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DECOHERENCE, CHAOS, QUANTUM–CLASSICAL CORRESPONDENCE AND THE ARROW OF TIME*

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The environment — external or internal degrees of freedom coupled to the object of interest — can, in effect, monitor some of its observables. As a result, the eigenstates of these observables decohere and behave like classical states. Continuous destruction of superpositions leads to the effective environment-induced superselection (einselection), which is beginning to be recognized as a key step in the transition from quantum to classical. We investigate it here in the context of quantum chaos. I show that the evolution of a chaotic macroscopic system is not just difficult to predict (requiring accuracy exponentially increasing with time) but quickly ceases to be deterministic in principle as a result of the Heisenberg uncertainty (which limits the available resolution). This happens after a time $t_\hbar$ which is only logarithmic in the Planck constant. For example, various components of the solar system are chaotic, with the Lyapunov timescales ranging from a bit more then a month (Hyperion) to millions of years (planetary system as a whole). On the timescale $t_\hbar$ the initial minimum uncertainty wavepackets corresponding to celestial bodies would be smeared over distances of the order of radii of their orbits into "Schrödinger cat-like" states, and the very concept of a "trajectory" would cease to apply. In reality, such paradoxical states are eliminated by decoherence which helps restore quantum–classical correspondence. The price for the recovery of classicality is the loss of predictability. In the classical limit (associated not with the smallness of $\hbar$, but with decoherence) the rate of increase of entropy is independent of the strength of the coupling to the environment, and equal to the sum of the positive Lyapunov exponents. I end by noting that the cost of information transfer between systems — of the action measured in units of $\hbar$'s per bit — decreases with the increasing size. This suggests why information may seem to be so irrelevant for classical dynamics, and yet is obviously so crucial at the quantum level.

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1. Introduction

Movements of planets have served as a paradigm of order and predictability since ancient times. This view was not seriously questioned until the time of Poincaré, who has initiated the enquiry into the stability of the solar system [1] and thus laid foundations of the subject of dynamical chaos. However, only recently and as a result of sophisticated numerical experiments questions originally posed by Poincaré are being answered. Two groups, using very different numerical approaches, have reported that solar system is chaotically unstable [2,3]. The characteristic *Lyapunov exponent* which determines the rate of divergence of neighboring trajectories in the phase space is estimated to be $\lambda = (4 \times 10^6)^{-1}$ [year^{-1}]. Fortunately (and in accord with the overwhelming experimental evidence) it is likely that this instability will not alter crucial characteristics of the orbits of planets such as their average distance from the sun (although eccentricities of the orbits may not be equally "safe" [3]). Rather, it is the location of the planet along its orbit which is exponentially susceptible to minute perturbations. Even trajectories of the massive outer planets alone appear to be exponentially unstable, although the Lyapunov exponent for that subsystem of the solar system is harder to estimate [2], and may correspond to a timescale as short as few million years, or as long as 30 Myr.

While the instability of the planetary system takes place on a relatively long timescale, there are celestial bodies which become chaotically unpredictable much more rapidly. Perhaps the best studied example is Hyperion, one of the moons of Saturn. Hyperion is shaped as an elongated ellipsoid. The interaction between its quadrupole moment and the gravitational field of Saturn leads to chaotic tumbling, which results in an exponential divergence on a timescale approximately equal to twice its 21 day orbital period [4]. There are also numerous examples of chaos in the asteroid belt (such as Chiron) with the exponential instability timescales of few hundred thousand years [5].

In spite of its obviously macroscopic characteristics solar system is, ultimately, undeniably *quantum*. This is simply because its constituents are a subject to quantum laws. The action associated with the solar system is, of course, enormous:

$$I \simeq \frac{GM_{\odot}M_{\rm J}}{R_{\rm J}} \times \tau_{\rm J} \simeq 1.2 \times 10^{51}[\text{erg s}]\,, \tag{1.1}$$

where the mass $M_{\rm J}$ and period $\tau_{\rm J}$ of Jupiter were used in the estimate. Given this order of magnitude of I and the smallness of the Planck constant ($\hbar = 1.055 \times 10^{-27}$ erg s), one might have anticipated that the dynamics of the solar system is a safe distance away from the quantum regime. However, and as a consequence of the chaotic character of its evolution, this is *not* the case.

I will begin by showing that the macrosopic nature of the system does not guarantee its classicality. Quantum theory demonstrates that the solar system – and every other chaotic system – is *in principle* indeterministic, and not just "deterministically chaotic": Classical predictability in the chaotic context would require an ever increasing accuracy of its initial conditions. This is possible in principle in classical physics. But, according to quantum mechanics, simultaneously increasing accuracy of position and momentum would violate the Heisenberg principle at some instant. This time is surprisingly short, and – at least for some of its components – definitely less than the age of the solar system.

Classicality is restored with the help of the environment - induced decoherence, which continuously destroys purity of the wavepackets. The resulting loss of predictability can be quantified through the rate of entropy production. For a decohering chaotic system we shall see that it is *(i)* independent of the strength of the coupling to the environment, and *(ii)* given by sum of the positive Lyapunov exponents. That is, *quantum* entropy production rate coincides with the Kolmogorov–Sinai entropy in open systems, even though its ultimate cause is the loss of the information to the correlations with the environment.

