+\dots+n_L}}\frac{\partial}{\partial a_{1\mu_1}}\dots\frac{\partial}{\partial a_{L\lambda_s}}\exp\left[i(a_1k_1+\dots+a_Lk_L)\right]\Big|_{a_i=0}. \tag{5}$$

To convert the integral

$$G^{(d)} = \prod_{i=1}^{L}\int d^dk_i\prod_{j=1}^{N}P^{\nu_j}_{\overline{k}_j,m_j}\exp i[a_1k_1+\dots+a_Lk_L], \tag{6}$$

into the α-representation we first transform all propagators into a parametric form

$$\frac{1}{(k^2-m^2+i\epsilon)^\nu} = \frac{i^{-\nu}}{\Gamma(\nu)}\int\limits_0^\infty d\alpha\ \alpha^{\nu-1}\exp\left[i\alpha(k^2-m^2+i\epsilon)\right]. \tag{7}$$

Using the d-dimensional Gaussian integration formula

$$\int d^dk\exp\left[i(Ak^2+2(pk))\right] = i\left(\frac{\pi}{iA}\right)^{\frac{d}{2}}\exp\left[-\frac{ip^2}{A}\right], \tag{8}$$

we evaluate integrals over loop momenta and obtain

$$\begin{aligned} G^{(d)} &= i^L\left(\frac{\pi}{i}\right)^{Ld/2}\prod_{j=1}^{N}\frac{i^{-\nu_j}}{\Gamma(\nu_j)}\int\limits_0^\infty\dots\int\limits_0^\infty\frac{d\alpha_j\alpha_j^{\nu_j-1}}{[D(\alpha)]^{d/2}} \\ &\times\exp i\left[\frac{Q(\{q_j\},\{\alpha_j\},\{a_j\})}{D(\alpha)}-\sum_{i=1}^{N}\alpha_i(m_i^2-i\varepsilon)\right], \end{aligned} \tag{9}$$

where D and Q are polynomials in α, a and q_iq_j. Differentiating (9) with respect to a_j we get by simple identifications the operator T:

$$\begin{aligned} T(\{q\},\{\partial\},\boldsymbol{d}^+) &= \frac{e^{-iQ(\{\{q\},\alpha,\{0\})\rho}}{i^{n_1+\dots+n_N}}\prod_{r=1}^{n_1}\frac{\partial}{\partial a_{1\mu_r}}\dots \\ &\prod_{s=1}^{n_L}\frac{\partial}{\partial a_{L\lambda_s}}e^{i[Q(\{q_j\},\alpha,\{a_j\})]\rho}\Big|_{\substack{a_j=0\\ \alpha_j=i\partial_j\\ \rho=(\frac{i}{\pi})^L\boldsymbol{d}^+}}. \end{aligned} \tag{10}$$

For any particular tensor integral operator T can be constructed by using computer algebra languages. To generate T for 2-3 loop tensor integrals of

rank 3-4 with the help of a short package written in FORM several minutes on a PC Pentium 90 are needed. To our opinion the evaluation of tensor integrals by the proposed method has several advantages in comparison with the traditional method. Firstly, in order to obtain tensor decomposition of the integral no contractions with external momenta and the metric tensor and no solution of a linear system of equations are needed. Secondly, it is easy to select the scalar coefficient of the particular tensor structure. Thirdly, a representation in terms of integrals with shifted d is very compact and may be useful for numerical calculations.

To obtain the final result scalar integrals with different indices and different shifts in d are to be evaluated. This problem can be solved using the method of generalized recurrence relations as proposed in [1].

