Vol. 29 (1998) ACTA PHYSICA POLONICA B No 12

BRIDGING THE DIMENSIONAL GAP: FROM KINK IN ONE DIMENSION TO CURVED DOMAIN WALL IN THREE DIMENSIONS* **

H. Arodź

Institute of Physics, Jagellonian University
Reymonta 4, 30-059 Cracow, Poland

(Received October 10, 1998)

Improved expansion in width is applied to a curved domain wall in non-relativistic dissipative $\lambda(\Phi^2 - v^2)^2$ model with real scalar order parameter Φ. Approximate analytic description of such a domain wall to the second order in the width is presented.

PACS numbers: 11.27.+d, 02.30.Mv

1. Introduction

Physics of domain walls and vortices has been a rather interesting field of experimental as well as theoretical research for quite a long time [1]. Example of a recent hot topic is production of such soft solitonic objects in rapid phase transitions — testing and refining a theoretical description proposed by Kibble [2] and Żurek [3]. Theoretical analysis of dynamics of domain walls and vortices is relatively difficult because pertinent field equations are nonlinear, and the most interesting solutions do not belong to weak field sector.

Among problems which have been discussed in literature is time evolution of single curved domain wall or vortex. It is commonly regarded as accessible only by a numerical analysis. Actually, there exist also analytical approaches which yield (an approximate) description of the time evolution: the classical effective action (CEA) method which has been developed in a series of papers starting from [4,5], examples of more recent works are [6–8], and a version of Hilbert–Chapman–Enskog method which we call the improved expansion in width (IEW). This latter approach has been developed

* Presented at the XXXVIII Cracow School of Theoretical Physics, Zakopane, Poland, June 1–10, 1998.

** Paper supported in part by KBN grant 2P03B 095 13.

in papers [9], with an inspiration coming from [10]. The two methods have been outlined and compared in paper [11]. These two analytical approaches are not simple, but neither is the purely numerical approach — this is just a reflection of the fact that dynamics of the curved domain wall or a vortex is nontrivial due to nonlinearity, many spatial dimensions, and many modes of involved fields. The numerical and analytical approaches should be regarded as equally important and complementary sources of information about the dynamics.

In the present paper we apply IEW method to a curved domain wall in a nonrelativistic dissipative system. Time evolution is governed by a diffusion type equation for which no simple action functional exists. Therefore, it is not clear how CEA scheme could be applied in this case, while, as it turns out, IEW method works quite well. Because our second goal is a presentation of the method, we consider a relatively simple system with scalar order parameter. An application to domain walls in nematic liquid crystals we will present elsewhere [12]. IEW method has also been applied to a vortex line, see [13]. Work on application to a disclination line in a nematic liquid crystal is in progress [14].

In our opinion IEW method has several attractive features, *e.g.*, it combines the old and elegant subject of differential geometry of surfaces in 3-dimensional space with nonlinear dynamics of the curved domain wall. Another interesting aspect is that IEW scheme relates properties of the domain wall to properties of one-dimensional kink. In this sense IEW method embodies the idea that the curved domain wall can be regarded as three-dimensional embedding of the one-dimensional kink.

The expansion in width is based on the idea that transverse profile of the curved domain wall considered in suitable coordinates (which are called comoving coordinates) differs from transverse profile of a planar domain wall by small corrections which are due to curvature of the domain wall, and that these corrections can be calculated perturbatively. There is a condition for applicability of such a perturbative scheme: the two main curvature radia of the domain wall should be much larger than its width. As we shall see below, turning that idea into a concrete calculational scheme requires some work, but that should be expected in any approach which tackles generic curved domain walls.

The plan of our paper is as follows. In Section 2 we introduce the comoving coordinate system. Section 3 is devoted to the presentation of the perturbative expansion. Several remarks are collected in Section 4.