Why is it so difficult to detect violations of the quantum–classical correspondence on the macroscopic level? We shall see throughout the paper that decoherence is caused by the information transfer between the system and the environment. And we shall close our arguments with an estimate indicating that the action per bit — the cost of information — decreases with the increase of the size of the system. Thus, decoherence is easier to accomplish (and harder to track) in the macroscopic domain.

2. Quantum predictability horizon: How the correspondence is lost

As a result of chaotic evolution, the patch in the phase space which corresponds to some regular (and classically "reasonable") initial condition becomes drastically deformed: Classical chaotic dynamics is characterized by the exponential divergence of trajectories. Moreover, conservation of the volume in the phase space in course of Hamiltonian evolution implies that the exponential divergence in some of the directions must be balanced by the exponential squeezing – convergence of trajectories – in the other directions. It is that squeezing which forces a chaotic system to explore quantum regime: As the wavepacket becomes narrow in the direction corresponding to momentum;

$$\Delta p(t) \; = \; \Delta p_0 \; \exp(-\lambda t)\,, \qquad (2.1)$$

(where Δp_0 is its initial extent in momentum, and λ is the relevant Lyapunov exponent) the position becomes delocalized: Wavepacket becomes coherent over the distance $\ell(t)$ which can be inferred from the Heisenberg's principle:

$$\ell(t) \geq (\hbar/\Delta p_0)\exp(\lambda t)\,. \qquad (2.2)$$

Coherent spreading of the wavepacket over large domains of space is disturbing in its own right. Moreover, it leads to a breakdown of the correspondence principle — predictions of the classical and quantum dynamics concerning some of the expectation values no longer coincide after a time $t_\hbar$ when $\ell(t)$ reaches the scale on which the potential is noticeably nonlinear.

Such scale χ can be usually defined by comparing the classical force (given by the gradient of the potential $\partial_x V$) with the leading order nonlinear contribution $\sim \partial_x^3 V$:

$$\chi \; \simeq \; \sqrt{\frac{\partial_x V}{\partial_x^3 V}}\,. \qquad (2.3)$$

For the gravitational potential $\chi \simeq R/\sqrt{2}$, where R is the size of the system (*i.e.*, a size of the orbit of the planet). The reason for the breakdown of the correspondence is that when the coherence length of the wavepacket reaches the scale of the nonlinearity,

$$\ell(t) \; \simeq \chi\,, \qquad (2.4)$$

the effect of the potential energy on the motion can be no longer represented by the classical expression for the force [8], $F(x) = \partial_x V(x)$, since it is not even clear where the gradient is to be evaluated for a delocalised wavepacket. As a consequence, after a time given by:

$$t_\hbar \; = \; \lambda^{-1} \ln \frac{\Delta p_0 \chi}{\hbar}\,, \qquad (2.5)$$

the expectation value of some of the observables of the system may even begin to exhibit noticeable deviations from the classical evolution.

This is also close to the time beyond which the combination of classical chaos and Heisenberg's indeterminacy make it impossible *in principle* to employ the concept of a trajectory. Over the time $\sim t_\hbar$ chaotic system will spread from a regular Planck-sized volume region in the phase space into a (possibly quite complicated) wavepacket with the dimensions comparable to the size of the system. This breakdown of correspondence can be investigated more rigorously by following evolution generated by the *Moyal* *bracket* (that is, a Wigner transform of the von Neumann equation for the time development of the density matrix).

Moyal bracket can be expressed through the familiar classical Poisson bracket as:

$$\{H, W\}_{\rm MB} \;=\; -i\sin(i\hbar\{H, W\}_{\rm PB})/\hbar\,. \tag{2.6}$$

Above, H is the Hamiltonian of the system, and W is an object in the phase space (*i.e.*, a probability distribution). In our quantum case, W will denote Wigner function — a Wigner transform of the density matrix.

When the potential V in H is analytic, Moyal bracket can be expanded in powers of the Planck constant. Consequently, evolution of W is given by:

$$\dot{W} \;=\; \{H, W\}_{\rm PB} + \sum_{n\geq 1} \frac{\hbar^{2n}(-)^n}{2^{2n}(2n+1)!}\partial_x^{2n+1}V(x)\partial_p^{2n+1}W(x,p)\,. \tag{2.7}$$

Correction terms above will be negligible when $W(x,p)$ is a reasonably smooth function of p — that is, when the higher derivatives of W with respect to momentum are small. However, Poisson bracket alone predicts that, in the chaotic system, they will increase exponentially quickly as a result of the "squeezing" of W in momentum, Eq. (2.1). Hence, after a logarithmic time quantum "corrections" will become comparable to the first classical term on the right hand side of Eq. (2.7). At that point Poisson bracket will no longer suffice as a generator of evolution. Phase space distribution will be coherently extended over macroscopic distances, and interference between the fragments of W will begin to play a role.

