Vol. 29 (1998) ACTA PHYSICA POLONICA B No 10

GEOMETRICAL APPROACH TO THE EVALUATION OF MULTILEG FEYNMAN DIAGRAMS*

Andrei I. Davydychev†

Department of Physics, University of Mainz
Staudingerweg 7, D-55099 Mainz, Germany
e-mail: davyd@thep.physik.uni-mainz.de

and Robert Delbourgo

Physics Department, University of Tasmania
GPO Box 252-21, Hobart, Tasmania, 7001 Australia
e-mail: Bob.Delbourgo@utas.edu.au

(Received July 20, 1998)

A connection between one-loop N-point Feynman diagrams and certain geometrical quantities in non-Euclidean geometry is discussed. A geometrical way to calculate the corresponding Feynman integrals is considered.

PACS numbers: 12.38.Bx

1. Introduction

As a rule, explicit results for diagrams with several external legs possess a rather complicated analytical structure. This structure can be better understood if one employs a geometrical interpretation of kinematic invariants and other quantities. For example, the singularities of the general three-point function can be described pictorially through a tetrahedron constructed out of the external and internal momenta. This method can be used to derive the Landau equations defining the positions of possible singularities [1] (see also in [2]) and a similar approach can be applied to the four-point function [3] too. Another known example of using geometrical ideas is the massless three-point function with arbitrary off-shell external momenta (see [4, 5]).

* Presented by A.I. Davydychev at the DESY Zeuthen Workshop on Elementary Particle Theory "Loops and Legs in Gauge Theories", Rheinsberg, Germany, April 19-24, 1998.

† Alexander von Humboldt fellow. On leave from the Institute for Nuclear Physics, Moscow State University, 119899 GSP Moscow, Russia.

In this paper, we briefly describe how some geometrical ideas can be used to calculate multileg Feynman diagrams. In particular, we show that there is a direct transition from the Feynman parametric representation to the geometrical description connected with an N-dimensional simplex. A more detailed discussion can be found in [6] (see also in [7]).

2. A simplex related to the N-point function

The scalar integral corresponding to the one-loop N-point function is

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \mathrm{d}^n q \prod_{i=1}^{N} \left[(p_i+q)^2 - m_i^2\right]^{-\nu_i} , \qquad (1)$$

where n is the space-time dimension and ν_i are the powers of the propagators. In general, it depends on $\frac{1}{2}N(N-1)$ momenta invariants k_{jl}^2 $(j<l)$, where $k_{jl} \equiv p_j - p_l$, and N masses m_i corresponding to the internal propagators. The Feynman parametric representation for the integral (1) reads

$$\begin{aligned} J^{(N)}(n;\nu_1,\ldots,\nu_N) &= \mathrm{i}^{1-2\Sigma\nu_i}\pi^{n/2}\, \Gamma\left(\sum \nu_i - \tfrac{n}{2}\right) \left[\prod \Gamma(\nu_i)\right]^{-1} \\ &\quad \times \int\limits_0^1 \ldots \int\limits_0^1 \prod \alpha_i^{\nu_i-1} \mathrm{d}\alpha_i \; \delta\left(\sum \alpha_i - 1\right) \\ &\quad \times \left[\sum \alpha_i^2 m_i^2 + 2 \sum_{j<l}\sum \alpha_j\alpha_l m_j m_l c_{jl}\right]^{n/2-\Sigma\nu_i} , \end{aligned} \qquad (2)$$

where

$$c_{jl} \equiv (m_j^2 + m_l^2 - k_{jl}^2)/(2m_j m_l). \qquad (3)$$

In the region between the corresponding two-particle pseudo-threshold, $k_{jl}^2 = (m_j - m_l)^2$, and the threshold, $k_{jl}^2 = (m_j + m_l)^2$, we have $|c_{jl}| < 1$, and therefore in this region they can be understood as cosines of some angles τ_{jl}, $c_{jl} = \cos\tau_{jl}$, with $c_{jl} = 1$ and $\tau_{jl} = 0$ at the pseudo thresholds, whereas at the threshold $c_{jl} = -1$ and $\tau_{jl} = \pi$. Note that the limits of integration in Eq. (2) can be extended from $(0,1)$ to $(0,\infty)$, since the actual region of integration is defined by the δ function. The expressions in other regions should be understood in the sense of analytic continuation, using (when necessary) the causal prescription for the propagators.