2. The comoving coordinates

We shall seek the curved domain wall solutions of the following equation

$$\gamma\frac{\partial\Phi}{\partial t}+\frac{\delta F}{\delta\Phi}=0\,, \tag{2.1}$$

where the free energy F has the form

$$F=\frac{1}{2}\int d^3x\left(\frac{\partial\Phi}{\partial x^\alpha}\frac{\partial\Phi}{\partial x^\alpha}+\lambda(\Phi^2-v^2)^2\right)\,. \tag{2.2}$$

Here Φ is a real scalar order parameter; $\lambda,\ v$ and γ are positive constants; and $(x^\alpha),\ \alpha=1,2,3$, are Cartesian coordinates in the usual R^3 space. From (2.1) and (2.2) we obtain the following equation for the rescaled dimensionless order parameter $\phi=\Phi/v$

$$\gamma\frac{\partial\phi}{\partial t}=\Delta\phi-2\lambda v^2\phi(\phi^2-1)\,. \tag{2.3}$$

The domain wall solutions of Eq. (2.3) smoothly interpolate between $\phi=-1$ on one side of the wall and $\phi=+1$ on the other side. Canonical example of such a solution is given by the formula

$$\phi_0(x^3)=\tanh\frac{x^3-a}{2l_0}\,, \tag{2.4}$$

where $l_0^{-2}=4\lambda v^2$ and a is an arbitrary constant. The presence of the constant a is due to translational invariance of Eq. (2.3). This particular solution represents planar static domain wall located at the plane $x^3=a$. Such domain wall is homogeneous along that plane. Its transverse profile is parametrized by x^3. Width of the wall is approximately equal to l_0, in the sense that for $|x^3-a|\gg l_0$ values of ϕ differ from $+1$ or -1 by exponentially small terms. The form (2.4) of $\phi_0(x^3)$ coincides with one-dimensional static kink present in one-dimensional version of the model defined by (2.1) and (2.2). The domain wall can be regarded as embedding of that kink in the 3-dimensional space R^3. Notice that somewhere inside any domain wall there is a surface on which ϕ vanishes. For example, in the case of planar domain wall (2.4) $\phi_0=0$ for $x^3=a$. Such surface is called the core of the domain wall.

The first step in our construction of the perturbative scheme consists in introducing special coordinates comoving with the domain wall. One coordinate, let us say ξ, parametrizes direction perpendicular to the domain wall, two other $(\sigma^1,\ \sigma^2)$ parametrize the domain wall regarded as a surface in the R^3 space. The comoving coordinates have been proposed in the context

of CEA method, [5]. We introduce one important modification: an auxiliary surface S, which is present in the definition of the comoving coordinates, is apriori independent of the domain wall. In literature on CEA method it is defined directly in terms of the domain wall — the most popular choice is that S coincides with the core. It turns out that the latter choice in general is not compatible with certain consistency conditions which appear in our approximation scheme. Transformations to comoving coordinates in the cases of relativistic domain walls and vortices in Minkowski space-time can be found in [9, 13], respectively. Below we introduce such coordinates for the nonrelativistic domain wall moving in the R^3 space.

After all these remarks let us finally define the comoving coordinates. We consider a smooth, closed or infinite surface S in the usual R^3 space. It is close to the core and its shape is similar to the shape of the domain wall. In particular one may assume that S coincides with the core at certain time t_0. Points of S are given by $\vec{X}(\sigma^i, t)$, where σ^i $(i = 1, 2)$, are two intrinsic coordinates on S, and t denotes the time — we allow for motion of S in the space. The vectors $\vec{X}_{,k}$, $k = 1, 2$, are tangent to S at the point $\vec{X}(\sigma^i, t)$[1]. They are linearly independent, but not necessarily orthogonal to each other. At each point $\vec{X}(\sigma^i, t)$ of S we also introduce a third vector $\vec{p}(\sigma^i, t)$ which is perpendicular to S, that is

$$\vec{p}\vec{X}_{,k} = 0.$$

We assume that $\vec{p}$ has unit length, $\vec{p}^2 = 1$. The three vectors $(\vec{X}_{,k}, \vec{p})$ form a local basis at the point $\vec{X}(\sigma^i, t)$ of S. With this basis given at each point of S, we introduce geometric characteristics of S: induced metric tensor on S