The timescale on which the quantum–classical correspondence is lost in a chaotic system can be also estimated (or, rather, bounded from above) by the formula [6,7]:

$$t_r \;=\; \lambda^{-1}\ln(I/\hbar)\,, \tag{2.8}$$

where I is the action which — for the solar system — we have already estimated, Eq. (1.1). It follows that, for the planetary system, quantum–classical correspondence should be significantly violated after approximately:

$$t_r \;\simeq\; 711\ [\text{Myr}]\,.$$

This is less than a fifth of the modest estimates of the age of Earth, and, presumably, a still smaller fraction of the actual age of the solar system. When we compute instead the value of $t_\hbar$, setting initial uncertainty in the momentum to $\Delta p_0 = \hbar/\Lambda_{\rm dB}(T) \simeq 10^9$ [g cm/s], where $\Lambda_{\rm dB}(T)$ is the thermal de Broglie wavelength of Jupiter at its present surface temperature of ~ 100 K, we can estimate almost identical:

$$t_\hbar \;\simeq\; 682\ [\text{Myr}]\,.$$

A similar calculation for Hyperion results in a much smaller (and, therefore, so much more disturbing):

$$t_\hbar \simeq 20 \text{ [yr]} .$$

Moreover, it should be pointed out that — in the macroscopic regime considered here — the above estimates are exceedingly *insensitive* to either the action or the initial momentum uncertainty. Both of these quantities appear under the logarithm. Therefore, the estimated time of the breakdown of quantum–classical correspondence does not change much in response to changes of I and Δp_0 (as long as they are not obviously unreasonable).

3. Solar system as a Schrödinger cat

We have seen above that a seemingly very secure prediction of quantum physics as applied to the solar system fails: According to Schrödinger equation, less than a billion years after its formation behavior of the solar system should be flagrantly non-classical, with the quantum states of celestial bodies spread over dimensions comparable with the sizes of their orbits, and with the planetary dynamics no longer in accord with the laws of Newton! Somehow, this does not seem to be the case. The source of the paradox is obvious: Chaotic dynamics increases the size of the coherent wavepacket with $\exp(\lambda t)$, so that it becomes comparable with the dimensions of the solar system after a time $t_\hbar$, which is but a fraction of its age. Similarly, and after only ~ 20 years the quantum state of Hyperion would be a coherent superposition involving macroscopically distinct orientations of its major axes.

There are parallels between our discussion above and the famous argument due to Schrödinger [10] in that a very macroscopic object (planet here, cat there) is forced, through the strict compliance with the laws of quantum mechanics, into a very non-local state, never encountered as an ingredient of our familiar "classical reality". The main difference between the two examples — Schrödinger cat and the chaotic quantum planet — is in the manner in which they are forced into the final superposition: Schrödinger cat either lives (or dies) as a result of decay of an unstable nucleus: An intermediate step in which a quantum state of the nucleus is *measured* (to determine the fate of the cat) is essential. Thus, in the case of the cat it was possible to entertain the notion that the (admittedly preposterous) final superposition of dead and alive cat could be avoided if the process of measurement was properly understood. This "way out" is no longer available in the case of celestial bodies we are discussing. They evolve into states which are nonlocal and flagrantly quantum simply as a result of dynamical evolution — the measurement plays absolutely no role in setting up the paradox. And the systems involved are certainly even more macroscopic than the cat. Moreover, if the reader considers the idea of putting a living being (cat) in a superposition especially tantalizing, this is certainly occurring also in the case considered here. For, in accord with the quantum arguments presented above, after a time $t_\hbar$ Earth would evolve into a state corresponding to a coherent superposition of all the seasons, as well as all of the hours of the day!

So what is the resolution of the above paradox? Let us us start with a few possibilities which may be tempting at the first sight, but which ultimately lead nowhere. To begin with, one might be worried that in the arguments above we have cut corners by considering just one spatial dimension and one momentum, while the solar system is inhabiting a multidimensional phase space. This is certainly true, but the squeezing in momentum and the resulting delocalization is unlikely to be alleviated by considering multidimensional phase space of all the celestial bodies for which it occurs. This is especially true for systems such as Hyperion or Chiron, which have a rather large Lyapunov exponent. Another more contrived possibility of avoiding the difficulty with quantum–classical correspondence at present would be to design an initial state which evolves into a "classical looking" state by the present epoch. This can be in principle done, but — as the reader is invited to verify — leads to initial states which are even more flagrantly quantum than those we were forced to consider above.

Finally and in desperation one might consider abandoning quantum theory for some other theory which is almost exactly like quantum theory (to pass all of the experimental tests) but contains either nonlinear corrections, or allows for underlying "hidden variable" dynamics, or, perhaps, introduces "collapse of the wavepacket" *ad hoc* at some fundamental level in order to get rid of the quantum nonlocality. All of these ideas face either serious experimental constraints (which render them useless for the purpose of making quantum theory look classical) or profound theoretical difficulties (such as a conflict with the Lorentz invariance), or both.

4. Decoherence, the quantum, and the classical

I shall instead contend that quantum theory is rigorously correct, but that the superposition principle cannot be applied naively, especially to the macroscopic objects. The failure of such simple-minded application of quantum principles to the classical domain is, however, itself a consequence of the unitarity of quantum evolution: Macroscopic objects are all but impossible to insulate from their environments. Consequently, external and internal degrees of freedom continuously "monitor" — that is, become correlated — with their state [11–13]. This is the process of decoherence [11–20].

We have no room here to develop theory of decoherence systematically and completely. A sketch with a few leads to the existing (and rapidly expanding) literature will have to suffice. The key point is the observation that in quantum mechanics information matters much more than in classical mechanics, where it can be acquired without influencing in any way the *actual* state of the system, which exists and evolves independently of what is known about it. This neat division between information and "physical reality" is impossible to implement in the quantum realm: Acquisition of information is equivalent to the establishment of a correlation, which in turn is reflected in the loss of the capacity for interference. Double slit experiment is a classic example. As soon as it is known through which slit the photon has passed, the possibility for interference is lost.