Let us consider a set of N-dimensional Euclidean "mass" vectors whose lengths are m_i. Let them be directed so that the angle between the j-th and the l-th vectors is τ_{jl}. If we denote the corresponding unit vectors as a_i (so that the "mass" vectors are $m_i a_i$), we get $(a_j \cdot a_l) = \cos\tau_{jl} = c_{jl}$. If we put all "mass" vectors together as emanating from a common origin, they, together with the sides connecting their ends, will define a *simplex* which is the *basic* one for a given Feynman diagram. In two dimensions, the simplex is just a triangle, whereas in three dimensions we get a tetrahedron. It is easy to see that the length of the side connecting the ends of the j-th and the l-th mass vectors is $\sqrt{k_{jl}^2}$, so we shall call it a "momentum" side. In total, the *basic* N-dimensional simplex has $\frac{1}{2}N(N+1)$ sides, among them N mass sides (corresponding to the masses $m_1, \ldots, m_N$) and $\frac{1}{2}N(N-1)$ momentum sides (corresponding to the momenta $k_{jl}, j < l$), which meet at $(N+1)$ vertices. Each vertex is a "meeting point" for N sides. There is one vertex where all mass sides meet, the *mass meeting point*, whereas all other vertices are meeting points for $(N-1)$ momentum sides and one mass side.

The matrix $\|c\| \equiv \|c_{jl}\|$ with the components (3) is nothing but the Gram matrix of the vectors $a_1, \ldots, a_N$. It is associated with many geometrical properties of the basic simplex. In particular, we need its determinant,

$$D^{(N)} \equiv \det \|c_{jl}\|. \tag{4}$$

The *content* (hyper-volume) of the N-dimensional simplex is given by

$$V^{(N)} = \frac{1}{N!} \left(\prod_{i=1}^{N} m_i \right) \sqrt{D^{(N)}} \ . \tag{5}$$

The number of $(N-1)$-dimensional hyperfaces is $(N+1)$. N of them correspond to the $(N-1)$-point functions, which can be obtained from the basic N-point function by shrinking one of the internal propagators in turn. The last hyperface contains only momentum sides and can be associated with the massless N-point function. The content of this $(N-1)$-dimensional *momentum* hyperface is

$$\frac{\Lambda^{(N)}}{(N-1)!} \ , \qquad \Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\|. \tag{6}$$

Using substitutions of variables similar to those described in Refs. [5,8], we can transform (2) into the following form:

$$J^{(N)}(n; \nu_1, \ldots, \nu_N) = 2\mathrm{i}^{1-2\Sigma\nu_i} \pi^{n/2} \, \Gamma\left(\sum \nu_i - \tfrac{n}{2}\right) \left[\prod \Gamma(\nu_i)\right]^{-1} \prod m_i^{-\nu_i}$$
$$\times \int\limits_0^\infty \ldots \int\limits_0^\infty \prod \alpha_i^{\nu_i - 1} \mathrm{d}\alpha_i \, \delta\left(\alpha^T \|c\| \alpha - 1\right) \left(\sum \frac{\alpha_i}{m_i}\right)^{\Sigma\nu_i - n} , \tag{7}$$

where

$$\alpha^T \|c\| \alpha \equiv \sum_{j=1}^{N} \sum_{l=1}^{N} c_{jl} \alpha_j \alpha_l = \sum \alpha_i^2 + 2 \sum\sum_{j<l} \alpha_j \alpha_l c_{jl}. \tag{8}$$

