

HOW USEFUL CAN KNOT AND NUMBER THEORY BE FOR LOOP CALCULATIONS?*

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We summarize recent results connecting multiloop Feynman diagram calculations to different parts of mathematics, with special attention given to the Hopf algebra structure of renormalization.

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1. Introduction

In this contribution I want to report on some recent developments in the area of multiloop Feynman diagram calculations. These developments touch widely separated areas of mathematics, ranging from knot theory to number theory [1] as well as from combinatorical Hopf algebras [2] to the realm of noncommutative geometry [3, 4] and the classification of operator algebras. At the same time, on the physics side, they touch areas as separated as the Ising model [5] in three dimensions, the β -function in ϕ^4 theory in four dimensions [6], and the recent identification of the unknown constant in the ρ -parameter of the Standard Model [7].

In David Broadhurst's recent work you will find results concerning the identification of transcendental numbers arising from such finite parts of Feynman diagrams with the hyperbolic covolume of hyperbolic knots [5, 7, 8]. Such results, generalizing the by now familiar identification of knots, numbers and counterterms [1, 6, 9–13], point towards a deep and yet to be clarified connection between QFT and (hyperbolic) geometry.

The recent results of Andrei Davydychev and Bob Delbourgo [14] identifying one-loop triangles and boxes with the geometry of the hyperbolic tetrahedron point in the same direction. All these results give testimony to

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the still premature and incomplete understanding of QFT which we still have more than half a century after its discovery. On the other hand they give hope that we begin to see some parts of the mathematics which underlies QFT.

Let me summarize the results achieved in recent years, and let me stress the most promising facts which deserve, in my opinion, future attention.

2. Knots, Numbers, Diagrams

Let us start with the consideration of a Feynman diagram which has a non-vanishing superficial degree of divergence, but has no subdivergent graphs. The coefficient of the divergence is then a well-defined number, and independent of the chosen regularization or renormalization scheme.

The study of such coefficients allows to observe some remarkable patterns: we can abstract from the nature of the particles realizing a given topology, and consider solely the topology of the graph as determined by internal propagators. Whenever the topology of two different graphs matches, even in different theories, we expect the same transcendental number from these graphs. In Fig. 1 we see some graphs and their topology which completely determines the transcendentals in their counterterms.

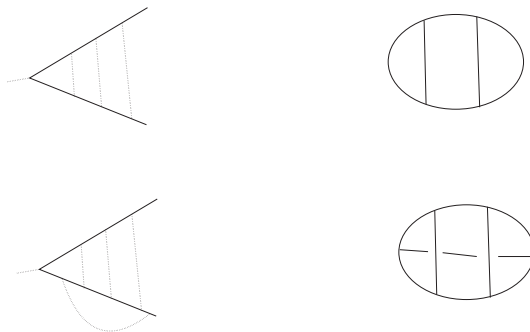


Fig.1. The topology of a diagram. We discard external lines and the different nature of internal propagators. The resulting graph solely encodes the topology of the diagram. This is sufficient information to determine the transcendental nature of the counterterm of the Feynman graph.

This leads to a knot-to-number dictionary. Fig. 2 summarizes some of the knot and number classes identified so far [6, 9]. Hence we can classify topologies by assigning knots to Feynman graphs [1, 6, 12], and find that graphs which deliver similar knots deliver similar transcendentals. This pattern works well enough to abstract conjectures about the nature of these

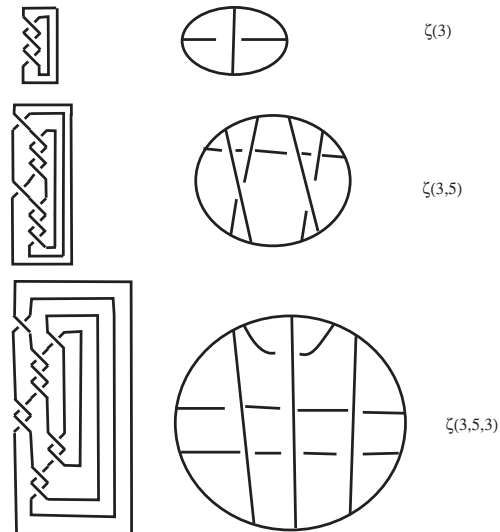


Fig. 2. Knots, Numbers, Diagrams. Some of the diagrams, knots and numbers which are identified by now. While ladder diagrams give rational counterterms, other topologies generate single, double and triple sums.

transcendental numbers [9, 15]. So far severe numerical tests have confirmed these conjectures [16].

