

DIFFERENTIAL RENORMALIZATION OF GAUGE THEORIES*

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The scope of constrained differential renormalization is to provide renormalized expressions for Feynman graphs, preserving at the same time the Ward identities of the theory. It has been shown recently that this can be done consistently at least to one loop for abelian and non-abelian gauge theories. We briefly review these results, evaluate as an example the gluon self-energy in both coordinate and momentum space, and comment on anomalies.

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1. Introduction

Differential renormalization (DR) is a renormalization method in coordinate space which provides finite Green functions without any intermediate regulator or counterterms [1]. It has been applied to a variety of problems (a complete list of references can be found in Ref. [2]). As initially formulated, DR is too general for it introduces many arbitrary constants in the process of renormalization, which have to be fixed at the end of the calculations requiring the fulfilment of the relevant Ward identities. A *constrained* version has been recently proposed which aims to fix this arbitrariness from the beginning and automatically fulfil the Ward identities [2–4]. In Section 2 we briefly review this constrained procedure. It has been explicitly shown to work at one loop for gauge theories (QED, Scalar QED and QCD are discussed in Refs. [3, 4, 2] and [5], respectively). The method is defined in coordinate space, but one can perform the calculations also in momentum space, using the Fourier transforms of the renormalized expressions in coordinate space. A computer code doing all operations automatically is

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available [6]. For illustration, we evaluate in Section 3 the one-loop gluon selfenergy in both coordinate and momentum space, and verify that it is transverse, as required by gauge invariance [7]. Finally we comment on anomalies in Section 4 and on future prospects in the conclusions.

2. Overview of constrained differential renormalization

Constrained differential renormalization (CDR) is essentially a set of rules which allow to expand any Feynman graph in coordinate space in a set of *basic functions* and at the same time determine the (universal) renormalization of these functions. *The renormalized graph is then defined as the corresponding linear combination of the renormalized basic functions.* There are two types of rules, those stating how to manipulate the singular diagrams and those fixing the renormalization of the basic functions. The former establish that all the algebra (including Dirac algebra in four dimensions) must be simplified first, treating the singular expressions as if they were well-defined. This includes contracting all possible Lorentz indices and using the Leibniz rule for derivatives. After this, the Feynman graph is written as a linear combination of total (external) derivatives of basic functions. These are products of propagators with all (internal) derivatives acting only on the last propagator:

$$\begin{aligned} & F_{m_1 m_2 \dots m_n}^{(n)}[\mathcal{O}^{(r)}](z_1, z_2, \dots, z_{n-1}) \\ & \equiv \Delta_{m_1}(z_1) \Delta_{m_2}(z_2) \cdots \mathcal{O}^{(r)z_1} \Delta_{m_n}(z_1 + z_2 + \cdots + z_{n-1}), \end{aligned} \quad (1)$$

where $\Delta_{m_i}(z_i) = \frac{1}{4\pi^2} \frac{m_i K_1(m_i z_i)}{z_i}$, with K_1 a modified Bessel function [8], is the propagator of a particle of mass m_i ¹ and $\mathcal{O}^{(r)z_1}$ is a differential operator of order r acting on the variable z_1 . $z_i = x_i - x_{i+1}$, $i = 1, \dots, n-1$ are the coordinate differences and we work in four-dimensional euclidean space. The rules of the second type reduce the degree of singularity (changing singular expressions by derivatives of less singular ones, which are prescribed to act formally by parts on test functions) and extend mathematical identities among tempered distributions to arbitrary functions. In four-dimensional renormalizable theories (in the Feynman gauge) there are only thirteen singular basic functions. They can have one (A), two (B), three (T) or four (Q) propagators, corresponding to the lines in Table I, and the singular behaviour at coincident points can be of order z^{-4} (logarithmic), z^{-5} (linear), z^{-6} (quadratic) and z^{-7} (cubic), corresponding to the columns in that table. A cubic singular behaviour only occurs in tadpole diagrams that vanish for symmetry reasons in usual theories. A detailed description of the CDR

¹ The expression of the propagator reduces to $\Delta(z_i) = \frac{1}{4\pi^2} \frac{1}{z_i^2}$ in the massless case.