Consider a special case $n = N$, $\nu_1 = \ldots = \nu_N = 1$. In this case, the integrand of the parametric integral in (7) is just the δ function. The integration extends over a part of a quadratic hypersurface defined by $\alpha^T \|c\| \alpha = 1$. We can make a rotation to the principal axes, $\alpha^T \|c\| \alpha \Rightarrow \sum \lambda_i \beta_i^2$, where $\lambda_1 \ldots \lambda_N = D^{(N)}$. Let us assume that all λ_i are real and positive, *i.e.* the hypersurface is an N-dimensional ellipsoid (if some of the λ's are negative, the analytic continuation should be used). Now we can rescale $\beta_i = \gamma_i / \sqrt{\lambda_i}$, and the ellipsoid becomes a hypersphere. All we need to calculate is the content of a part of this hypersphere which is cut out (in the space of γ_i) by the images of the hyperfaces restricting the region where all α_i are positive (in the space of α_i). This content, $\Omega^{(N)}$, can be understood as the N-dimensional solid angle subtended by the above-mentioned hyperfaces.

The following statement can be proved (see in [6]): The content of the N-dimensional solid angle $\Omega^{(N)}$ in the space of γ_i is equal to that at the mass meeting point of the basic N-dimensional simplex. Moreover, the angles between the corresponding hyperfaces in the space of γ_i and those in the basic simplex are the same. Therefore, the result can be expressed as

$$J^{(N)}\left(N; 1, \ldots, 1\right) = \mathrm{i}^{1-2N} \; \pi^{N/2} \; \frac{\Gamma\left(N/2\right)}{N!} \; \frac{\Omega^{(N)}}{V^{(N)}} \, . \tag{9}$$

We see that $\Omega^{(N)}$ is indeed the only thing which is to be calculated, since $V^{(N)}$ is known through Eq. (5).

Moreover, $\Omega^{(N)}$ is nothing but the content of a *non-Euclidean* $(N-1)$-dimensional simplex calculated in the spherical (or hyperbolic, depending on the signature of the eigenvalues λ_i) space of constant curvature. The sides of this non-Euclidean simplex are equal to the angles τ_{jl}. Therefore, the problem of calculating Feynman integrals is intimately connected with the problem of calculating the content of a simplex in non-Euclidean geometry.

In the general case, when $\Sigma \nu_i \neq n$, we need some modification of the above transformations (see Ref. [6]). In particular, when $\nu_1 = \ldots = \nu_N = 1$ (but $N \neq n$) the result generalizing Eq. (9) reads

$$J^{(N)}(n; 1, \ldots, 1) = \mathrm{i}^{1-2N} \pi^{n/2} \; \Gamma\left(N - \tfrac{n}{2}\right) \; \frac{m_0^{n-N} \; \Omega^{(N;n)}}{N! \; V^{(N)}}, \tag{10}$$

with

$$\Omega^{(N;n)} \equiv \int_{\Omega^{(N)}} \!\!\ldots\! \int \frac{\mathrm{d}\Omega_N}{\cos^{n-N} \theta}. \tag{11}$$

Geometrically, θ can be understood as the angle between the "running" vector of integration and the direction of the height of the basic simplex, H_0.

Denoting the angle between H_0 and the i-th mass side as τ_{0i}, we get

$$\cos\tau_{0i} = \frac{m_0}{m_i}\,, \qquad m_0 \equiv |H_0| = \left(\prod_{i=1}^{N} m_i\right)\sqrt{D^{(N)}/\Lambda^{(N)}}, \tag{12}$$

with $\Lambda^{(N)}$ defined by Eq. (6).

Furthermore, we can use the height H_0 to split the basic N-dimensional simplex into N rectangular ones, each time replacing one of the mass sides, m_i, by H_0 ($|H_0| = m_0$). In this way, we split $\Omega^{(N)}$ into N parts $\Omega_i^{(N)}$. Therefore, the Feynman integral (10) can be presented as

$$J^{(N)}(n;1,\ldots,1) = \sum_{i=1}^{N} \frac{V_i^{(N)}}{V^{(N)}}\, J_i^{(N)}(n;1,\ldots,1), \tag{13}$$

where $J_i^{(N)}$ denotes the integral associated with the i-th rectangular simplex, whilst $V_i^{(N)}$ is the known content of this simplex.