$$g_{ik} = \vec{X}_{,i}\vec{X}_{,k}\,,$$

and the extrinsic curvature coefficients

$$K_{il} = \vec{p}\vec{X}_{,il}\,.$$

$(i, k, l = 1, 2)$. They appear in Gauss–Weingarten formulas

$$\vec{X}_{,ik} = K_{,ik}\vec{p} + \Gamma^l_{ik}\vec{X}_{,l}\,, \quad \vec{p}_{,k} = -g^{il}K_{lk}\vec{X}_{,i}\,. \tag{2.5}$$

Here the matrix (g^{ik}) is by definition the inverse of the matrix (g_{kl}), *i.e.* $g^{ik}g_{kl} = \delta^i_l$, and Γ^l_{ik} are Christoffel symbols constructed from the metric g_{ik}. The two by two matrix (K_{ik}) is symmetric. Two eigenvalues k_1, k_2 of the matrix (K^i_j), where $K^i_j = g^{il}K_{lj}$, are called extrinsic curvatures of S at the point $\vec{X}$. The local curvature radia are defined as $R_i = 1/k_i$.

[1] We use the compact notation $f_{,k} \equiv \partial f/\partial\sigma^k$.

The comoving coordinates $(\sigma^1, \sigma^2, \xi)$ at the time t are introduced by the following formula

$$\vec{x} = \vec{X}(\sigma^i, t) + \xi \vec{p}(\sigma^i, t). \tag{2.6}$$

Here $\vec{x} = (x^\alpha), \alpha = 1, 2, 3$, are the usual Cartesian coordinates in the space R^3. ξ is the coordinate in the direction perpendicular to S. Notice that this direction has very simple parametrization — the r.h.s. of formula (2.6) is a linear function of ξ. We will use a compact notation for the comoving coordinates: $(\sigma^1, \sigma^2, \xi) = (\sigma^\alpha)$, with α=1,2,3 and $\sigma^3 = \xi$. The coordinates (σ^α) are just a special case of curvilinear coordinates in R^3. The corresponding metric tensor $G_{\alpha\beta}$ in R^3 has the following components:

$$G_{33} = 1, \;\; G_{3k} = G_{k3} = 0, \;\; G_{ik} = N_i^l g_{lr} N_k^r \,,$$

where

$$N_i^l = \delta_i^l - \xi K_i^l \,,$$

$i, k, l, r = 1, 2$. Simple calculations give

$$\sqrt{G} = \sqrt{g} N \,,$$

where $G = \det(G_{\alpha\beta})$, $g = \det(g_{\alpha\beta})$ and $N = \det(N_k^i)$ is given by the following formula

$$N = 1 - \xi K_i^i + \frac{1}{2}\xi^2 (K_i^i K_l^l - K_l^i K_i^l) \,.$$

Components $G^{\alpha\beta}$ of the inverse metric tensor have the form

$$G^{33} = 1, \;\; G^{3k} = G^{k3} = 0, \;\; G^{ik} = (N^{-1})_r^i g^{rl} (N^{-1})_l^k \,,$$

where

$$(N^{-1})_r^i = \frac{1}{N} \left((1 - \xi K_l^l) \delta_r^i + \xi K_r^i \right) .$$

We see that dependence of $G_{\alpha\beta}$ on the transverse coordinate ξ is explicit, and that σ^1, σ^2 appear through the tensors g_{ik}, K_r^l which characterise the surface S.

In general the coordinates (σ^α) have certain finite region of validity. In particular, the range of ξ is given by the smallest positive $\xi_0(\sigma^i, t)$ for which $N = 0$. It is clear that such ξ_0 increases with decreasing extrinsic curvature coefficients K_i^l, reaching infinity for the planar domain wall. We assume that the surface S (and the domain wall) is smooth enough, so that outside of that region there are only exponentially small tails of the domain wall which give negligible contributions to physical characteristics of the domain wall.