This allows to determine a search base of transcendental numbers in which a diagram will evaluate. This knowledge suffices to turn a numerical answer for a diagram into an analytic one, as was demonstrated for the first time in [6]. The rational weights with which the transcendentals seen in counterterms contribute depend on the specific nature of the theory realizing a given topology. It is a fascinating and still open problem to understand how these weights can be derived from the representation theory of the Lorentz group.

Meanwhile, we understand that the presence of extra symmetries like gauge symmetries [11] or supersymmetry [17] can annihilate the presence of certain transcendentals in the final result.

3. The Hopf algebra of renormalization

But then, if the above described primitive Feynman diagrams have such wonders in store, what about general Feynman diagrams, with full-fledged subdivergences? There are clearly patterns in the study of subdivergent graphs which are hardly accessible with standard techniques of perturbation theory [18].

It was the question of how to disentangle a general Feynman graph in terms of primitive graphs which lead to the discovery of a Hopf algebra structure to which we now turn [2]. We change from the notion of parenthesized words, used in [2], to the notion of rooted trees. This notion turned out to be very convenient in the recent thorough investigation of the mathematical role played by the Hopf algebra of renormalization [19]. Fig. 3 summarizes some basic notions, as developed in [2] and, with particular emphasis given to rooted trees, in [19].

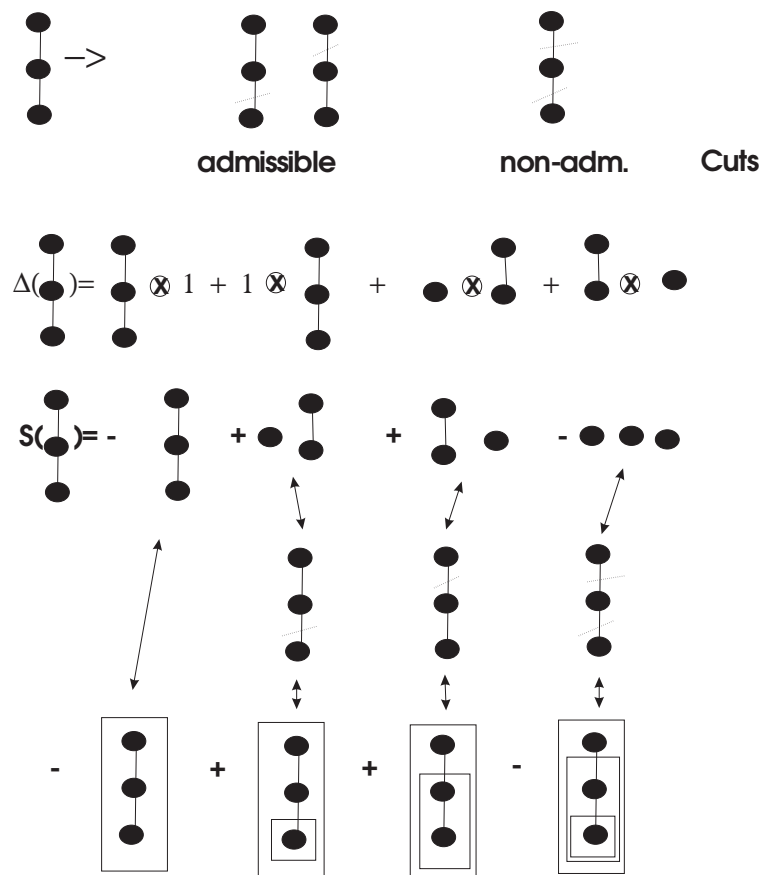


Fig. 3. The Hopf algebra of rooted trees. We define it using admissible cuts on the trees, and give the coproduct Δ in terms of admissible cuts. An admissible cut allows for at most one single cut in any path from any vertex to the root [19]. Note that the antipode S can be formulated in terms of all cuts [19], and that cuts can be represented by boxes on the tree in the indicated manner. The sign is determined by $(-1)^{n_c}$, where n_c is the number of boxes.