3. Some examples

For the two-point function, the basic simplex is a triangle with the sides m_1, m_2 and $\sqrt{k_{12}^2}$. Furthermore, $V^{(2)} = \frac{1}{2}m_1 m_2 \sin\tau_{12}$, $\Omega^{(2)} = \tau_{12}$ and $\Lambda^{(2)} = k_{12}^2$. In two dimensions, from (9) we obtain the well-known result

$$J^{(2)}(2;1,1) = \frac{\mathrm{i}\pi}{m_1 m_2}\,\frac{\tau_{12}}{\sin\tau_{12}}, \tag{14}$$

In four dimensions, introducing dimensional regularization [9], we get

$$J^{(2)}(4-2\varepsilon;1,1) = \mathrm{i}\pi^{2-\varepsilon}\Gamma(\varepsilon)\frac{m_0^{1-2\varepsilon}}{\sqrt{\Lambda^{(2)}}}\,\left\{\Omega_1^{(2;4-2\varepsilon)} + \Omega_2^{(2;4-2\varepsilon)}\right\}, \tag{15}$$

with (see, *e.g.*, in [10])

$$\Omega_i^{(2;4-2\varepsilon)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{2-2\varepsilon}\theta} = 2\tan\tau_{0i}\ \ {}_2F_1\left(\begin{array}{c} 1/2,\ \varepsilon \\ 3/2 \end{array}\middle| -\tan^2\tau_{0i}\right), \tag{16}$$

where τ_{01} and τ_{02} are defined in Eq. (12), $\tau_{01}+\tau_{02}=\tau_{12}$.

For the three-point function, the three-dimensional basic simplex is a tetrahedron with three mass sides (the angles between these mass sides are τ_{12}, τ_{13} and τ_{23}) and three momentum sides. The volume of this tetrahedron is defined by Eq. (5) at $N=3$. Furthermore, $\Omega^{(3)}$ is the usual solid angle at the vertex derived by the mass sides. Its value can be defined as the area of a part of the unit sphere cut out by the three planar faces adjacent to the vertex; in other words, this is the area of a spherical triangle corresponding to this section. The sides of this spherical triangle are obviously equal to the angles τ_{12}, τ_{13} and τ_{23} while its angles, ψ_{12}, ψ_{13} and ψ_{23}, are equal to those between the plane faces. The area of this spherical triangle is

$$\Omega^{(3)} = \psi_{12} + \psi_{13} + \psi_{23} - \pi = 2 \arctan \left(\frac{\sqrt{D^{(3)}}}{(1+c_{12}+c_{13}+c_{23})} \right) . \tag{17}$$

Finally, the result

$$J^{(3)}(3;1,1,1) = - \frac{\mathrm{i}\pi^2}{2m_1 m_2 m_3} \frac{\Omega^{(3)}}{\sqrt{D^{(3)}}} \tag{18}$$

corresponds to one obtained in [11] in a different way.

If we consider the four-dimensional three-point function, the only (but very essential!) difference is that we should divide the integrand by $\cos\theta$. We split the spherical triangle with the sides τ_{12}, τ_{13} and τ_{23} into three spherical triangles, corresponding to the solid angles of rectangular tetrahedra. Calculating the corresponding integrals, we obtain the result in terms of the dilogarithms, or the Clausen function (see *e.g.* in [12]).

For the four-point function, the corresponding four-dimensional simplex has four mass sides and six momentum sides. It has five vertices and five three-dimensional hyperfaces. Four of these hyperfaces are the *reduced* ones, corresponding to three-point functions, whereas the fifth one is the momentum hyperface. This four-dimensional simplex is completely defined by its mass sides m_1, m_2, m_3, m_4 and six "planar" angles between them, $\tau_{12}, \tau_{13}, \tau_{14}, \tau_{23}, \tau_{24}$ and τ_{34}. The content (hyper-volume) of this simplex is given by Eq. (5) at $N = 4$, with $D^{(4)} = \det \|c_{jl}\|$.