The comoving coordinates are utilised to write Eq. (2.3) in a form suitable for calculating the curvature corrections. Laplacian Δ in the new coordinates has the form

$$\Delta\phi = \frac{1}{\sqrt{G}}\frac{\partial}{\partial\sigma^\alpha}\left(\sqrt{G}G^{\alpha\beta}\frac{\partial\phi}{\partial\sigma^\beta}\right).$$

The time derivative on the l.h.s. of Eq. (2.3) is taken under the condition that all x^α are constant. It is convenient to use time derivative taken at constant σ^α. The two derivatives are related by the formula

$$\frac{\partial}{\partial t}|_{x^\alpha} = \frac{\partial}{\partial t}|_{\sigma^\alpha} + \frac{\partial\sigma^\beta}{\partial t}|_{x^\alpha}\frac{\partial}{\partial\sigma^\beta},$$

where

$$\frac{\partial\xi}{\partial t}|_{x^\alpha} = -\vec{p}\dot{\vec{X}}, \quad \frac{\partial\sigma^i}{\partial t}|_{x^\alpha} = -(N^{-1})^i_k g^{kr}\vec{X}_{,r}(\dot{\vec{X}} + \xi\dot{\vec{p}}),$$

the dots stand for $\partial/\partial t|_{\sigma^i}$. The final step consists in rescaling the transverse variable ξ

$$\xi = 2l_0 s.$$

The dimensionless variable s measures the distance from the surface S in the unit $2l_0$. Equation (2.3) transformed to the comoving coordinates with ξ rescaled as above has the following form

$$\begin{aligned} &2\gamma l_0^2\left(\tfrac{\partial\phi}{\partial t}|_{\sigma^\alpha} - \tfrac{1}{2l_0}\vec{p}\dot{\vec{X}}\tfrac{\partial\phi}{\partial s} - (N^{-1})^i_k g^{kr}\vec{X}_{,r}(\dot{\vec{X}} + 2l_0 s\dot{\vec{p}})\tfrac{\partial\phi}{\partial\sigma^i}\right) \\ &= \tfrac{1}{2}\tfrac{\partial^2\phi}{\partial s^2} + \phi - \phi^3 + \tfrac{1}{2N}\tfrac{\partial N}{\partial s}\tfrac{\partial\phi}{\partial s} + 2l_0^2\tfrac{1}{\sqrt{g}N}\tfrac{\partial}{\partial\sigma^j}\left(G^{jk}\sqrt{g}N\tfrac{\partial\phi}{\partial\sigma^k}\right), \end{aligned} \tag{2.7}$$

which is convenient for construction of the expansion in the width.

3. The expansion in the width

We seek domain wall solutions of Eq. (2.7) in the form of expansion with respect to l_0, that is

$$\phi = \phi_0 + l_0\phi_1 + l_0^2\phi_2 + \dots. \tag{3.1}$$

Inserting formula (3.1) in Eq. (2.7) and keeping only terms of the lowest order ($\sim l_0^0$) we obtain the following equation

$$\frac{1}{2}\frac{\partial^2\phi_0}{\partial s^2} + \phi_0 - \phi_0^3 = 0. \tag{3.2}$$

It has the well-known kink solutions

$$\phi_0 = \tanh(s - s_0)\,,$$

which formally have the same form as the planar domain walls (2.4). In the remaining part of the paper we shall consider curvature corrections to the simplest solution

$$\phi_0 = \tanh s\,. \tag{3.3}$$

Notice that ϕ_0 interpolates between the vacuum solutions ± 1. Therefore, the corrections $\phi_k,\ k \geq 1$, should vanish in the limits $s \to \pm\infty$.