TABLE I

Singular basic functions for renormalizable gauge theories in four dimensions in the Feynman gauge. Lines are ordered according to the number of propagators and columns according to the degree of singularity. The function A , that appears in tadpoles, is defined as $A_m[\mathcal{O}] = \mathcal{O}\Delta_m(z)\delta(z)$.

		$A_m[1]$	$A_m[\partial_\mu]$
$B_{m_1 m_2}[1]$	$B_{m_1 m_2}[\partial_\mu]$	$B_{m_1 m_2}[\square]$ $B_{m_1 m_2}[\partial_\mu \partial_\nu]$	
$T_{m_1 m_2 m_3}[\square]$ $T_{m_1 m_2 m_3}[\partial_\mu \partial_\nu]$	$T_{m_1 m_2 m_3}[\square \partial_\mu]$ $T_{m_1 m_2 m_3}[\partial_\mu \partial_\nu \partial_\rho]$		
$Q_{m_1 m_2 m_3 m_4}[\square \square]$ $Q_{m_1 m_2 m_3 m_4}[\square \partial_\mu \partial_\nu]$ $Q_{m_1 m_2 m_3 m_4}[\partial_\mu \partial_\nu \partial_\rho \partial_\sigma]$			

rules and the renormalized expressions of these singular basic functions in both coordinate and momentum space are given in Ref. [2]. With those expressions and the decomposition described above one can systematically calculate any one-loop amplitude. A computer program performing all operations automatically is available. Its description can be found in Ref [6].

3. The gluon selfenergy

To show how the method works, let us evaluate the gluon selfenergy in QCD. We shall first do the calculation in coordinate space, and afterwards perform a Fourier transform to obtain the corresponding momentum space expression. To simplify the resulting expressions, we shall consider the case of massless quarks, although the massive case is analogous. The lagrangian and Feynman rules in coordinate space can be found in Ref. [5]. The four contributing diagrams are depicted in Fig. 1. Their “bare” expressions in coordinate space, in the Feynman gauge, are

$$\begin{aligned} \Pi_{\mu\nu}^{ab(A)} = & \frac{1}{2}g^2 N_c \delta^{ab} \Delta(x) \left[\delta_{\mu\sigma} (D_\rho - \partial_\rho) + \delta_{\sigma\rho} (\partial_\mu - \overleftarrow{\partial}_\mu) + \delta_{\rho\mu} (\overleftarrow{\partial}_\sigma - D_\sigma) \right] \\ & \times \left[\delta_{\sigma\nu} (\partial_\rho - D_\rho) + \delta_{\nu\rho} (D_\sigma - \overleftarrow{\partial}_\sigma) + \delta_{\rho\sigma} (\overleftarrow{\partial}_\nu - \partial_\nu) \right] \Delta(x), \end{aligned} \quad (2)$$

$$\Pi_{\mu\nu}^{ab(B)} = -6g^2 N_c \delta^{ab} \delta_{\mu\nu} \Delta(x) \delta(x), \quad (3)$$

$$\Pi_{\mu\nu}^{ab(C)} = -g^2 N_c \delta^{ab} \partial_\mu \Delta(x) \partial_\nu \Delta(x), \quad (4)$$

$$\Pi_{\mu\nu}^{ab(D)} = -\frac{1}{2}g^2 N_f \delta^{ab} \text{Tr}[\not{\partial} \Delta(x) \gamma_\mu \not{\partial} \Delta(x) \gamma_\nu], \quad (5)$$

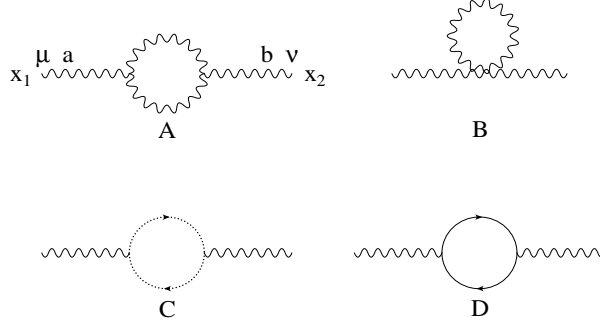


Fig. 1. Feynman diagrams contributing to the gluon selfenergy.

