

$\mathcal{O}(\alpha)$  1-LOOP RADIATIVE CORRECTIONS  
TO  $e^+e^- \rightarrow 4f^*$ 

ALESSANDRO VICINI

DESY, Platanenallee 6, D-15738 Zeuthen (Germany)

*(Received July 17, 1998)*

The status of the calculation of the  $\mathcal{O}(\alpha)$  1-loop radiative corrections to  $e^+e^- \rightarrow 4f$  and some technical progresses are presented.

PACS numbers: 12.15.Ji

**1. Introduction**

The phase 2 of LEP, running above the  $W$  pair production threshold, is providing a very important piece of information concerning the gauge structure of the Standard Model (SM). It is possible that in the future a new  $e^+e^-$  collider with a center of mass energy of about 1 TeV will improve further our knowledge of this sector of elementary particle physics. The measurement of the mass of the  $W$  boson, the nature of the tri- and quadrilinear gauge boson vertices, the investigation of the mechanism of symmetry breaking and the study of the properties of the top quark are some of the main issues under investigation. Thanks to the clean environment characteristics of  $e^+e^-$  collisions, it should be possible to perform precision-studies of all these topics. In order to obtain a sensible comparison between experiments and theory, it is mandatory that the theoretical predictions include in the most complete and accurate way the effects of the SM 1-loop radiative corrections, which are known to be potentially sizeable.

The full electroweak 1-loop corrections to on-shell  $W$  pair production have been calculated in the past [1], but this is not sufficient to analyze in detail the processes at energies above the production threshold, because of the instability of the  $W$ 's. We can distinguish, in the process  $e^+e^- \rightarrow W^+W^- \rightarrow 4f$ , two kinds of radiative corrections: those affecting either the  $W$  pair production or the  $W$ 's decay, and those that connect these

---

\* Presented at the DESY Zeuthen Workshop on Elementary Particle Theory "Loops and Legs in Gauge Theories", Rheinsberg, Germany, April 19-24, 1998.

subprocesses by means of the exchange of a virtual particle. The former are usually called *factorizable*, while the latter are *non-factorizable*. At the moment only some classes of universal leading QED and QCD corrections are available for the process  $e^+e^- \rightarrow W^+W^- \rightarrow 4f$ . Recently a first estimate of *non-factorizable* QED corrections has been presented, in the double-pole approximation [2]. It is well known that the *non-factorizable* corrections yield terms which are potentially large, growing like  $\log \frac{s}{m_W^2}$  and thus will be important at the energies of a Next-Linear-Collider. An estimate [1, 3] of the effect of the missing 1-loop corrections in  $e^+e^- \rightarrow 4f$  shows that they could modify the presently used approximations by 1.5% at LEP2 energies, up to 22% at  $\sqrt{s} = 2$  TeV. It is therefore mandatory to take into account also these *non-factorizable* corrections.

## 2. Layout of the calculation

The reaction indicated with  $e^+e^- \rightarrow 4f$  includes a large set of different processes which have been classified in [4]. We have chosen to investigate in detail the channel  $e^+e^- \rightarrow \mu^- \bar{\nu}_\mu u \bar{d}$ , because of its particular experimental interest.

We are calculating the exact  $\mathcal{O}(\alpha)$  1-loop electroweak radiative corrections to this process. We distinguish virtual and real corrections; the latter contain diagrams where a photon is emitted from each charged lines and from the tri-linear gauge boson vertex. We neglect the masses of all the fermions and therefore all the diagrams in which a scalar particle couples to a fermion vanish. In the virtual sector loops with up to six internal lines appear. We generate all the relevant diagrams using the program FeynArts [5]. We classify the diagrams upon the number of points of the loop integrals. We find 30 diagrams with a 6-point function, 85 with a 5-point function, 200 with a 4-point function, 435 with a 3-point function, 90 diagrams with a full self-energy insertion, and finally 71 real bremsstrahlung diagrams.

The large number of Feynman diagrams which contribute to the amplitude, about nine hundred, is one of the main problems to perform the complete calculation. For this reason the so called ‘‘pole expansion’’ has been introduced in the past, in order to identify some simpler gauge-invariant subsets of contributions. This expansion is done about the on-shell  $W$  pair production threshold: the leading terms (double-pole) contain two resonant  $W$  propagators; the next-to-leading ones (single-pole) are those with only one resonant  $W$  propagator, while all the others are considered to be sub-leading. It is useful to stress that double-pole, single-pole and sub-leading terms are not distinguishable from an experimental point of view, *i.e.* they do not have a direct physical interpretation. The double-pole term allows to estimate the size of these corrections at LEP2 energies [2], but should

become inaccurate when higher energies are considered. Essentially only QED-like corrections, in the so called “extended soft photon approximation”, contribute to this first term. An exact calculation is needed, in order to check the range of validity of the double-pole approximation and to evaluate the radiative corrections at higher energies, where also the purely weak corrections become relevant. It is well known [4] that in the exact calculation there is no recipe to separate QED from non-QED corrections in a  $SU(2)$  gauge-invariant way, because of the mixing between the photon and the  $Z$  boson. We are therefore forced to consider sets of diagrams that are quite larger than those necessary in the double-pole approximation.