3. Generalized recurrence relations

In order to obtain recurrence relation for integrals we use an identity:

$$\prod_{i=1}^{L} \int d^d k_i \frac{\partial}{\partial k_{r\mu}} \left\{ R_{\{\mu\}} \left(\{k\}, \{q\} \right) \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} \right\} \equiv 0, \tag{11}$$

where R is an arbitrary tensor polynomial. After performing the differentiation two different representation for scalar products can be used:

$$\text{(a) } k_i q_j = \frac{1}{2}(k_i^2 + q_j^2 - (k_i - q_j)^2)\,,$$

$$\text{(b) } q_{j\mu} \int k_{i\mu} d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j} = q_{j\mu} T_\mu(\{q\}, \{\partial\}, \boldsymbol{d}^+) \int d^d k_i \prod_{j=1}^{N} P^{\nu_j}_{\overline{k}_j, m_j}\,. \tag{12}$$

By using all possible combinations of these representations we produce many relations connecting integrals with changed exponents of scalar propagators and changed values of the space-time dimension. For the same scalar product both (a) and b) can be used. Combining the different relations one can try to find the most optimal set of relations for the reduction of the concrete class of integrals to the minimal set of basic integrals.

In fact, in the traditional method of integration by parts only representation (a) for scalar products is used. Our derivation is more general and it includes integration by parts method [4] as a particular case. Recurrence relations connecting integrals with shifted d cannot be obtained from the traditional method of integration by parts.

To illustrate the difference we consider the one-loop propagator type integral with massive particles:

$$I^{(d)}_{\nu_1\nu_2} = \int \frac{d^d k_1}{[i\pi^{d/2}]} P^{\nu_1}_{k_1,0} P^{\nu_2}_{k_1-q,m}. \tag{13}$$

From the traditional method of integration by parts two relations can be derived:

$$\begin{aligned} 2\nu_2 m^2 I^{(d)}_{\nu_1\ \nu_2+1} + \nu_1 I^{(d)}_{\nu_1-1\ \nu_2+1} + \nu_1(m^2-q^2) I^{(d)}_{\nu_1+1\ \nu_2} \\ -(d-2\nu_2-\nu_1) I^{(d)}_{\nu_1\nu_2} &= 0, \\ \nu_1 I^{(d)}_{\nu_1+1\ \nu_2-1} - \nu_2 I^{(d)}_{\nu_1-1\ \nu_2+1} + \nu_1(m^2-q^2) I^{(d)}_{\nu_1+1\ \nu_2} \\ +\nu_2(m^2+q^2) I^{(d)}_{\nu_1\ \nu_2+1} + (\nu_2-\nu_1) I^{(d)}_{\nu_1\nu_2} &= 0. \end{aligned} \tag{14}$$

The integral $I^{(d)}_{\nu_1\nu_2}(q^2,0,m^2)$ is proportional to the Gauss hypergeometric function [6]:

$$\begin{aligned} I^{(d)}_{\nu_1\nu_2}(q^2,0,m^2) &= (-1)^{\nu_1+\nu_2} \frac{\Gamma(\nu_1+\nu_2-\frac{d}{2})\Gamma(\frac{d}{2}-\nu_1)}{(m^2)^{\nu_1+\nu_2-\frac{d}{2}}\Gamma(\frac{d}{2})\Gamma(\nu_2)} \\ &\quad \times {}_2F_1\left[\begin{matrix} \nu_1, \nu_1+\nu_2-\frac{d}{2}\,; \\ \frac{d}{2}\,; \end{matrix} \ \frac{q^2}{m^2} \right]. \end{aligned} \tag{15}$$

As is well known there are fifteen relations of Gauss between contiguous functions ${}_2F_1$. Substituting (15) into (14) one can find correspondence between the recurrence relations (14) and only six relations of Gauss. The reason is obvious — in the (14) relations the third parameter of ${}_2F_1$ in (15) does not change and therefore all corresponding relations for contiguous functions cannot be reproduced. If we include into consideration also generalized recurrence relations we cover all fifteen relations.

Usually the number of generalized recurrence relations for a given kind of integrals is quite substantial. For their effective applications we must find answers to the following questions:

- How can one use the generalized recurrence relations?
- How to find the minimal set of recurrence relations?
- How to find the minimal set of basic integrals or how to determine the number of basic integrals?