Information transfer which accompanies decoherence has the same nature as the one encountered in quantum measurements, or for that matter, in quantum computation. In either case it is useful to represent it with an elementary logical gate — so-called "controlled not" or a "`c-not`" — which reversibly copies a single bit of information between two (two-state) quantum systems, a "control" and a "target":

$$(a|0\rangle \; + \; b|1\rangle)_{\rm c} \; |0\rangle_t \;\; \longrightarrow \;\; a|0\rangle_{\rm c}|0\rangle_t \; + \; b|1\rangle_{\rm c}|1\rangle_t \, . \tag{4.1}$$

In short, a quantum `c-not` is an obvious generalization of a classical gate (also occasionally known as an "exclusive or" or "`xor`"), which flips the state of the target bit when the control is in a state "1", and does nothing otherwise. The analogy between Eq. (4.1) and the von Neumann model of a quantum measurement [21] is obvious: Control bit acts as a measured system, forcing the apparatus (target bit) to measure its state.

The state on the right hand side of Eq. (4.1) is, however, not the resolution of the measurement problem, but, rather, its cause: When $a = -b = 1/\sqrt{2}$ it is in fact identical to the states encountered in the Bohm's version of the Einstein–Podolsky–Rosen experiment [22,23]. The correlation established is a quantum entanglement, with all its seemingly paradoxical consequences, which deny to each of the two systems involved the right to possess a state prior to a measurement [23–25]. Decoherence converts quantum entanglement into classical correlation by virtue of allowing the environment to carry out additional measurements on the to-be-classical system [11,12]. Its consequence is then a disentanglement — correlations between the system and the apparatus weaken to their classically allowed form.

Decoherence can then be conveniently "caricatured" (if not quite "represented") by means of the `c-not` like gates transferring information about the to-be-classical observable to the environment. In the process of decoherence information flows from the to-be-classical observable of the decohering entity to the environment. Decoherence is a purely quantum effect — as we have already noted, information does not matter in classical dynamics. In the symbolism of `c-not`'s decoherence can be conveniently contrasted with the more familiar consequence of the coupling to the environment — noise — in which the state of the environment becomes inscribed on the observable of interest. In case of decoherence, the system is the control, and the environment in the target which is perturbed in a manner dependent on the state of the system. In the case of noise, the roles are reversed.

The direction of the information flow depicted in the `c-not`s depends on the observables involved. This is an important consequence of the quantum nature of the information transfer. The reader can verify this by re-writing action of the `c-not` in the complementary basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, and checking that in the new basis the roles of the control and target are reversed. This illustrates the connection between the loss of phase coherence and "reduction of the state vector" — while the environment is "measuring" a certain observable A, its (Fourier) complement is busy "measuring" the state of the environment, and storing the information in the phases between the eigenstates of A — in the correlations with the states $|+\rangle$ and $|-\rangle$.

In idealised examples of decoherence (*i.e.*, in absence of the self-Hamiltonian) preferred observable is selected by the interaction Hamiltonian with the environment — it satisfies (or, at least, approximates) the commutation relation [11,12]:

$$[H_{\rm int}, A] \;=\; 0\,. \tag{4.2}$$

However, in more realistic circumstances involving dynamical evolution Eq. (4.2) defines only the "instantaneous" preferred (pointer) observable. Long-term predictability (which is a convenient and natural criterion of the more elusive "classicality") is maximized by the states which are least perturbed by the environment in spite of the incessant rotation between the observables and their complements [20,26,27].

A system — such as a harmonic oscillator or, for that matter, a chaotic quantum system — is then described by an effective *master equation* [28–31], which continuously transforms pure states into mixtures. The rate at which this happens is set in part by the coupling, but the nature of the initial state plays the decisive role. States which become least mixed are then most predictable and can be regarded as most classical. High-temperature master equation for a particle interacting with the thermal excitations of the environment composed of harmonic oscillators is a convenient and often studied example [29–31]. Density matrix $\rho(x, x')$ of the system in the position representation evolves in this case according to:

$$\dot{\rho} = \overbrace{\underbrace{-\frac{i}{\hbar}[H,\rho]}_{\dot{p}=-\text{FORCE}=\nabla V}}^{\text{von Neumann eq.}} - \overbrace{\underbrace{\gamma(x-x')(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'})\rho}_{\dot{p}=-\gamma p}}^{\text{relaxation}} - \overbrace{\underbrace{\frac{2m\gamma k_{\mathrm{B}}T}{\hbar^2}(x-x')^2\ \rho}_{\text{classical phase space}}}^{\text{decoherence}} . \tag{4.3}$$

Above, H is the effective Hamiltonian of the system (*i.e.*, with the potential renormalised to recognise the influence of the environment), and γ is the relaxation rate. Interaction Hamiltonian was assumed to couple coordinate x of the system with the coordinates of the environment oscillators. When the oscillators are collectively represented by a field $\phi(q)$, the coupling can be taken to have a form [30]:

$$H_{\mathrm{int}} = \varepsilon x \dot{\phi}(q,t)\,,$$

in which case effective viscosity $\eta = 2m\gamma = \varepsilon^2/2$.

