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PARTICLE-HOLE ASYMMETRY IN THE BCS THERMODYNAMICS* **

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It has been shown that the particle-hole asymmetry (PHA) of DOS leads to the first-order phase transition, a small deviation from the Luttinger theorem, and to very strange behaviour of subcritical specific heat. Because of the accuracy of the BCS thermodynamics in the thermodynamic limit (Bogolubov) it is strange that in trying to strengthen the theory while taking into account the tendency of DOS, we are in fact causing the deterioration of the theory. The answer lies in the retardation of the electron-phonon interaction for low temperature superconductors. Hence, if some elements of the BCS theory are applied for HTSC, it becomes necessary to be very careful in the question of thermodynamic properties. Moreover, the criteria of stability of the superconducting state has been formulated, at constant p and V as well, for one-component superconductors and isotropic Fermi superfluids. These criteria are free of the strong connection with the BCS model, they are purely thermodynamical. It is also shown that for the superconducting/superfluid Fermi systems the specific heat at constant p and V differ substantially, in contrast to any other low-temperature systems.

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1. Introduction. The outline of the problem

The basic purpose of this paper is to explain how to correlate the band structure with the BCS pairing mechanism [1]. More exactly, it will be done for the truncated BCS Hamiltonian, being the result of the elimination of the phonon amplitudes by some averaging procedure. In the original paper [1] the density of states (DOS) is used which is energy-independent

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in the pairing area $|\varepsilon - \mu| \leq \hbar\omega_D$, where ε denotes the electron energy, μ — its chemical potential and $\hbar\omega_D$ — the Debye energy. On the other hand, it is clear that the chemical potential cannot be a reference point for DOS, because μ is the thermodynamic function. This pseudoparadox is solved here by the particle-hole symmetry (PHS) (for the formal introduction *cf.* [2], *cf.* also [3]). The thermodynamics, at least for isotropic systems is fully determined by DOS, [4]. In this case, PHS means that $\nu(\varepsilon_F - \varepsilon) = \nu(\varepsilon - \varepsilon_F)$, where ε_F is the Fermi energy and ν the DOS, say per unit volume, as a function of the energy. If this equality is not fulfilled, then we deal with PHA. It is clear that this equality cannot be fulfilled in the case of the paper [1], because there electron energy has the bottom, *i.e.* such ε_0 that $\nu(\varepsilon) = 0$ for $\varepsilon < \varepsilon_0$, but does not have the top. The exact PHS is possible only in the finite system of bands such that the function $\nu(\varepsilon)$ has the center of symmetry and the Fermi energy coincides exactly with this point. Because the distribution of particles over states has the the form:

$$n(\varepsilon) = \tfrac{1}{2}[1 - \xi \tanh(E_\xi/2T)/E_\xi]\,, \tag{1}$$

where $E_\xi^2 = \xi^2 + \Delta^2$, $\xi \equiv \varepsilon - \mu$, T is the temperature and Δ — the isotropic energy gap, thus at half-filling of the band μ is exactly equal to ε_F and is temperature-independent. It is true for superconducting, $\Delta \neq 0$, as well for normal systems, $\Delta = 0$. In this case, the symmetric cut-off, with respect to ε_F, of the pairing area has been assumed. On the other hand, it is easy to show that for generalized DOS applied in [1], but still symmetric with respect to ε_F, $\mu = \varepsilon_F + O[\exp(-\hbar\omega_D)/T]$ and last term is usually neglected in the weak-coupling theory. Hence, we apply the cut-off used in [1], because $\mu = \varepsilon_F$. These considerations illustrate the idea which can be called *zeroth principle of theoretical physics*. Namely, if some quantity is equal to zero or two quantities are equal to one another, then we are forced to explain this fact in words. Now we know that PHS is responsible for the formula $\mu = \varepsilon_F$, and for the normal and superconducting systems as well.

As it is clear from the summary, the presented topic leads to quite unexpected results. In the oral presentation, it is a unique privilege of a lecturer to pull a rabbit out of a hat. It is possible even in a short presentation [5]. But, I think, it is improper in the systematic approach, to the lecture.