We propose to solve these problems by transforming the system of generalized recurrence relations into an (in general overdetermined) system of linear

partial differential equations (PDE) and we apply algebraic methods being developed by mathematicians for their analysis. Several algorithms and different computer algebra packages for the analysis of linear systems of PDE exist. In our opinion the most adequate will be the algorithm "Standard Form" by Reid [2, 3, 7].

4. Transformation of generalized recurrence relations into a system of l inear PDEs

The system of generalized recurrence relations can be transformed into a linear system of PDEs by several methods. Two procedures turns out to be most convenient.

In the first approach recurrence relations for integrals with arbitrary powers of propagator indices- ν_j are considered. If a term with $\nu_i - 1$ occurs in the recurrence relation then one should make the substitution

$$\nu_i \to \nu_i + 1.$$

Propagators with 'shifted' indices are represented as:

$$\frac{1}{[(k_i - q_s)^2 - m_l^2]^{\nu_l + r}} = \frac{\Gamma(\nu_l)}{\Gamma(\nu_l + r)} \frac{\partial^r}{\partial m_l^{2r}} \frac{1}{[(k_i - q_s)^2 - m_l^2]^{\nu_l}}. \tag{16}$$

Integrals I^d with different shifts of the parameter of the space-time dimension d must be considered as different functions, *i.e.*:

$$I^d = V^1, \quad I^{d+2} = V^2, \quad I^{d+4} = V^3, \ldots \tag{17}$$

These substitutions allow one to transform the system of recurrence relations into a linear system of PDEs for the vector function $V \equiv \{V^1, V^2, \ldots\}$.

In the second approach one considers a system of recurrence relations for integrals with particular integer values of $\nu_i > 0$. First, one should derive recurrence relations for the given integral $I^{(d)}_{\nu_1 \ldots \nu_N}$ with arbitrary ν's. From relations obtained, one needs to derive recurrence relations for the particular sets of integer values of $\nu_i > 0$ by increasing the sum of indices $S = N, N+1, N+2, \ldots$ ($S = \sum_{i=1}^{N} \nu_i$). In this case the system of linear PDEs will include integrals corresponding to graphs with a different number of lines. These integrals are to be considered as different functions. As in the previous case scalar propagators with $\nu_i > 1$ must be represented as

$$\frac{1}{[(k_i - q_s)^2 - m_l^2]^{r+1}} = \frac{1}{\Gamma(r+1)} \frac{\partial^r}{\partial m_l^{2r}} \frac{1}{[(k_i - q_s)^2 - m_l^2]}, \tag{18}$$

and integrals with shifts in d must be considered as different functions.

As was mentioned before the system of linear PDEs for a given integral will include the original as well as simpler integrals obtained by contracting lines in the original one. For integrals with contracted lines one should also write the system of equations. For all these integrals one can consider different sets of ν's which can be classified according to the value of S.

The analysis with help of Standard Form should be started with the system for the simplest non-zero integrals obtained from the original integral by contracting as much lines as possible. Applying Standard Form algorithms the minimal set of recurrence relations (leading derivatives) and minimal set of basic integrals (parametric derivatives) for the simplest integral will be found. This information will be used to investigate more complicated integrals having more lines. Using the terminology of [7] integrals with contracted lines can be treated as classification functions.

The second approach to our opinion is more suitable for considering integrals occurring in calculating Feynman diagrams. In the first approach with arbitrary non-integer indices one can expect more integrals in the basic set.

5. Algorithm Standard Form

As we already mentioned the algorithm Standard Form (SF) was proposed by Reid in [2, 3] and was implemented as `Maple` package in [7]. In this section a short review of the SF algorithm will be given. More detailed information can be found in [2, 3, 7].

In order to compare different differential equations and different terms in a differential equation one should introduce total ordering which will be denoted as $>_s$. The concept of total ordering is very important.