Equations for the corrections $\phi_k,\ k \geq 1$, are obtained by expanding the both sides of Eq. (2.7) and equating terms proportional to l_0^k. They can be written in the form

$$\hat{L}\phi_k = f_k\,, \tag{3.4}$$

where

$$\hat{L} = \frac{1}{2}\frac{\partial^2}{\partial s^2} + 1 - 3\phi_0^2 = \frac{1}{2}\frac{\partial^2}{\partial s^2} + \frac{3}{\cosh^2 s} - 2\,, \tag{3.5}$$

and f_k depends on the lower order contributions $\phi_l,\ l < k$. Straightforward calculations give

$$f_1 = \partial_s\phi_0(K_r^r - \gamma\vec{p}\dot{\vec{X}})\,, \tag{3.6}$$

$$f_2 = 3\phi_0\phi_1^2 + 2s\partial_s\phi_0 K_j^i K_i^j + \partial_s\phi_1(K_r^r - \gamma\vec{p}\dot{\vec{X}})\,, \tag{3.7}$$

$$\begin{aligned} f_3 = {} & 2\gamma(\partial_t\phi_1 - g^{kr}\vec{X}_{,r}\dot{\vec{X}}\partial_k\phi_1) + 6\phi_0\phi_1\phi_2 + \phi_1^3 \\ & + 2s\partial_s\phi_1 K_j^i K_i^j - 2s^2\partial_s\phi_0 K_r^r\left((K_i^i)^2 - 3K_j^i K_i^j\right) \\ & - \frac{2}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k\phi_1) + \partial_s\phi_2(K_r^r - \gamma\vec{p}\dot{\vec{X}})\,, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} f_4 = {} & 2\gamma(\partial_t\phi_2 - 2sg^{ik}\dot{\vec{p}}\vec{X}_{,k}\partial_i\phi_1) - 2\gamma g^{jk}\vec{X}_{,k}\dot{\vec{X}}(\partial_j\phi_2 + 2sK_j^i\partial_i\phi_1) \\ & + 3\phi_0\phi_2^2 + 6\phi_0\phi_1\phi_3 + 3\phi_1^2\phi_2 + 2s\partial_s\phi_2 K_j^i K_i^j \\ & - 4s^3\partial_s\phi_0\left((K_r^r)^4 - (K_s^r K_r^s)^2 - 2(K_r^r)^2 K_j^i K_i^j\right) \\ & - 2s^2\partial_s\phi_1 K_r^r\left((K_i^i)^2 - 3K_j^i K_i^j\right) - \frac{2}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k\phi_2) \\ & - \frac{8s}{\sqrt{g}}\partial_j(\sqrt{g}K^{jk}\partial_k\phi_1) + 4sg^{jk}\partial_j K_r^r\partial_k\phi_1 + \partial_s\phi_3(K_r^r - \gamma\vec{p}\dot{\vec{X}})\,, \end{aligned} \tag{3.9}$$

where $\partial_t = \partial/\partial t,\ \ \partial_i = \partial/\partial\sigma^i$.

Notice that all Eqs. (3.4) for ϕ_k are linear. The only nonlinear equation in our perturbative scheme is the zeroth order equation (3.2).

Now comes the crucial point: operator $\hat{L}$ has a zero-mode, that is a normalizable function $\psi_0(s)$ such that

$$\hat{L}\psi_0 = 0\,.$$

This function can be obtained by differentiating $\phi_0(x^3)$ given by formula (2.4) with respect to $a/2l_0$ and putting $a = 0$,

$$\psi_0 = \frac{1}{\cosh^2 s}\,. \tag{3.10}$$

This zero-mode owes its existence to invariance of Eq. (3.2) with respect to translations in s, therefore it is often called the translational zero-mode. Let us multiply Eqs. (3.4) by $\psi_0(s)$ and integrate over s. Because

$$\int\limits_{-\infty}^{\infty} ds\psi_0\hat{L}\phi_k = 0\,,$$

we obtain the consistency (or integrability) conditions

$$\int\limits_{-\infty}^{\infty} ds\psi_0(s)f_k(s) = 0\,. \tag{3.11}$$

The operator $\hat{L}$ appears also in the expansion in width for relativistic domain walls [9]. Using standard methods [9,15] one can obtain the following formula for vanishing in the limits $s \to \pm\infty$ solutions ϕ_k of Eqs. (3.4):

$$\phi_k = G[f_k] + C_k(\sigma^i, \tau)\psi_0(s)\,, \tag{3.12}$$

where

$$G[f_k] = -2\psi_0(s)\int\limits_0^s dx\psi_1(x)f_k(x) + 2\psi_1(s)\int\limits_{-\infty}^s dx\psi_0(x)f_k(x)\,. \tag{3.13}$$