The four-dimensional four-point function can be exhibited as (*cf.* Eq. (9))

$$J^{(4)}(4;1,1,1,1) = \tfrac{1}{12}\, \mathrm{i}\pi^2 \, \frac{\Omega^{(4)}}{V^{(4)}} = \frac{2\, \mathrm{i}\pi^2}{m_1 m_2 m_3 m_4} \frac{\Omega^{(4)}}{\sqrt{D^{(4)}}} . \tag{19}$$

So, the main problem is how to calculate $\Omega^{(4)}$.

In four dimensions, $\Omega^{(4)}$ is the value of the four-dimensional generalization of the solid angle at the mass meeting point of the simplex. In the spherical case, it can be defined as the volume of a part of the unit hypersphere which is cut out from it by the four three-dimensional reduced hyperfaces, each hyperface involving three mass sides of the simplex. This hyper-section is a three-dimensional spherical tetrahedron whose six sides (edges) are equal to the angles τ_{jl}. In the hyperbolic case, this is a hyperbolic tetrahedron whose volume can be obtained by analytic continuation.

Unfortunately, there are no *simple* relations like (17) which might make it possible to express the volume of a spherical (or hyperbolic) tetrahedron in terms of its sides or dihedral angles. In fact, calculation of this volume in an elliptic or hyperbolic space is a well-known problem of non-Euclidean geometry (see *e.g.* in [14]). A standard way to solve this problem, say in spherical space, is to split an arbitrary tetrahedron into a set of birectangular ones. The volume of a birectangular tetrahedron is known and can be expressed in terms of Lobachevsky or Schläfli functions which can be related to dilogarithms or Clausen function (see in [15]). Different ways of splitting the non-Euclidean tetrahedron can be used to reduce the number of dilogarithms (or related functions) involved (*cf.* in [12, 13]).

4. Conclusion

We have shown that there is a direct link between Feynman parametric representation of a one-loop N-point function and the basic simplex in N-dimensional Euclidean space. In the case $N = n$ (where n is the space-time dimension), the result for the Feynman integral turns out to be proportional to the ratio of an N-dimensional solid angle at the meeting point of the mass sides to the content of the N-dimensional basic simplex. For the four-dimensional four-point function, the representation (7) provides a very interesting connection with the volume of the non-Euclidean (spherical or hyperbolic) tetrahedron.

In the general case ($N \neq n$), the height of the basic simplex, H_0, plays an essential role in the calculation of the integrals. It is used to split the basic Euclidean simplex into N rectangular simplices. When $N < n$, this splitting simplifies the calculation of separate integrals. When $N = n + 1$, each integral $J_i^{(N)}$ (see Eq. (13)) corresponding to one of the resulting rectangular tetrahedra can be reduced to an $(N-1)$-point function (*cf.* also in [11, 16]).

In the resulting expressions, all arguments of functions arising possess a straightforward geometrical meaning in terms of the dihedral angles, etc. In particular, this is quite useful for choosing the most convenient kinematic variables to describe the N-point diagrams. We suggest that this approach can help in understanding the geometrical structure of loop integrals with several external legs, as well as the structure of phase-space integrals. We also note a connection with 3-loop vacuum graphs in three dimensions [17].

A.D. is grafeful for hospitality to the Physics Department, University of Tasmania, where this work was started. A.D.'s research was essentially supported by the Alexander von Humboldt Foundation, and partly by the grants INTAS–93-0744, RFBR–98-02-16981 and Volkswagen–I/73611. A.D. is grateful to the organizers (DESY-Zeuthen) for partial support of his participation in the Rheinsberg workshop.

REFERENCES

[1] L.D. Landau, *Nucl. Phys.* **13**, 181 (1959); G. Källen, A. Wightman, *Mat. Fys. Skr. Dan. Vid. Selsk.* **1** (No.6), 1 (1958).