For the PHS case, $\delta\mu \equiv \mu_s - \mu_n$, where μ_s and μ_n denote the chemical potentials of the superconducting and the normal phases, respectively, is exactly equal to zero. In the case of PHA $\delta\mu = O(\Delta^2)$. In fact, it cannot be $O(\Delta)$, as a result of gauge invariance of the theory. On the other hand, there is no special reason that the first order term of the series development with respect to Δ^2 of μ_s vanishes. The last quantity is obtained from the particle number balance, *i.e.* from the condition that DOS multiplied by Eq.(1) and integrated over the energy gives the real particle number. The

attempt to do $\delta\mu = O(\Delta^4)$ leads to a condition imposed on the parameters which describe PHA and the coupling constant. This condition is slightly different at low temperatures, $T \ll T_c$, where T_c is the critical temperature, and subcritical temperatures, as we will see.

Let us do a little thermodynamics. If $\delta\mu$ is equal to zero then $\delta\Omega \equiv \delta(-pV)$ is the difference of the free energies in the superconducting and normal states, [4]. It is convenient to express $\delta\Omega$ by the function $f(T,\Delta)$ so that $\delta\Omega = -(\partial N/\partial\mu)_n f(T,\Delta)$, where N is the particle number and the subscript "n" denotes that we deal with the derivative in the normal state, at constant V and T. Taking the difference of the free energies per particle, one finds that:

$$\delta\mu(T,\rho) - (\partial \ln\rho/\partial\mu)_n f(T,\Delta) < 0, \tag{2}$$

[4], where the inequality is the stability condition of the superconducting phase at constant V, $\rho = N/V$. The quantity $\delta\mu$ was taken at constant ρ, because the free energy is the thermodynamic potential at constant V and the particle number is conserved. In superconductors, in a vanishing external magnetic field, the lattice plays the role of the piston, sustaining the constant volume, *cf. e.g.* [6], at least for usual superconductors.

The quantity $\delta\Omega$ is always negative at $\Delta \neq 0$ and DOS constant in the pairing area, [2, 6]. This quantity preserves its form, characteristic for energy-independent DOS, provided that PHA terms of DOS do not tend to infinity too strongly if the energy tends to infinity. It is so, because in the integral determining $\delta\Omega$, subintegral function, tends suitably quickly to zero and the integral does not need any cut-off. When the PHA term of DOS is linear with respect to $\varepsilon - \varepsilon_F$, the influence of such a term on $\delta\Omega$ vanishes as a result of bilateral symmetry with respect to $\varepsilon - \varepsilon_F$. Such PHA terms will be considered in a later part of this work.

Let us consider a system at constant p. Formally, the vast majority of experiments in low-temperature solid state physics are performed with a constant p. On the other hand, in some cases solids preserve their form and, hence, the experiment is performed with a constant V.

For the electrons, a constant p denotes, as yet very hypothetical case of soft lattice, irresistible with respect to the external pressure. On the other hand, the experiments on the superfluid ^{3}He are possible though more difficult, at a constant p.

At a constant p, the difference of the thermodynamic potentials in the superconducting and normal states, taken per particle is $\delta\mu(T,p)$, [4]. We have:

$$\delta\mu(T,p) \equiv \mu_s(T,\rho) - \mu_n(T,\rho') = \delta\mu(T,\rho) + (\partial\mu/\partial\rho)_n\delta\rho + O(\Delta^4),$$

where $p_s(\rho) = p_n(\rho')$ and, consequently, $\delta\rho = \rho - \rho'$. Because now $\delta\Omega =$

$-p\delta V = vpV\delta\rho$, $v = \rho^{-1}$, as the variation appears at constant p, thus

$$\delta\mu(T,p) = \delta\mu(T,\rho) - f(T,\Delta)(\partial \ln p/\partial\mu)_n < 0 \tag{3}$$

is the stability condition of the superconducting state at constant p. In the proof of Eq.(3), the fact that the density of the normal Fermi system integrated over the virtual Fermi energy up to the actual one, gives the pressure of the system with the accuracy $O(T^2)$, [4]. At the weak coupling limit, this correction is actually quite unimportant.