An arbitrary superpositions of localized wavepackets can in principle exist in the Hilbert space, but, as a result of environmental monitoring — they are exceedingly unstable in practice: Continuous monitoring enforces *environment — induced superselection* [12]: Only some — relatively few — of the quantum states which can exist in principle are capable of surviving the interaction with the environment more or less intact. Which states can survive depends on the form of the interaction with the environment [11]. The general rule is that the states which are localized in the monitored observables are most stable [11,12,20,26,27]. Moreover, when the Hamiltonian of interaction is a function of some observable, then the environment is most effective in monitoring it. This singles out the preferred *pointer states* [11,17]. They usually turn out to be localized in position, since the interactions tend to depend on the distance [12].

Such tendency towards localization in position can be characterized by the time it takes for the two fragments of the wavepacket separated in space by the distance δx to loose quantum coherence. *Decoherence time* [14]:

$$\tau_D(\delta x) \;=\; \gamma^{-1}\left(\frac{\Lambda_{\mathrm{dB}}(T)}{\delta x}\right)^2 \tag{4.4}$$

can be computed from the third term of Eq. (4.3). It is proportional to the relaxation time $\tau_R = 1/\gamma$ (which determines the rate at which the system looses energy due to the interaction with the environment), but it is much faster: For a one gram object at the room temperature the ratio of the thermal de Broglie wavelength $\Lambda_{\mathrm{dB}}(T) = \hbar/\sqrt{2mk_{\mathrm{B}}T}$ to the separation $\delta x = 1$ cm is approximately 10^{-20}. Hence, in the above example, and under the circumstances in which thermal excitations dominate the process of decoherence (an assumption which allows one to derive Eq. (4.3) and the simple expression for the decoherence time, Eq. (4.4), but which does not effect the conclusion about its nearly instantaneous onset for macroscopic objects) $\tau_D \simeq 10^{-40}/\gamma$.

It follows that quantum coherence may be (and, for macroscopic objects, it is) lost exceedingly rapidly even when the relaxation time is very large. Preliminary experimental indication which corroborates these theoretical expectations are just at hand, following beautiful microwave cavity experiments [31]. More detailed studies of various aspects of decoherence are likely to follow in the wake of the ion trap "Schrödinger cat" experiments [32].

For the microscopic objects (such as an electron) and/or for microscopic separations, estimate of τ_D may become comparable to τ_R, and — when the isolation from the environment is sufficient — can be much larger than the characteristic dynamical timescales. The quantum nature of the evolution would then manifest itself unimpeded. But for Jupiter or Hyperion the opposite — *reversible classical limit* [17,20] with $\tau_D \lll t_{\mathrm{dynamical}} \lll \tau_R$ — is going to be enforced.

5. Exponential instability *vs.* decoherence

In a quantum chaotic system weakly coupled to the environment the process of decoherence briefly sketched above will compete with the tendency for coherent delocalization, which occurs on the characteristic timescale given by the Lyapunov exponent λ. Exponential instability will attempt to spread the wavepacket to the "paradoxical" size, while monitoring by the environment will attempt to limit its coherent extent. The two processes can be expected to fight one another into a standstill when their rates will become comparable:

$$\tau_D(\delta x)\ \lambda\ \simeq\ 1\,. \tag{5.1}$$

As the decoherence rate depends on δx, this equation can be solved for the critical, steady state coherence length, which turns out to be $\ell_{\mathrm{c}} \sim \varLambda_{\mathrm{dB}}(T) \times \sqrt{\lambda/\gamma}$.

A more careful analysis can be based on the combination of the Moyal bracket we have introduced in Section 2 with the master equation approach to decoherence we have just sketched. In many cases (including the situation of large bodies immersed in the environment of photons, rarefied gases, *etc.*, which are all present in the interplanetary medium) an effective approximate equation can be derived and translated into the phase space by performing Wigner transform on Eq. (4.3). Then:

$$\begin{aligned}\dot{W} \;&=\; \{H,W\}_{\mathrm{PB}} + 2\gamma\partial_p pW + D\partial_p^2 W \\ &\quad + \sum_{n\geq 1}\frac{\hbar^{2n}(-)^n}{2^{2n}(2n+1)!}\partial_x^{2n+1}V(x)\partial_p^{2n+1}W(x,p)\,. \end{aligned} \tag{5.2}$$

The second term causes relaxation, and, in the macroscopic limit, it can be made very small without decreasing the effect of decoherence caused by the third, diffusive term. Its role is to destroy quantum coherence of the fragments of the wavefunction between spatially separated regions. Thus, in effect, this decoherence term can assure that the Poisson bracket is always accurate: Diffusion in momentum prevents the Wigner function from becoming too finely structured in momentum, which — as we have seen early on in the paper, is the cause of the failure of the correspondence principle in chaotic quantum systems. In case of the thermal environment diffusion coefficient $D = \eta k_{\rm B} T$, where η is viscosity. The competition between these two effects — squeezing due to the chaotic instability and spreading due to diffusion — leads to a standoff reached when the structure of the Wigner function reaches critical scale

$$\sigma_{\rm c} \;=\; \sqrt{\frac{2D}{\lambda}}\,, \tag{5.3}$$

in momentum. This translates into the critical (spatial) coherence length of:

$$\ell_{\rm c} \;=\; \hbar/\sqrt{\frac{\lambda}{2D}} \;=\; \Lambda_{\rm dB}(T) \times \sqrt{\frac{\lambda}{2\gamma}}\,. \tag{5.4}$$

This nearly coincides with the quick estimate, Eq. (5.1).