[2] S. Mandelstam, *Phys. Rev.* **115**, 1742 (1959); R.E. Cutkosky, *J. Math. Phys.* **1**, 429 (1960); R.J. Eden, P.V. Landshoff, D.I. Olive, J.C. Polkinghorne, *The analytic S-matrix*, Cambridge UP, 1966.

[3] R. Karplus, C.M. Sommerfield, E.H. Wichmann, *Phys. Rev.* **114**, 376 (1959); A.C.T. Wu, *Mat. Fys. Medd. Dan. Vid. Selsk.* **33** (No.3), 1 (1961).

[4] J.S. Ball and T.-W. Chiu, Phys. Rev. **D22**, 2550 (1980); A.I. Davydychev, J. Phys. **A25**, 5587 (1992); H.J. Lu and C.A. Perez, preprint SLAC-PUB-5809 (1992).

[5] A.I. Davydychev, J.B. Tausk, *Phys. Rev.* **D53**, 7381 (1996).

[6] A.I. Davydychev, R. Delbourgo, *J. Math. Phys.*, to appear, hep-th/9709216.

[7] N. Ortner, P. Wagner, *Ann. Inst. Henri Poincaré (Phys. théor.)* **63**, 81 (1995).

[8] R. Scharf, Doctoral Thesis, Würzburg (1994); R. Scharf, J.B. Tausk, *Nucl. Phys.* **B412**, 523 (1994).

[9] G. 't Hooft, M. Veltman, *Nucl. Phys.* **B44**, 189 (1972); C.G. Bollini, J.J. Giambiagi, *Nuovo Cimento* **12B**, 20 (1972).

[10] U. Nierste, D. Müller, M. Böhm, *Z. Phys.* **C57**, 605 (1993); F.A. Berends, A.I. Davydychev, V.A. Smirnov, *Nucl. Phys.* **B478**, 59 (1996); E. Remiddi, *Nuovo Cimento* **110A**, 1435 (1997).

[11] B.G. Nickel, *J. Math. Phys.* **19**, 542 (1978).

[12] G. 'tHooft and M. Veltman, *Nucl. Phys.* **B153**, 365 (1979); A. Denner, *Fortschr. Phys.* **41**, 307 (1993).

[13] A. Denner, U. Nierste, R. Scharf, *Nucl. Phys.* **B367**, 637 (1991).

[14] N.I. Lobatschefsky, *Imaginäre Geometrie*, Kasaner Gelehrte Schriften, 1836, Übersetzung mit Anmerkungen von H. Liebmann, Leipzig, 1904; L. Schläfli, *Quart. J. Math.* **3**, 54 (1860); **3**, 97 (1860); *Gesammelte matematische Abhandlungen*, Band II, Birkhäuser, Basel 1953.

[15] H.S.M. Coxeter, *Quart. J. Math.* **6**, 13 (1935); E.B. Vinberg, *Uspekhi Mat. Nauk* **48** (No.2), 17 (1993) [*Russian Math. Surveys* **48** (No.2), 15 (1993)]; R. Kellerhals, in *Structural Properties of Polylogarithms*, ed. L. Lewin, AMS Math. Surveys and Monographs, vol. 37, (1991), p.301.

[16] L.M. Brown, *Nuovo Cimento* **22**, 178 (1961); F.R. Halpern, *Phys. Rev. Lett.* **10**, 310 (1963); G. Källén and J. Toll, *J. Math. Phys.* **6**, 299 (1965); B. Petersson, *J. Math. Phys.* **6**, 1955 (1965); D.B. Melrose, *Nuovo Cimento* **40A**, 181 (1965); W.L. van Neerven, J.A.M. Vermaseren, *Phys. Lett.* **B137**, 241 (1984); Z. Bern, L. Dixon, D.A. Kosower, *Phys. Lett.* **B302**, 299 (1993); *Nucl. Phys.* **B412**, 751 (1994).

[17] D.J. Broadhurst, preprint OUT-4102-74 (1998), hep-th/9806174.