Let us go on to our bands and DOS. Besides the classical BCS model, with the pairing appearing in the narrow band of electrons, $|\varepsilon - \varepsilon_F| < \hbar\omega_D$, we will also consider a band of fully paired electrons. Such a band was introduced for the consideration of electrons paired in a narrow band characteristic for HTSC [7,8]. Theorists working with itinerant low-temperature electrons have been corrupted by the easy simplicity of the thermodynamics of the degenerate Fermi gas, and still have the tendency to express the subsequent corrections to the main result by subsequent derivatives of DOS. Unfortunately, it does not appear for the gas with pairing, because of the nonexponential tail of the particle distribution function (1). Hence, DOS per unit volume will be chosen in the form:

$$\begin{array}{ll} 2N(0)(1+a\varepsilon), & (i) \\ 2N(\varepsilon_F)[1+a(\varepsilon-\varepsilon_F)], & (ii) \end{array} \tag{4}$$

for the band considered in refs [7,8], and for the band considered in Ref. [1], respectively. In the case *(i)*, it is convenient to choose the bandwidth as our unit of energy; hence $a \geq -1$. On the other hand, in the model *(ii)*, DOS at $|\varepsilon - \varepsilon_F| \geq \hbar\omega_D$ is quite arbitrary and $|a|\hbar\omega_D \leq 1$. PHS in the model *(i)* denotes that $\varepsilon_F = 1/2$ and $a = 0$, whereas in *(ii)* only that $a = 0$. It is easy to see that at $a \neq 0$ the quantity contains the nonanalitycal term $O(\Delta^2 \ln \Delta)$, enhancing the first term common to Eqs (2,3) with respect to the second ones. Disregarding this, in refs [7,8] the erroneous result concerning μ_s was obtained.

It is worth emphasizing that Eqs (2,3) as well as the inequalities therein were obtained in a purely thermodynamic base, besides a concrete model of superconductivity. The inequalities (2,3) should be fulfilled for superconductivity to be stable at constants V and p, respectively. Usually, in the theory of HTSC, theorists treat the superconductivity as a noble state, in contrast to the normal one, that it would stable besides any inequality to be fulfilled. This author knows only three papers discussing the stability of the superconducting versus normal state: the paper [1] and two his own [9,10].

At $T = 0$, PHA leads to $\mu_s = \varepsilon_F + O[\Delta^2(a \ln \Delta + b)]$, where b is proportional to $(1 - 2\varepsilon_F)$ for *(i)*, and vanishes for the *(ii)* model, respectively. It

denotes, under PHA, the small deviation from the Luttinger theorem [11]. This theorem, proved for normal Fermi liquids, extended its validity to superconducting Fermi liquids with the S-pairing [12] and also on the pseudoisotropic Fermi liquids with the P-pairing, [13]. This pairing, BWV, has been determined in the papers [14, 15]. In the papers [12, 13], the result is a consequence of PHA. To the best of the author knowledge, the fact that we deal with the violation of the Luttinger theorem for DOS (4), *(ii)*, has been remarked in the note [14] though without the words PHA. This author, independently of Ref. [3], remained in the position of Monsieur Jourdain who did not know that he spoke in prose.

In fact, we deal with the interaction Hamiltonian in a separable form $-\lambda B^{+}B/V$, where λ is the coupling constant, V the volume of the system. Also, we deal with:

$$B \equiv \sum_{\boldsymbol{p}} a_{\boldsymbol{p}\uparrow} a_{-\boldsymbol{p}\downarrow}, \tag{5}$$

where the summation over $\boldsymbol{p}$ runs over the momenta in the pairing area. In the formula (5), the operator $a_{\boldsymbol{p}\uparrow}$ is the annihilation operator of the fermion with the momentum $\boldsymbol{p}$ and the spin up, $+$ denotes the Hermitean conjugate and the arrow $\downarrow$ – the spin down. Hence, we are restricted to the S-pairing from the point of view of momenta and spins as well.

The extension of this result to other types of pairing as well as nonseparable interactions is very easy. In this case, the equation for the chemical potential is firmed from the equality of the particle density, ρ, to the product of the functions (1) and (4) integrated over the whole range of one-particle energies. In this case, the gap energy is equal to Δ only in the pairing area; out of it it should be set equal to zero [1, 6]. Note that in the case *(i)* the entire band is the pairing area. The gap equation, for our separable interaction, has the form of the equality of the inverse coupling constant, λ^{-1}, and the integral over the pairing area of the product of the function (4) and the function $\tanh(E_\xi/2T)/E_\xi$. Hence, it is convenient to introduce the dimensionless coupling constant, $\lambda N(0)$, *(i)*, and $\lambda N(\varepsilon_F)$, *(ii)*. In this case, $N(0)$, $N(\varepsilon_F)$ passes to unity whereas λ – into the dimensionless coupling constant, κ. It is easy to see that the equations for the chemical potential, value of the energy gap, and the thermodynamic potential Ω preserve the same form for the pseudoisotropic BWV pairing.