Here $\psi_0(s)$ is the zero-mode (3.10) and

$$\psi_1(s) = \frac{1}{8}\sinh(2s) + \frac{3}{8}\tanh s + \frac{3}{8}\frac{s}{\cosh^2 s} \tag{3.14}$$

is the other solution of the homogeneous equation

$$\hat{L}\psi = 0\,.$$

The second term on the r.h.s. of formula (3.12) obeys the homogeneous equation $\hat{L}\phi_k = 0$. It vanishes when $s \to \pm\infty$.

One can worry that ϕ_k, $k \geq 1$, given by formulas (3.12), (3.13) do not vanish when $s \to \pm\infty$ because the second term on the r.h.s. of formula (3.13) is proportional to ψ_1 which exponentially increases in the limits $s \to \pm\infty$. However, the integrals

$$\int_{-\infty}^{s} dx \psi_0 f_k$$

vanish in that limit, see the consistency conditions (3.11). Moreover, qualitative analysis of Eq. (2.7) shows that $f_k \sim$ (polynomial in s) $\times \exp(-2|s|)$ for large $|s|$, hence those integrals behave like (polynomial in s) $\times \exp(-4|s|)$ for large $|s|$ ensuring that ϕ_k exponentially vanish when $|s| \to \infty$.

We have explicitly solved Eqs. (3.4). The solutions (3.12) contain as yet arbitrary functions $C_k(\sigma^i, t)$, and also $\vec{X}(\sigma^i, t)$ giving points of the comoving surface S — K^i_l, g_{ik} follow from $\vec{X}$. It turns out that the conditions (3.11) are so restrictive that they essentially fix these functions. The consistency condition with $k = 1$ is equivalent to

$$\gamma \vec{p} \dot{\vec{X}} = K^r_r \,, \tag{3.15}$$

where we have used formulas (3.6), (3.10). Thus we have obtained equation for $\vec{X}$. It is of the same type as Allen–Cahn equation [16], but in our approach it describes motion of the auxiliary surface S only. Equation (3.15) should be compared with Nambu–Goto equation for a relativistic membrane obtained in the relativistically invariant version of our model [9]. We shall see below that the remaining consistency conditions do not give more restrictions for $\vec{X}$ at least up to the fourth order — they can be saturated by the functions $C_k(\sigma^i, t)$. We expect that this is true to all orders but we have not attempted to provide a proof.

Let us now proceed with the discussion of the perturbative corrections: Taking into account the condition (3.15) we have $f_1 = 0$. Therefore

$$\phi_1 = \frac{C_1(\sigma^i, t)}{\cosh^2 s} \,. \tag{3.16}$$

Equation (3.4) with $k = 1$ does not provide more information.

The second order contribution ϕ_2 is given by formula (3.12) with $k = 2$. Using the results (3.15), (3.16) from the first order we obtain the following formula

$$\phi_2 = \psi_2(s) C_1^2(\sigma^i, t) + \psi_3(s) K^i_j K^j_i + \frac{C_2(\sigma^i, t)}{\cosh^2 s} \,, \tag{3.17}$$

where

$$\psi_2(s) = -\frac{\sinh s}{\cosh^3 s}\,,$$
$$\psi_3(s) = \frac{-4}{\cosh^2 s}\int\limits_0^s dx \frac{x\psi_1(x)}{\cosh^2 x} + 4\psi_1(s)\int\limits_{-\infty}^s dx \frac{x}{\cosh^4 x}\,.$$

$\psi_3(s)$ can easily be evaluated, *e.g.*, numerically. Also the higher order corrections involve only rather simple integrals of elementary functions.