Fig. 4 gives a diagrammatic explanation of this Hopf algebra, and shows how the combinatorics of the forest formula derive from this Hopf algebra. The figure explains that the local counterterm appears as a fundamental operation in the Hopf algebra. It corresponds to the coinverse, the antipode. It reveals that the calculus of renormalization can be completely understood as derived from an underlying Hopf algebra structure. This includes the case of overlapping divergences, which can be resolved in terms of this Hopf algebra using either Schwinger Dyson equations [2], powercounting arguments and differential equations for bare Green functions [19], or algebraic methods [20].

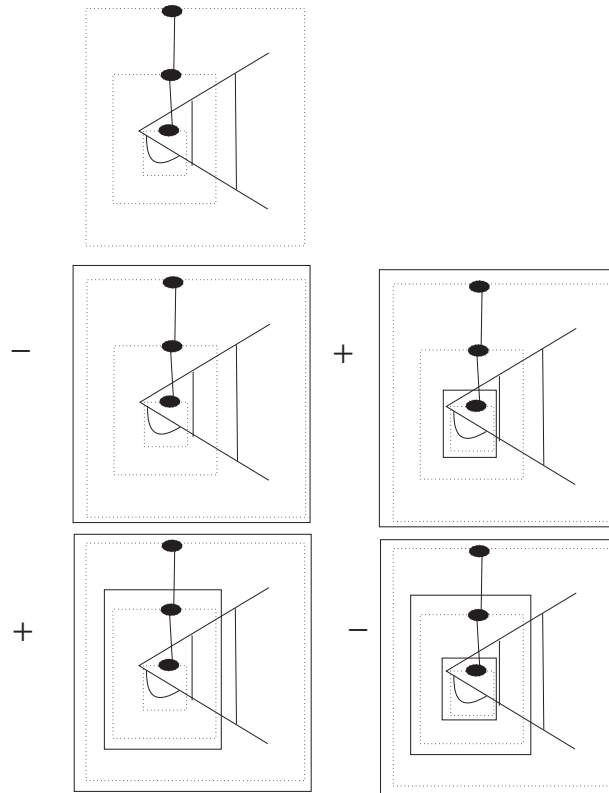


Fig. 4. The Hopf algebra of renormalization. We indicate how to assign a decorated tree to a diagram. On such trees we establish the above Hopf algebra structure. Each black box corresponds to a cut on the tree, and these cuts are in one to one correspondence with the forest structure. We calculate the antipode on the tree, and represent the results on Feynman diagrams, to find that the antipode corresponds to the local counterterm.

The universal structure [19] of the Hopf algebra of [2] gives hope that other diagrammatic expansions, for example asymptotic expansions, can be interpreted in terms of such algebraic structures as well.

Let us also stress that the appearance of this Hopf algebra structure in QFT establishes a link to recent developments in mathematics. Quite the same Hopf algebra turned up in the work of Alain Connes and Henri Moscovici [4]. The precise relation is now clarified [19]. This gives a conceptual backing to the structure of local quantum field theories which ought to be thoroughly investigated in the future.

4. Hopf algebra and four-term relations

Let us close these remarks with a simple observation which demonstrates the usefulness of the Hopf algebra in the understanding of the appearance of transcendentals in Feynman diagrams.

The results of [21, 22] point towards algebraic four-term relations between Feynman diagrams. Clearly, to fully understand all relations between counterterms of diagrams, some more work is needed. One might hope to construct a basis of diagrams which have to be calculated, and then hopes to be able to determine all other diagrams by knowing relations between diagrams.

The simplest example of a four-term relation could appear at three loops, as at least two chords are necessary, by definition. It is given in Fig. 5. Relations between these four topologies are obscured by the fact that there will be subdivergences present.

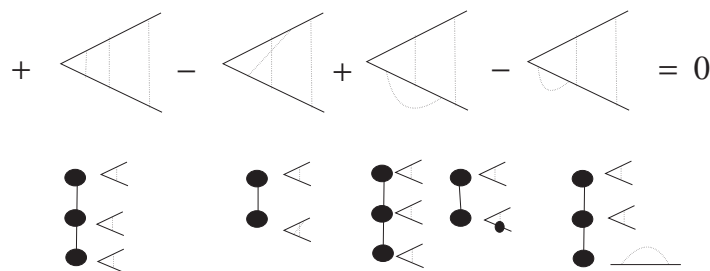


Fig. 5. Disentangling Feynman Diagrams in terms of primitive elements. At the three loop level, a modified four-term relation still has only two contributions which are of $\zeta(3)$ topology. They are provided by the second and third graph above. The third graph decomposes into two trees, when we decompose internal vertex corrections into primitive elements. Decomposing them into primitive elements is in accordance with a decomposition into transcendental and rational parts. The transcendental parts cancel out as demanded by the four-term relation.