For instance, for a planet of the size of Jupiter, chaotic instability on the four million year timescale and the consequent delocalization would be easily halted even by a very rarefied medium (0.1 atoms/cm^3, comparable to the density of interplanetary gas in the vicinity of massive outer planets) at a temperature of 100K (comparable to their surface temperature): The resulting $\ell_{\rm c}$ is of the order of 10^{-29} cm! Thus, decoherence is exceedingly effective in preventing the packet from spreading — $\ell_{\rm c} \lll \chi$, by an enormous margin. Hence, the paradox we have described in the first half of the paper has little chance of materializing in the macroscopic realm.

The example of quantum chaos in the solar system is a dramatic illustration of the effectiveness of decoherence, but its effects are, obviously, not restricted to celestial bodies: Schrödinger cats, Wigners friends, and, generally, all of the systems which are in principle quantum but sufficiently macroscopic will be forced to behave in accord with classical mechanics as a result of the environment-induced superselection [11,12].

This incredible efficiency of the environment in monitoring (and, therefore, localizing) states of quantum objects is actually not all that surprising. Dittrich and Graham have shown that either continuous quantum measurements or dissipation can help erase discord between quantum and classical evolutions in chaotic systems [34]. And we know (through direct experience) that photons are capable of maintaining an excellent record of the location of Jupiter (or any other macroscopic body). This must be the case, since we obtain our visual information about the rest of the Universe by intercepting a minute fraction of the reflected (or emitted) radiation with our eyes.

Given the efficiency of decoherence, can one ever expect to observe quantum manifestations of chaos? Characteristic coherence length $\ell_{\rm c}$ establishes the scale below which different fragments of the quantum wavepacket will be able to retain their relative phases. Hence, the system confined to less than $\ell_{\rm c}$ can exhibit quantum behavior. When the size of the system can be characterized by the scale of the nonlinearity χ (as is often the case), this argument results in the inequality:

$$\chi \ll \ell_{\rm c} \tag{5.5}$$

as a condition for quantum chaotic evolution. This requirement is obviously impossible to satisfy for the celestial bodies, but it should hold for sufficiently well isolated objects of atomic masses and microscopic dimensions.

6. The arrow of time: a price of classicality?

Decoherence is caused by the continuous measurement-like interactions between the system and the environment. Measurements involve transfer of information, and decoherence is no exception: The state of the environment acquires information about the system. For an observer who has measured the state of the system at some initial instant the information he will still have about it at some later time will be influenced (and, in general, diminished) by the subsequent interaction between the system and the environment. When the observer and the system monitor the same set of observables, information losses will be minimized. This is in fact the idea behind the *predictability sieve* [20] — an information-based tool which allows one to look for the einselected effectively classical states under quite general circumstances. When, however, the state implied by the information acquired by the observer either differs right away from the preferred basis selected by the environment, or — as will be the case here — evolves dynamically into such a "discordant" state, environment will proceed to measure it in the preferred basis, and, from the observers point of view, information loss will ensue.

The loss of information can be quantified by increase of the von Neumann entropy:

$$\mathcal{H} = -{\rm Tr}\, \rho \ln \rho\,, \tag{6.1}$$

where ρ is the reduced density matrix of the system. We shall now focus on the rate of increase of von Neumann entropy in a dynamically evolving system subject to decoherence. As we have seen before, decoherence restricts spatial extent of the quantum coherent patches to the critical coherence length ℓ_c, Eq. (5.3). A coherent wavepacket which overlaps a region larger than ℓ_c will decohere rapidly, on a timescale shorter than the one associated with the classical predictability loss rate given by the Lyapunov exponent λ. Such wavepacket will deteriorate into a mixture of states each of which is coherent over an area of dimension ℓ_c by $\sigma_c = \hbar/\ell_c$. Consequently, the density matrix can be approximated by an incoherent sum of reasonably localized and approximately pure states. When N such states contribute more or less equally to the density matrix, the resulting entropy is of the order of $\mathcal{H} \simeq \ln N$. Coherence length ℓ_c determines the resolution of the "environmental monitoring" on a chaotic quantum system. That is, by making an appropriate measurement on the environment one could in principle localize the system to within ℓ_c. As the time goes on, the initial phase space patch characterizing observers information about the state of the system will be smeared over an exponentially increasing range of the coordinate, Eq. (2.2).

When the evolution is reversible, such stretching does not matter, at least in principle: It is matched by the squeezing of the probability density in the complementary directions (corresponding to negative Lyapunov exponents). Moreover, in the quantum case folding will result in the interference – telltale signature of the long range quantum coherence, best visible in the structure of the Wigner functions.