The evaluation of a matrix element with such a large number of terms represents a challenge from the computational point of view: the expressions are very lengthy, the CPU time needed to evaluate them is long and the propagation of the errors due to the numerical inaccuracy of the routines becomes an important element in the analysis of the results.

In the virtual sector our strategy can be summarized in the following way:

- (i) a simplification procedure is applied to each diagram; this procedure is optimized to reduce as much as possible the rank of the tensor integral and to minimize the number of generated terms. For each Feynman diagram the output is a combination of tensor integrals, simpler than the initial one; we still find rank-1 6-point functions and up to rank-2 4- and 5-point functions. This simplification is done using a Mathematica package and the result is afterwards automatically translated in a Fortran 90 subroutine.
- (ii) Even if this is possible, we do not try to generate analytical expressions for each diagram, because we want to keep the result as much as possible in a compact form. We perform the reduction from tensor to scalar integrals and the evaluation of the latter in a completely numerical way. We rely upon the FF package [6] to evaluate the basic scalar integrals, while we have written new routines to reduce 5- and 6-point functions to a combination of 4-point functions and to perform the covariant decomposition of the tensor integrals. The problem of the stability and of the accuracy of the results must be discussed in detail.

Since we take the fermions to be massless, we write the amplitudes using the helicity states formalism. For each state the amplitude is evaluated numerically, its modulus squared is taken and finally the sum over all the helicity states is performed. The integration over the final state phase space will be done using a Monte Carlo technique.

### 3. Simplification procedure

The expressions of the virtual corrections involve tensor loop integrals with up to rank-3 4-, 5- and 6-point functions. Through a covariant decomposition we can write each integral as a combination of all the possible tensorial structures which can be formed using the metric tensor and the momenta which appear in the integral. The number of possible tensorial structures increases with the rank and with number of points of the function, while at the same time the accuracy in the determination of the corresponding scalar coefficients decreases. It is therefore quite obvious that the reduction of the rank and of the number of points of the integrals can be very convenient to evaluate the diagram in a faster and more precise way.

To reach this goal we have implemented an observation by Pittau [7], concerning some elementary properties of the Dirac algebra. Each diagram is characterized by the presence of three external fermion lines, connected by the exchange of a gauge boson. In most of the cases each line contains one power of the integration momentum, which can not be simplified by any means because of the Dirac structure. Our goal is to rearrange the expression in order to factor out the integration momenta of the Dirac lines, in the form of scalar products. The scalar products can be re-expressed in terms of the propagators present in the integral and therefore simplified.

If this method is applied blindly, it is possible to generate a large number of terms starting from one single expression, simply using a not adequate rule of the Dirac algebra. We have implemented the procedure [7] in a Mathematica package optimized to keep the final number of terms as small as possible and to reduce the maximal rank of the tensorial structures as much as possible.

Despite of the fact that this procedure still generates several terms from one single initial Feynman diagram, we have obtained two results:

- (i) the number of scalar coefficients which are calculated in the covariant decomposition is anyway smaller by a factor 12 in average with respect to the number of coefficients of the decomposition of the initial rank-3 integral;
- (ii) the absence of rank-3 tensor integrals allows to reach a higher numerical accuracy, as it will be clarified later.

### 4. Reduction of scalar 5- and 6-point functions

The reduction of a 5-point function to a combination of five 4-point functions has already been described in detail in [8]. We implement this formula for the general case of four independent external momenta, but we

notice that there are some kinematic configurations where the result might become unstable and inaccurate. This instability is related to the presence of several Gram determinants in the reduction formula. The evaluation of a Gram determinant (or of a determinant in general) becomes difficult from the numerical point of view, when the difference of two large numbers occurs, which might lead to a sensible precision loss. The problem can be rephrased saying that the Gram determinant is given by the product of the eigenvalues of the matrix; when one of them becomes very small with respect to the others and very close to zero, still the product of the eigenvalues can be non-vanishing. If the value of the vanishing eigenvalue goes below the precision of the machine, then the result can be non-vanishing but completely arbitrarily wrong. A good way to prevent the appearance of numerical instabilities is to evaluate the eigenvalues of the Gram matrix and check whether some of them are below the precision of the machine. If this is the case, a solution alternative to the one proposed in [8] is used.