Let us introduce several sandstorm:

$$\begin{aligned} a &= (a_1, \ldots, a_m), \\ D_a V^p &= \frac{\partial^{a_1+\ldots+a_m}}{\partial x_1^{a_1} \ldots \partial x_m^{a_m}} \\ \mathrm{ord}(a) &= a_1 + \ldots a_m \geq 0, \qquad \text{order of the derivative} \end{aligned} \tag{19}$$

We say that $D_a V^p >_s D_b V^q$ if

(i) $ord(a) > ord(b)$
or *(ii)* $ord(a) = ord(b)$, and $p < q$,
or *(iii)* $ord(a) = ord(b)$, $p = q$ and $a >_{\mathrm{lex}} b$,

where the lexicographical ordering $>_{\mathrm{lex}}$ is defined by $a >_{\mathrm{lex}} b$ iff the first nonzero $a_k - b_k > 0$. This is Tresse ordering [8]. Other orderings can be used in the SF package.

For the reduction of Feynman diagrams to the minimal set of basis integrals we will use the ordering:

(i) $p > q$
or *(ii)* $p = q$ and $ord(a) > ord(b)$,
or *(iii)* $ord(a) = ord(b)$, p=q and $a >_{\rm lex} b$.

This ordering allows one to exclude all functions with shifts in d from the minimal set of differential equations.

Input for the SF algorithm may be any linear system of PDEs for the vector function $V = \{V^1, ..., V^p\}$ depending on m independent variables $x_1, ..., x_m$. In our case V^q will be scalar integral, the same integral with different shifts in d and integrals obtainable from the given integral by contracting lines. As independent variables $x_1, ..., x_m$ one can take masses or their ratios. We assume that all lines of the diagram have different masses.

The main result of the application of the SF algorithm will be a set of leading derivatives

$$D_b V^p = f_b^p, \tag{20}$$

where f_b^p are explicitly given functions of x_i and derivatives of V^q such that

(i) the derivative on the l.h.s. of (20) is strictly higher in ordering $>_s$ than any derivative on the r.h.s. of (20)

(ii) no derivative appears on both the l.h.s. and r.h.s. of (20)

(iii) the derivatives on the l.h.s. and r.h.s. of (20) are all distinct

(iv) no derivative in (20) is a nontrivial derivative of any derivative in the l.h.s. of (20).

(v) the integrability conditions of (20) are identically satisfied modulo all lexicographic substitutions which follow from (20).

To clarify the last statement we remind the reader the definition of integrability (consistency) conditions. For any distinct pair of equations

$$D_a V^p = f_a^p, \quad D_b V^p = f_b^p, \tag{21}$$

and for any $c = (c_k) \in N^m$, $c_k \geq \max\{a_k, b_k\}$ the consistency conditions are:

$$D_c V^p - D_c V^p = D_{c-a} f_a^p - D_{c-b} f_b^p = 0. \tag{22}$$

The standard form of a linear system of PDEs is achieved by repeating the following process until all conditions of the standard form are satisfied:

(i) isolate, and solve for, the highest order (leading) derivatives of each equation

(ii) substitute back from *(i)* throughout the rest of the system

(iii) append new equations resulting from the integrability conditions.

The output of the Standard Form algorithm (20) can be identified with our recurrence relations as follows:

- the set of equations for leading derivatives corresponds to a minimal set of recurrence relations
- parametric derivatives are our basic integrals
- the dimension of the solution space of a system is equal to the number of different parametric derivatives in the r.h.s. of (20).

Higher order derivatives (integrals corresponding to scalar diagrams with dots on lines and/or shifts in d) can be obtained by differentiating the set of leading derivatives (20). In the Standard Form package [7]) there is a special subroutine for reducing higher derivatives to parametric ones. Thus, in principle, after tensor reduction this subroutine can be directly applied to the evaluation of Feynman diagrams.

Using the Standard Form package we performed classification of the generalized recurrence relations for the one-loop propagator type integrals and for the simplest two-loop vertex integral with four lines.

Unfortunately the Standard Form package [7] (written in `Maple`) is not powerful enough and cannot be used for the calculation of real diagrams in gauge theories.

We believe that the implementation of the Standard Form algorithm in other computer languages will give better performance and can be used in practical calculations.

In the present paper we gave only a short description of the general scheme for evaluating arbitrary Feynman diagrams. Details and developments of the algorithms presented will be published elsewhere.

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