Narrow wavepackets, and, especially, small-scale interference fringes are however exceedingly susceptible to the monitoring by the environment. Thus, the situation changes dramatically as a result of decoherence: In a chaotic quantum system the number of independent eigenstates of the density matrix will increase as:

$$N \simeq \ell(t)/\ell_c \simeq \frac{\hbar}{\Delta p_0 \ell_c} \exp(\lambda t) \,. \tag{6.2}$$

Consequently, von Neumann entropy will grow at the rate:

$$\dot{\mathcal{H}} = \frac{d}{dt} \ln \left(\frac{\ell(t)}{\ell_c} \right) \simeq \lambda \,. \tag{6.3}$$

This result is expected to be valid — strictly speaking — only asymptotically, in the decoherence - imposed classical limit $\chi \gg \ell_c$. It is a "corollary" of our discussion, and perhaps even its key result [8]. Decoherence will help restore the quantum–classical correspondence. But we have now seen that this will happen at a price: Loss of information is an inevitable consequence of the eradication of the "Schrödinger cat" states which were otherwise induced by the chaotic dynamics. They disappear because the environment is "keeping an eye" on the phase space, monitoring the location of the system with an accuracy set by ℓ_c.

Throughout this paper we have "saved" on notation, using "λ" to denote (somewhat vaguely) the rate of divergence of the trajectories of the hypothetical chaotic system. It is now useful to become somewhat more precise. A Hamiltonian system with $\mathcal{D}$ degrees of freedom will have in general many ($\mathcal{D}$) pairs of Lyapunov exponents with the same absolute value but with the opposite signs. These global Lyapunov exponents obtain by averaging local Lyapunov exponents, which are the eigenvalues of the local transformation, and which describe the rates at which a small patch centered on a trajectory passing through a certain location in the phase space is being deformed. The averaging of the local exponents into the global ones is achieved by following the trajectory of the system for a sufficiently long time.

The evolution of the Wigner function in the phase space is governed by the local dynamics. However, over the long haul, and in the macroscopic case, the patch which supports the probability density of the system will be exponentially stretched. The stretching and folding will produce a phase space structure which differs from the classical probability distribution because of the presence of the interference fringes, with the fine structure on the scale of the order $\hbar/\ell^{(i)}(t)$. In an isolated system this fine structure will saturate only when the envelope of the Wigner function will fill in the available phase space volume [35]. Monitoring by the environment destroys these small scale interference fringes and keeps W from becoming narrower than σ_c in momentum. As a result — and in accord with Equation (6.3) above — the entropy production will asymptote to the rate given by the sum of the positive Lyapunov exponents:

$$\dot{\mathcal{H}} = \sum_{i=1}^{\mathcal{D}} \lambda_+^{(i)} . \tag{6.4}$$

This result [8] is at the same time familiar and quite surprising. It looks familiar because it coincides with the Kolmogorov–Sinai formula for the entropy production rate for a *classical* chaotic system. It is surprising because it is independent of the strength of the coupling between the system and the environment, even though the process of decoherence (caused by the coupling to the environment) is the ultimate source of entropy increase.

We emphasize that this rate of entropy production can be expected only in the decoherence-imposed classical limit, that is when the scale of nonlinearities χ (which also coincides with the scale on which the folding takes place) is much greater than the scale ℓ_c on which the wave function is coherent, and, by the same token, where σ_c is much less than the range of momenta traversed by the system. Moreover, we have had to assume that the system is far from equilibrium, so that $\dot{\mathcal{H}}$ does not saturate. This assumption can be implemented in an elegant manner by considering the algorithmic information content of the record of evolution [36].

The conjectured independence of the entropy production rate on the strength of the coupling is indeed remarkable, and leads one to suspect that the cause of the arrow of time may be traced to the same phenomena which are responsible for the emergence of classicality in chaotic dynamics, and elsewhere (*i.e.*, in quantum measurements). In a sense, this is of course not a surprise: Von Neumann knew that the measurements are irreversible [21]. And Zeh [37] emphasized the close kinship between the irreversibility of the "collapse" in quantum measurements, and in the second law, and cautioned against circularity of using one to solve the other. What is however surprising is that the classical-looking result has ultimately quantum roots, and that these roots are so well hidden from view — that in the $\chi\sigma_{\rm c} \gg \hbar$ limit the entropy production rate depends solely on the classical Lyapunov exponents.

Environment may not enter explicitely into the entropy production rate, Eq. (6.4), but it will help determine when this asymptotic formula becomes valid. The Lyapunov exponents will "kick in" as the dimensions of the patch begin to exceed the critical size in the corresponding directions, $\ell^{(i)}(t)/\ell_{\rm c}^{(i)} > 1$. The instant when that happens will be set by the strength of the interaction with the environment, which determines $\ell_{\rm c}$. This "border territory" may be ultimately the best place to test the transition from quantum to classical. One may, for example, imagine a situation where the above inequality is comfortably satisfied in some directions in the phase space, but not in the others. In that case the rate of the entropy production will be lowered to include only these Lyapunov exponents for which decoherence is effective.

Serious numerical reconnaissance into this quantum–classical border territory is just at hand: Sarkar and his collaborators report [38] that the entropy production rate indeed saturated at a plateau, but note that the level of that plateau is; *(i)* generally below the value estimated from Eq. (6.4), and; *(ii)* that $\dot{\mathcal{H}}$ seems to depend on the strength of the coupling with the environment. That the rate of entropy production should asymptote to zero in the quantum limit, $\chi\sigma_{\rm c} \ll \hbar$, Eq. (5.5), is not unexpected: Quantum evolution in this limit becomes unitary. Thus, an increase of entropy with the increase of the coupling strength (and the resulting increase of $\sigma_{\rm c}$) is anticipated. It would be most interesting is to determine the behavior in the transition region — before the classical limit is attained — and to study the manner in which it is traversed.