The presence of null eigenvalues indicates that the rank of the matrix is lower than the maximal one, *i.e.* in the case of the Gram determinant that some of the external momenta are linearly dependent. If this is the case an observation is in order for the 5-point function. We define the scalar 5-point function

$$E \equiv \int d^4k \prod_{l=1}^5 [(k + q_l)^2 - m_l^2 + i\varepsilon]^{-1}, \tag{1}$$

where  $k + q_l$  and  $m_l$  are the momentum and the mass of the particle running in the  $l^{th}$  propagator, and  $\varepsilon > 0$ . From the 5-point function it is possible to obtain five different 4-point functions dropping each time one of the propagators:

$$D(j) \equiv \int d^4k \prod_{l=1, l \neq j}^5 [(k + q_l)^2 - m_l^2 + i\varepsilon]^{-1}. \tag{2}$$

We want to write the 5-point function  $E$  as a combination of these five 4-point functions  $D(j)$ :

$$E = \sum_{i=1}^5 c_i D(i). \tag{3}$$

This is a linear system of six equations with five incognitae:

$$\begin{aligned} \sum_{i=1}^5 c_i &= 0, \\ \sum_{i=1}^5 c_i q_i^\mu &= 0, \quad (\mu = 0, 1, 2, 3) \\ \sum_{i=1}^5 c_i (q_i^2 - m_i^2) &= 1. \end{aligned} \tag{4}$$

When all the  $q_i^\mu$  are linearly independent, this system has in general no solution. Instead when the  $q_i^\mu$  are linearly dependent we have a linear system of five equations with five incognitae, which in general has a solution. The evaluation of the  $c_i$  can be done very precisely using standard numerical techniques. We have two complementary methods to evaluate the 5-point function and cover the whole phase space: in the "regular" region the general formula [8], while close to the singular points the expression (3). The switch from one to the other is controlled by a threshold value which is chosen to be  $10^{-p+1}$ , where  $p$  is the number of significant digits of the machine. Depending on this value the eigenvalues of the Gram matrix are considered to be or not to be vanishing.

Similar arguments apply to the scalar 6-point function, defined as

$$F \equiv \int d^4k \prod_{l=1}^6 [(k + q_l)^2 - m_l^2 + i\varepsilon]^{-1}. \quad (5)$$

From the 6-point function it is possible to obtain six different 5-point functions dropping each time one of the propagators.

$$E(j) \equiv \int d^4k \prod_{l=1, l \neq j}^6 [(k + q_l)^2 - m_l^2 + i\varepsilon]^{-1}. \quad (6)$$

We want to write the 6-point function  $F$  as a combination of these six 5-point functions

$$F = \sum_{i=1}^6 s_i E(i). \quad (7)$$

A system of equations identical to (3) can be written:

$$\begin{aligned} \sum_{i=1}^6 s_i &= 0, \\ \sum_{i=1}^6 s_i q_i^\mu &= 0, \quad (\mu = 0, 1, 2, 3) \\ \sum_{i=1}^6 s_i (q_i^2 - m_i^2) &= 1. \end{aligned} \quad (8)$$

We have now six equations and six incognitae and in general it is always possible to find a solution to this problem.

It is important to notice that the decomposition (7) applies both to the scalar and to the tensor 6-point integrals. The fictitious problem of an always vanishing Gram determinant for the tensorial decomposition of a 6-point tensor integral does not exist, because it is always possible to reformulate the problem in terms of 5-point functions.

### 5. Covariant decomposition of the tensor integrals

The technique to perform a covariant decomposition of a 1-loop tensor integral is well known since a long time [9]. The problem of the numerical stability of the results is also well known, and is related to the presence of Gram determinants in the denominator of the coefficients of each tensorial structure (*cf.* [6]). The zeroes of these determinants can lead to instabilities of the result. The corresponding singularities are in most of the cases spurious, *i.e.* without a physical meaning, and cancel in the final result. From the numerical point of view this cancellation is far from trivial.

As we have already noticed above, the Gram determinant vanishes when the external momenta are linearly dependent. As a consequence, in that particular configuration the basis of tensorial structures for the decomposition becomes smaller than the most general one. The strategy of the calculation is: *i)* evaluate the eigenvalues of the Gram matrix and check whether at least one is vanishing; if this is not the case then the most general basis is used, otherwise the elements of the smaller basis must be determined; *ii)* once the restricted basis has been calculated, the standard technique can be applied safely, because by construction the Gram determinant does not vanish and there cannot be spurious singularities. If we consider for example a rank-1 3-point function  $C^\mu$  and the kinematic configuration where

$$q_3^\mu = a q_1^\mu + b q_2^\mu, \tag{9}$$

we find that the coefficients  $C_i$  ( $i = 1, 2, 3$ ) individually contain singularities, while the  $C'_i$  ( $i = 1, 2$ ) do not:

$$\begin{aligned} C^\mu &= \int d^4k k^\mu \prod_{i=1}^3 [(k + q_i)^2 - m_i^2]^{-1} = C_1 q_1^\mu + C_2 q_2^\mu + C_3 q_3^\mu, \\ &= [C_1 + a C_3] q_1^\mu + [C_2 + b C_3] q_2^\mu, \\ &\equiv C'_1 q_1^\mu + C'_2 q_2^\mu. \end{aligned} \tag{10}$$

The direct evaluation of the  $C'_i$ s turns out to be more stable and accurate.