The consistency condition (3.11) with $k = 2$ does not give any restrictions — it can be reduced to the identity $0 = 0$. More interesting is the next condition, that is the one with $k = 3$. It can be written in the form of the following inhomogeneous equation for $C_1(\sigma^i, t)$

$$\gamma(\partial_t C_1 - g^{kr}\vec{X}_{,r}\dot{\vec{X}}\partial_k C_1) - \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k C_1) - K^i_j K^j_i C_1$$
$$= \frac{1}{2}(\frac{\pi^2}{6} - 1)K^r_r\left((K^i_i)^2 - 3K^i_j K^j_i\right)\,. \tag{3.18}$$

This equation determines C_1 provided that we fix initial data for it. The consistency condition coming from the fourth order ($k = 4$) is equivalent to the following homogeneous equation for C_2

$$\gamma(\partial_t C_2 - g^{kr}\vec{X}_{,r}\dot{\vec{X}}\partial_k C_2) - \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}g^{jk}\partial_k C_2) - K^i_j K^j_i C_2 = 0\,. \tag{3.19}$$

The perturbative scheme presented above is not quite straightforward. Therefore we would like to add several explanations. The formulas presented above give a whole family of domain wall solutions. To obtain one concrete domain wall solution we have to choose initial position of the auxiliary surface S. Its positions at later times are determined from Eq. (3.15). We also have to fix initial values of the functions C_1, C_2 and to find the corresponding solutions of Eqs. (3.18), (3.19). The approximate domain wall solution ϕ is given by formula (3.17). Notice that we are not allowed to choose the initial profile of the domain wall arbitrarily because the dependence on the transverse coordinate s is explicitly given by formulas (3.3), (3.16), (3.17). Any choice of the initial data gives an approximate domain wall solution. Of course such a choice should not lead to large perturbative corrections at least in certain finite time interval. Therefore one should require that at the initial time $l_0 C_1 \ll 1, l_0^2 C_2 \ll 1, l_0 K^i_j \ll 1$. The domain wall is located close to the surface S because for large $|s|$ the perturbative contributions vanish and the leading term $\tanh s$ is close to one of the vacuum values ± 1.

The presented formalism is invariant with respect to changes of coordinates σ^1, σ^2 on S. In particular, in a vicinity of any point $\vec{X}$ of S we can choose the coordinates in such a way that $g_{ik} = \delta_{ik}$ at $\vec{X}$. In these coordinates Eq. (3.15) has the form

$$\gamma v = \frac{1}{R_1} + \frac{1}{R_2}\,, \tag{3.20}$$

where v is the velocity in the direction perpendicular to S and R_1, R_2 are the main curvature radia of S at the point $\vec{X}$.

Let us present a simple example: take S to be a sphere of radius R. Then $R_1 = R_2 = -R(t)$, $v = \dot{R}$ and Eq. (3.20) gives

$$R(t) = \sqrt{R_0^2 - \frac{4}{\gamma}(t - t_0)},$$

where R_0 is the initial radius. Our approximate formulas are expected to be meaningful as long as $R(t)/l_0 \gg 1$. Equations (3.18), (3.19) reduce to

$$\gamma\partial_t C_1 - \frac{1}{R^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta C_1) + \frac{1}{\sin^2\theta}\partial_\phi^2 C_1\right) - \frac{2}{R^2}C_1 = 2(\frac{\pi^2}{6} - 1)\frac{1}{R^3}\,,$$
$$\gamma\partial_t C_2 - \frac{1}{R^2}\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta C_2) + \frac{1}{\sin^2\theta}\partial_\phi^2 C_2\right) - \frac{2}{R^2}C_2 = 0\,.$$

In the last equation θ, ϕ are the usual spherical coordinates on S (we apologize for using letter ϕ also in this meaning). If we take the simplest initial data at $t = t_0$, namely $C_1 = C_2 = 0$, then the last equation implies that $C_2 = 0$ also for $t > t_0$ while $C_1 > 0$. The Cartesian coordinate frame is located at the center of the sphere S and $\vec{p}$ is the outward normal to S; $s = (r - R(t))/2l_0$, where r is the radial coordinate in R^3.