As we will see in the model *(ii)*, the expression of the gap at $T = 0$, $\Delta(0)$, through κ and $\hbar\omega_D$ remains at $a \neq 0$ as it does for $a = 0$. It is a result of the bilateral symmetry with respect to $|\varepsilon - \varepsilon_F|$. Such a situation does not appear in the model *(i)*, even if we take into account that DOS on the Fermi surface per unit spin and volume is not in this case $N(0)$ but $N(0)(1+a\varepsilon_F)$. Note that ε_F is in this instance the variable determined by the value of doping. The ratio $T_c/\Delta(0)$, where T_c is the critical temperature such that

$\Delta(T_c) = 0$, still preserves its value given in Ref. [1] if one neglects the strong coupling corrections. In the subcritical region, the order parameter, expressed in terms of T_c and the parameter a preserves its classical form [1,6] with accuracy up to $O[(aT_c)^2]$ in the expansion coefficients. This result, in the light of the theory of similarity and dimension, one can understand due to the fact that aT_c is the single dimensionless parameter. On the other hand, the square appears by the virtue of the gauge invariance. The parameter a can be of the order of $1/\hbar\omega_D$ or $1/\varepsilon_F$, provided that the electron-phonon interaction influences or does not influences, respectively, the band energy. Hence, the correction $O[(aT_c)^2]$ is negligibly small in the weak coupling limit. As a result, in the subcritical region $\Delta^2 = O(\tau)$, where $\tau \equiv (T_c - T)/T_c$. In turn, because $\delta\mu = O(\Delta^2)$, the subcritical $f(T, \Delta)$ is $O(\tau^2)$, [1, 6], and the formulae (2) and (3) represent the thermodynamical potentials, per particle, at constant V and p, respectively, thus we deal with the first order phase transition.

For such quantities as these, the subcritical stability condition is $\delta\mu(T, \rho) < 0$, at constant V and p as well [2, 3]. Hence, simple thermodynamics, [4], shows that the latent heat of the transition from the superconducting to the normal state is always negative at PHA by virtue of the stability conditions. The same sign of the latent heat appears in the phase transition of superconductors but in the external magnetic field, *cf. e.g.* [6]. In this case, because of the Meissner effect and the full penetration of the normal metal by the *dc* magnetic field, the condition of the transition has the form $\delta f = H_c^2(T)/8\pi$, [6]. Here δf denotes the variation $s \to n$ of the thermodynamic potentials per unit volume and $H_c(T)$ – temperature dependent critical field. The negative sign of the latent heat is caused here by diminishing $H_c(T)$ as a function of temperature. The subcritical stability condition, $\delta\mu = O(\Delta^2) < 0$, has a nonanalitycal dependence on T_c, leading to the inverse coupling constant κ in the coefficient at Δ^2 in $\delta\mu$. That and the negative sign of the term at τ^2 in the subcritical expansion of Δ^2, [1, 6], lead to serious troubles with the subcritical specific heat of the system. Namely, if the parameter a is $O(1/\hbar\omega_D)$ or $O(1/\varepsilon_F)$ as it should be, then because of the proportionality of $\delta\mu$ to $1/\kappa$, the total specific heat of the Fermi system can attain negative values in the subcritical region. Note that it is easier for this to occur at very small values of κ. The negative specific heat is possible only for systems with long-range forces, where the sum of the energies of the macroscopic parts of the system differs from the total energy. For our system, that contradicts the thermodynamics. The negative specific heat corresponds to the maximum of the thermodynamic potential, not to the minimum as should be the case [4]. If we represent the canonical statistical sum as a Laplace transform of DOS of the whole system, then the specific heat at constant V is equal to the squared dispersion

energy divided by T^2, [4], provided that we are able to differentiate within the integral. As a result of the Leibniz theorem, this is possible only if DOS of an entire system suffers the discontinuity.