The switch from the general to the restricted basis of tensorial structures is controlled by a value which depends on the precision of the machine and on the highest rank of the tensor integrals present in the calculation. Depending on this value, the eigenvalues of the Gram matrix are considered to be or not to be vanishing. In general one would choose for this threshold the smallest possible value compatible with the precision of the machine, but the presence of higher rank tensor integrals requires some caution.

In our calculation we have at most rank-2 tensors, but the following discussion can be easily generalized. At some intermediate step of the decomposition of a rank-2 integral, one has to evaluate also the corresponding

rank-1 integral. If the Gram determinant of the rank-2 integral has one (or more) vanishing eigenvalue, one would like to find, for consistency, that the same thing occurs also for the rank-1 integral. In a different language one would like to verify in both integrals that the external momenta have some linear dependence and that the set of independent momenta is the same in both cases. We have found that the two routines provide consistent results if we use a value  $\bar{x}_2$  for the rank-2 integral and a value  $\bar{x}_1 = \sqrt{\bar{x}_2}$  for the rank-1 one. This choice can be easily understood, if one reads the explicit expressions of the two Gram matrices. In the configuration in which one of the eigenvalues of the rank-1 matrix goes to zero like  $\lambda$ , the corresponding one in the rank-2 matrix goes to zero like  $\lambda^2$ .

With this procedure we make an error every time that two momenta which are not exactly linearly dependent are considered to be in such relation among themselves. In this case we neglect a contribution which is proportional to the threshold value  $\bar{x}_{1,2}$  that we use. It is therefore important to keep  $\bar{x}_{1,2}$  as small as possible.

From the previous argument it should be also clear why it is very important to avoid the presence of rank-3 tensor integrals: for the above-mentioned consistency reasons one should choose three different values of the threshold parameter  $\bar{x}_{1,2,3}$  to evaluate rank-1,-2,-3 integrals, with the relations  $\bar{x}_1 \simeq (\bar{x}_3)^{1/3}$  and  $\bar{x}_2 \simeq (\bar{x}_3)^{2/3}$ . If we would work in double precision, we should choose  $\bar{x}_1 \simeq 10^{-2} - 10^{-3}$  and therefore we would introduce rather large numerical errors.

I want to thank F. Jegerlehner, K. Kolodziej and S. Herrlich, with whom I am collaborating to complete this calculation, and the organizers for the very nice and stimulating atmosphere of the workshop.

## REFERENCES

- [1] M. Böhm *et al.*, *Nucl. Phys.* **B304**, 463 (1988); J. Fleischer, F. Jegerlehner, M. Zralek, *Z. Phys.* **C42**, 409 (1989); W. Beenakker, K. Kolodziej, T. Sack, *Phys. Lett.* **B258**, 469 (1991); W. Beenakker, F.A. Berends, T. Sack, *Nucl. Phys.* **B367**, 287 (1991); K. Kolodziej, M. Zralek, *Phys. Rev.* **D43**, 3619 (1991); J. Fleischer, F. Jegerlehner, K. Kolodziej, *Phys. Rev.* **D47**, 380 (1993).
- [2] W. Beenakker, F.A. Berends, A.P. Chapovsky, *Phys. Lett.* **B411**, 203 (1997) and *Nucl. Phys.* **B508**, 17 (1997); A. Denner, S. Dittmaier, M. Roth, hep-ph/9710521 and hep-ph/9803306.
- [3] A. Denner, S. Dittmaier, hep-ph/9706388.
- [4] Physics at LEP2, editors G. Altarelli, T. Sjöstrand and F. Zwirner, CERN 96-001.
- [5] J. Küblbeck, M. Böhm, A. Denner, *Comput. Phys. Commun.* **60**, 165 (1990).

- [6] G.J. van Oldenborgh, J.A.M. Vermaseren, *Z. Phys.* **C46**, 425 (1990).
- [7] R. Pittau, *Comput. Phys. Commun.* **104**, 23 (1997).
- [8] W.L. van Nerveen, J.A.M. Vermaseren, *Phys. Lett.* **B137**, 241 (1984).
- [9] G. Passarino, M. Veltman, *Nucl. Phys.* **B160**, 151 (1979).