To overcome this problem we disentangle graphs into primitive elements of the Hopf algebra of renormalization. Let us give an idea how this might help to achieve a full understanding of a (modified) four-term relation between diagrams.

We realize the four topologies in Yukawa theory. Assume we expect that a four-term relation holds between the four graphs given in the figure, possibly modified by terms including four-point couplings. Such a modified relation is suggested by a detailed study of the results in [21]. These two extra terms, as well as two terms of the original four-term relation, are of ladder topology.

The two remaining terms have the $\zeta(3)$ topology, but a different structure as rooted trees. Now let us decompose the one involving the one-loop vertex correction as indicated in the figure. This corresponds to a decomposition into primitive elements.

Thus, for the four-term relation to hold, the only possible way is that the terms $\sim \zeta(3)$ cancel on the level of antipodes. This is indeed the case, as indicated in the figure, and will be reported in greater detail elsewhere. Here, it suffices to note that a decomposition into primitive elements of the Hopf algebra agrees with a decomposition into the transcendental content of the various terms.

5. Conclusions

In this paper we succinctly reviewed a large body of results. Most of these results were gained through an investigation of Feynman diagrams as entities of interest in their own right.

Altogether, the underlying idea that the numbers seen in the calculation of Feynman diagrams are not that arbitrary, but on the contrary determined by their topology, underlies most of the results considered. The relation to braid-positive knots comes as a surprise in the exploration of such ideas, and still awaits its final explanation.

The fact that there is a Hopf algebra structure underlying renormalization theory points towards a beautiful connection between local quantum field theory and the theory of operator algebras, which, in modern terminology, is called non-commutative geometry.

If some of these ideas come to fruition in the future, we arrive at a picture of QFT which is quite different from what we find in textbooks these days.

Let us start with an arbitrary Feynman graph. We first construct its decomposition into primitive graphs, as determined by the Hopf algebra structure. These primitive graphs come as letters in which our field theory is formulated. We determine a minimal basis in these letters, guided by

relations between graphs as indicated by the study of a (modified) four-term relation.

The elements of this basis are Feynman graphs which are connected to braid-positive knots, evaluating to counterterms expressed in transcendental numbers, which, in the best of all worlds, we could infer from the knowledge of a knot-to-number dictionary gained from empirical data or some deeper insight in the future. This dictionary must not stop at the level of counterterms. The before mentioned results of David Broadhurst on the finite part of diagrams [5, 8] indicate that the story continues.

If we then solve one further problem, how to determine the rational numbers which come as coefficients of these transcendentals, dependent on the spin representation of the involved particles, we approach an understanding of local QFT which can be considered satisfactory.

Being in an optimistic mode, one might even wonder if a decomposition in terms of primitives might have something to say about the asymptotic nature of the perturbative expansion? The perturbation series is asymptotic at best, but it might well be much better behaved if we disentangle it into rational contributions, contributions $\sim \zeta(3)$, and so on, having in mind that each of these numbers seems to refer to a different topology, which we might be forbidden to add up naively.

All of these properties seem to be restricted to a local 'point-like' QFT. Indeed, it seems that most of the successes attributed to theories based on extended objects are present due to the fact that these theories notoriously avoid the subtleties imposed on us by the presence of quantum fields localized at a point. These subtleties are reflected by the presence of UV-divergent integrals. But then a careful study of the properties of these UV divergences indicate that they encapsulate an enormously rich structure at high loop orders, pointing towards the mathematics which might be needed to extend our understanding of QFT in the future.

It is a pleasure to thank the organizers for a lively and stimulating workshop. As always, I have to thank David Broadhurst for an uncountable number of discussions and all his enthusiasm, and I thank Alain Connes for guiding me into the realm of noncommutative geometry and for lot of illuminating discussions. I thank the DFG for support by a Heisenberg fellowship.

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