where $x \equiv z_1 = x_1 - x_2$ is the coordinate difference, N_c the number of colours and N_f the number of quark flavors. The partial derivatives $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$ act on the next propagator, while $\overleftarrow{\partial}_\alpha$ act on the previous one. D_α are partial derivatives acting on the external gluons. Using the Leibniz rule, they reduce to minus total derivatives of the amputated expression. We first expand Eqs. (2-5) in basic functions:

$$\begin{aligned} \Pi_{\mu\nu}^{ab(A)} = & \frac{1}{2}g^2 N_c \delta^{ab} \{ (2\partial_\mu \partial_\nu - 5\delta_{\mu\nu} \square) B[1] + 5(\partial_\mu B[\partial_\nu] + \partial_\nu B[\partial_\mu]) \\ & + 2\delta_{\mu\nu} \partial_\sigma B[\partial_\sigma] - 10B[\partial_\mu \partial_\nu] - 2\delta_{\mu\nu} B[\square] \} , \end{aligned} \quad (6)$$

$$\Pi_{\mu\nu}^{ab(B)} = -6g^2 N_c \delta^{ab} \delta_{\mu\nu} A[1] , \quad (7)$$

$$\Pi_{\mu\nu}^{ab(C)} = -g^2 N_c \delta^{ab} \{ \partial_\mu B[\partial_\nu] - B[\partial_\mu \partial_\nu] \} , \quad (8)$$

$$\begin{aligned} \Pi_{\mu\nu}^{ab(D)} = & -2g^2 N_f \delta^{ab} \{ \partial_\mu B[\partial_\nu] + \partial_\nu B[\partial_\mu] - \delta_{\mu\nu} \partial_\sigma B[\partial_\sigma] - 2B[\partial_\mu \partial_\nu] \\ & + \delta_{\mu\nu} B[\square] \} . \end{aligned} \quad (9)$$

Then, we replace the basic functions by their renormalized expressions in Table 2 of Ref. [2], namely

$$A^R[1] = 0 , \quad (10)$$

$$B^R[1] = -\frac{1}{64\pi^4} \square \frac{\log x^2 M^2}{x^2} , \quad (11)$$

$$B^R[\partial_\mu] = \frac{1}{2} \partial_\mu B^R[1] , \quad (12)$$

$$B^R[\square] = 0 , \quad (13)$$

$$B^R[\partial_\mu \partial_\nu] = \frac{1}{3} (\partial_\mu \partial_\nu - \frac{1}{4} \delta_{\mu\nu} \square) B^R[1] + \frac{1}{288\pi^2} (\partial_\mu \partial_\nu - \delta_{\mu\nu} \square) \delta(x) , \quad (14)$$

where the derivatives are prescribed to act formally by parts and M is the renormalization scale. The result for the gluon selfenergy is the sum of

Eqs. (6)–(9) with $A[1]$ and $B[\mathcal{O}]$ replaced by $A^R[1]$ and $B^R[\mathcal{O}]$:

$$\begin{aligned} \Pi_{\mu\nu}^{abR} = & -\frac{1}{144\pi^2} g^2 \delta^{ab} (\partial_\mu \partial_\nu - \delta_{\mu\nu} \square) \\ & \times \left[(15N_c - 6N_f) \frac{1}{4\pi^2} \square \frac{\log x^2 M^2}{x^2} + (2N_c - 2N_f) \delta(x) \right]. \end{aligned} \quad (15)$$

The result is transverse ($\partial_\mu \Pi_{\mu\nu}^{abR} = 0$), as required by gauge invariance [7].