Detailed maps of specific entropy production rates exhibit a quantum–classical correspondence which is remarkable both for the general similarity and because of the detailed differences. Another interesting result of that investigation is a generalization of the formula for entropy production to the case of significant damping. Obviously, much more remains to be done, both analytically and numerically, in anticipation of the experimental investigations which seem to be a matter of near future [39]. The goal of this pursuit is eminently worthwhile; nothing less than the elusive origins of the "arrow of time".

7. Decoherence, einselection, and the interpretation of quantum theory

The significance of the efficiency of decoherence goes beyond the example of the solar system or the task of reconciling quantum and classical predictions for classically chaotic systems. Every degree of freedom coupled to the environment will suffer loss of quantum coherence. Objects which are more macroscopic are generally more susceptible. In particular, the "hardware" responsible for our perceptions of the external Universe and for keeping records of the information acquired in course of our observations is obviously very susceptible to decoherence: Neurons are strongly coupled to the environment, and are definitely macroscopic enough to behave in an effectively classical fashion. That is, they have decoherence timescale many orders of magnitude smaller than the relatively sluggish timescale on which they can exchange and process information. As a result, in spite of the undeniably quantum nature of the fundamental physics involved, perception and memory have to rely on the decohered (and, therefore, effectively classical) degrees of freedom.

An excellent illustration of the seriousness of the constraint imposed on the information processing by decoherence comes from the recent discussions of the possibility of implementation of quantum computation: Decoherence is viewed as perhaps the most serious threat to the ability of a quantum information processing system to carry out a superposition of computations [40]. Yet, precisely such an ability to "compute" in an arbitrary superposition would be necessary for an observer to be able to "perceive" an arbitrary quantum state. Moreover, in the external Universe only these observables which are resistant to decoherence and which correspond to "pointer states" are worth recording: Records are valuable because they allow for predictions, and resistance to decoherence is a precondition to predictability [17,20].

An interesting insight into the fragility of quantum states of macroscopic systems comes from the realization that the average price of a bit of information decreases with the increasing size of the system. The reason for this is easy to understand. The least action required to correlate perfectly two

N-state quantum systems;

$$\left(\sum_{k=1}^{N}\alpha_k\ |c_k\rangle\right)|t_0\rangle \longrightarrow \sum_{k=1}^{N}\alpha_k\ |c_k\rangle|t_k\rangle\,, \tag{7.1}$$

which play roles of the control and target of Eq. (4.1) is given by:

$$\langle I\rangle = \sum_{k=1}^{N}|\alpha_k|^2 \arccos|\langle t_0|t_k\rangle|\,. \tag{7.2}$$

A successful correlation of the control and target requires orthogonality. Thus, at least in the limit of large N the average action should be:

$$\langle I_N\rangle \approx \frac{\pi}{2}\,. \tag{7.3}$$

This estimate can be lowered a little when N is small: For $N=2\langle I_2\rangle=\pi/4$ [11].

So, a fraction of a Planck per correlating (or entangling) operation, Eq. (7.1), seems to be a fair estimate of the least amount of action needed to bring the two systems into perfect correlation. One can be of course wasteful, and use more action, but it is impossible to be more frugal.

The information exchanged as a result of such an interaction is given by the von Neumann entropy of the reduced density matrix of one of the two systems. Hence;

$$\mathcal{H}\ \leq\ \log_2 N\,. \tag{7.4}$$

The most efficient transfer of information takes place when the inequality is saturated. In such a case, the *specific action* $\imath$ — action per bit of transferred information — can be as low as:

$$\imath\ = <I>/\mathcal{H} \simeq \frac{\pi}{2\log_2 N}\,. \tag{7.5}$$

The price of the bit decreases when they are traded wholesale — when the size of the system increases.

Logarithmic decrease of $\imath$ with the dimensionality of the Hilbert space N may seem far from dramatic. However, N increases exponentially with the physical size of the system (*i.e.*, with the number of spins, particles, *etc.*) Hence, $\imath$ will be of order $\hbar/4$ for small systems, but it will be infinitesimally small in the macroscopic domain.

We note that the scaling we have obtained does not depend on the assumption that the initial state was pure. Therefore, the price of a classical correlation (*i.e.*, information about the einselected pointer states) will behave in an essentially identical manner.

The action per bit $\imath$ we have estimated is the least required. Actual interactions will not be this efficient. Yet, it seems plausible that the trend exhibited by $\imath$ is indicative of the price of information in general. Therefore, in the macroscopic domain, the information seems to come for free, and it is easy not to notice that information carries a price — action — that it is physical, and that there is no information without (physical) representation.

The above argument — carried out more carefully elsewhere [44] — allows one to see the macroscopic-classical link in a new light. It is too early to claim that all the issues arising in the context of the transition from quantum to classical have been settled with the help of decoherence. Decoherence and einselection are, however, rapidly becoming a part of the standard lore [41,42]. Where expected, they deliver classical states, and — as we have seen above — guard against violations of the correspondence principle. The answers which emerge may not be to everyone's liking, and do not really discriminate between the Copenhagen Interpretation and the Many Worlds approach. Rather, they fit within either mold, providing the key missing elements — *i.e.*, the quantum–classical border postulated by Bohr, and the scheme for defining branches required by Everett [43,44].

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