4. Remarks

We would like to make several general remarks about the expansion in width.

1. We have used l_0 as a formal expansion parameter. It is a dimensionful quantity, hence it is hard to say whether its value is small or large. What really matters is smallness of the corrections $l_0\phi_1$, $l_0^2\phi_2$. This is the case if $l_0 C_1 \ll 1$, $l_0^2 C_2 \ll 1$ and $l_0 K_j^i \ll 1$, as seen from formulas (3.1), (3.16) and (3.17).

2. Notice that an assumption that S coincides with the core for all times in general would not be compatible with the expansion in width. If we

assume that $C_1 = 0 = C_2$ at certain initial time t_0, Eq. (3.18) implies that $C_1 \neq 0$ at later times unless the r.h.s. of it vanishes. Then, it follows from formulas (3.1), (3.3) and (3.16) that ϕ does not vanish at $s = 0$, that is on S.

3. The question of convergence of the expansion (3.1) has not been analysed. Actually we think that the expansion can turn out to be convergent, in spite of the fact that more frequent in field theory are asymptotic expansions. Moreover, this problem seems to be within the reach of the present day mathematical techniques.

4. Finally, we would like to stress that we have abandoned effects which come from perturbations of the exponential tails of the domain wall. For example, if we have a domain wall in the form of cylinder with very large radius and small height (and with rounded edges), then the top and bottom parts are flat, and according to Eq. (3.20) they do not move. In our approximation the domain wall shrinks from the sides where the mean curvature $1/R_1+1/R_2$ does not vanish. Now, in reality the top and bottom parts interact with each other. This interaction is very small only if the two flat parts are far from each other. We have neglected it altogether assuming the $\tanh s$ asymptotics for large s. Thus, our approximate solution takes into account only the effects of curvature.

I would like to thank the organizers for the possibility to present this work, and for stimulating atmosphere during the School.

REFERENCES

[1] For introduction to the topic see, e.g., P.M. Chaikin, T.C. Lubensky, *Principles of Condensed Matter Physics*, Cambridge University Press, 1995.

[2] T.W.B. Kibble, *J. Phys.* **A9**, 1387 (1976); *Phys. Rep.* **67**, 183 (1980).

[3] W. Żurek, *Nature* **317**, 505 (1985); *Acta Phys. Pol.* **B24**, 1301 (1993); *Phys. Rep.* **276**, 178 (1996).

[4] H.B. Nielsen, P. Olesen, *Nucl. Phys.* **B61**, 45 (1973).

[5] D. Förster, *Nucl. Phys.* **B81**, 84 (1974).

[6] R. Gregory, *Phys. Lett.* **B206**, 199 (1988).

[7] S.M. Barr, D. Hochberg, *Phys. Rev.* **D39**, 2308 (1989).

[8] M. Anderson, F. Bonjour, R. Gregory, J. Stewart, *Phys. Rev.* **D56**, 8014 (1997).

[9] H. Arodź, *Nucl. Phys.* **B450**, 174 (1995); *Nucl. Phys.* **B509**, 273 (1998).

[10] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland Publ.Comp., Amsterdam 1987, Chapt.8,§7.

[11] H. Arodź, talk given at the Conference "Particles, Fields and Gravitation", Łódź, April 15–19, 1998. To be published in the series American Institute of Physics Conference Proceedings.

[12] H. Arodź, in preparation.

[13] H. Arodź, *Nucl. Phys.* **B450**, 189 (1995); H. Arodź, A.L. Larsen, in preparation.

[14] H. Arodź, R. Pełka, work in progress.

[15] G.A. Korn, T.M. Korn, *Mathematical Handbook*, IInd Edition, McGraw-Hill Book Comp., New York 1968, Chapt.9.3-3.

[16] S.M. Allen, J.W. Cahn, *Acta Metall.* **27**, 1085 (1979); A.J. Bray, *Adv. Phys.* **43**, 357 (1994).