$\Pi_{\mu\nu}^{abR}$ is less singular than the bare $\Pi_{\mu\nu}^{ab}$ and admits a finite Fourier transform in four dimensions:

$$\hat{\Pi}_{\mu\nu}^{abR}(p) = \int d^4x e^{ix \cdot p} \Pi_{\mu\nu}^{abR}(x). \quad (16)$$

This momentum space gluon selfenergy is easily obtained acting with the external derivatives by parts on the exponential, as prescribed by DR, and Fourier transforming $\frac{\log x^2 M^2}{x^2}$ [1] and $\delta(x)$:

$$\begin{aligned} \hat{\Pi}_{\mu\nu}^{abR} = & -\frac{1}{144\pi^2} g^2 \delta^{ab} (p_\mu p_\nu - \delta_{\mu\nu} p^2) \\ & \times \left[(15N_c - 6N_f) \log \frac{\hat{M}^2}{p^2} - 2N_c + 2N_f \right], \end{aligned} \quad (17)$$

where $\hat{M} = 2M/\gamma_E$ and $\gamma_E = 1.781\dots$ is Euler's constant. In more involved cases including masses one can use the Fourier transforms of basic functions in Appendix B of Ref. [2].

Alternatively, one can do all the computations in momentum space, once the Fourier transforms of basic functions are known. As explained in Ref. [6], every step in the calculation in coordinate space can be translated into momentum space.

4. Anomalies

When dealing with singular expressions, anomalies can appear at the quantum level. In some cases (spurious anomalies) the symmetry may be restored with finite local counterterms, while in others the anomaly cannot be avoided and has physical relevance. This is the case of the ABJ triangular anomaly [9]. In general, the source of anomalies in CDR is the non-commutation of renormalization with the contraction of Lorentz indices, which comes from the fact that the rules we have imposed are incompatible with this operation [2–4]. Let us briefly discuss the case of the ABJ anomaly in QED. In the one-loop triangular diagram of one axial and two

vector currents, the trace of six Dirac matrices and one γ_5 appears. Although this trace is unique in four dimensions, it can be written in different ways that lead to different expansions in (singular) basic functions (each of them with different contractions of the internal derivatives). The final results for different expansions differ by finite pieces, and in all cases at least one of the three independent Ward identities is broken. We have checked that the democratic form,

$$\begin{aligned} \text{Tr}[\gamma_5 \gamma_\lambda \gamma_a \gamma_\nu \gamma_b \gamma_\mu \gamma_c] = & 4 (\epsilon_{\lambda a \nu b} \delta_{\mu c} - \epsilon_{\lambda a \nu \mu} \delta_{bc} + \epsilon_{\lambda a \nu c} \delta_{b\mu} \\ & + \epsilon_{\lambda a b \mu} \delta_{\nu c} + \epsilon_{\lambda \nu b c} \delta_{a\mu} - \epsilon_{\lambda a b c} \delta_{\mu\nu} + \epsilon_{\lambda a \mu c} \delta_{\nu b} - \epsilon_{\lambda \nu \mu c} \delta_{ba} + \epsilon_{\lambda b \mu c} \delta_{a\nu} \\ & + \epsilon_{a \nu b \mu} \delta_{\lambda c} + \epsilon_{a \nu \mu c} \delta_{\lambda b} - \epsilon_{a \nu b c} \delta_{\mu\lambda} + \epsilon_{\nu b \mu c} \delta_{\lambda a} + \epsilon_{\nu b \mu \lambda} \delta_{ca} - \epsilon_{a b \mu c} \delta_{\lambda\nu}), \end{aligned} \quad (18)$$

preserves the vector Ward identities, giving the correct anomaly in the axial current [3]. A detailed discussion of anomalies within constrained differential renormalization will be presented elsewhere.

5. Conclusions

Constrained differential renormalization is a new version of DR which renders renormalized amplitudes fulfilling automatically the Ward identities of symmetric theories. It has been explicitly shown to work at one loop for renormalizable gauge theories. Here we have verified that the one-loop gluon selfenergy renormalized with this method satisfies the transversality condition implied by gauge invariance, and have briefly discussed how the ABJ triangular anomaly is recovered. One must also wonder about higher order calculations. They require generalizing the renormalization rules (in particular, the treatment of integrated internal points must be specified) as well as the set of basic functions. This issue is under study.

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