Because $f(T, \Delta)$ is $O(\Delta^2)$ at low temperatures and $O(\Delta^4)$ at subcritical temperatures whereas $\delta\mu$ is always $O(\Delta^2)$, there exists a region in which inequalities (2) and (3) are fulfilled at low T but are not fulfilled at subcritical T. Hence, the superconductivity is stable at low T, but ceases to be stable at some $T < T_c$. In such cases, the unique reaction of the system is to attain the minimum of the proper thermodynamic potential. It results in the jump from $\Delta \neq 0$ to $\Delta = 0$, to the normal system. This jump is accompanied by the jump of μ. We deal with the first order phase transition including the jump of the order parameter and the impossibility for "s" and "n" phases to coexist. Such kinds of transitions were called the *sudden death* of superconductivity in football/soccer terms [10]. A full description of this effect has been performed in Ref. [5]. The jumps of the order parameter Δ have been observed in several cases, [16, 17]. They were observed either directly [16] or via the jump of the penetration depth of the magnetic field *e.g.* in In_3Au. That was caused by nuclear ferromagnetism appearing in milikelvin temperatures [17]. Of course, it is rather impossible for the jump of the penetration depth to appear without the the jump of Δ. To summarize our results on the jump of the order parameter and the negative subcritical specific heat we can say that at a large variety of parameters, a, κ and ε_F the system suffers one of these diseases, with the last being the most serious.

Unfortunately, we are unable to simply reject our result because of unsatisfactory approximation methods. It was shown by Bogolubov [18] that the BCS solution [1] is exact in the thermodynamic limit, N, $V \to \infty$ so that $N/V \to \text{const} \neq 0$ (*cf.* also the monograph [19]). Note that in the proof [18, 19], the interaction Hamiltonian can even be a nonseparable one, and there is no condition under which DOS is constant or fulfills PHS in the area of pairing. It is easy to understand why this coincidence occurs following the ideas of Bogolubov, Zubarev and Tserkovnikov, [20], for our simple interaction Hamiltonian. Our interaction Hamiltonian can be represented in the form:

$$-\lambda\langle B\rangle(B^+ + B)/V - \lambda\langle B\rangle^2/V - \lambda(B^+ - \langle B\rangle)(B - \langle B\rangle)/V\,,$$

where $\langle B\rangle$ is some real number. Next, adding the interaction Hamiltonian in the above form the band energy of the electrons, one should perform the thermodynamic perturbation procedure for the grand partition function, with respect to the last term. In turn, $\langle B\rangle$ is chosen so that it is equal to the thermodynamic grand partition average of the operators B and B^+, with the Hamiltonian being the sum of the two first terms above plus the band term. Note that it is possible to make the average value of the operators B

and B^+ equal to one another using the gauge transformation of the Fermi operators. If this is chosen, the quantities

$$a^+_{\boldsymbol{p}\uparrow}a_{\boldsymbol{p}\uparrow} - a^+_{-\boldsymbol{p}\downarrow}a_{-\boldsymbol{p}\downarrow},$$

i.e. seniority operators, [21], commute with the basic Hamiltonian in our perturbation procedure, and the perturbation terms, in all orders, will not contribute to the potential Ω, because they do not diverge as V does in the thermodynamic limit. This is so, because for our basic Hamiltonian the thermodynamical Wick's theorem still holds [22]. Moreover the unique averages unequal to zero are: $\langle a^+_{\boldsymbol{p}\sigma}a_{\boldsymbol{p}\sigma}\rangle$, $\sigma = \uparrow, \downarrow$, and $\langle a_{\boldsymbol{p}\sigma}a_{-\boldsymbol{p}-\sigma}\rangle$, $\langle a^+_{-\boldsymbol{p}-\sigma}a^+_{\boldsymbol{p}\sigma}\rangle$. These selection rules diminish the number of summations over $\boldsymbol{p}$ in the parametrically ordered powers of our perturbation term. In turn, it leads to the prevailing of the terms V in the denominator and to our conclusion. It is very easy to extend this proof to nonseparable interactions.

Let us explain why the "exact" theory constitutes to deteriorate if the energy dependence of DOS is taken into account. As it was mentioned, the unpleasant effects connected with the specific heat become relatively stronger if $\kappa \to 0$. Hence, the strong coupling is not a proper reason. Our results are exact for the instantaneous interaction, without any retardation, *i.e.* at $\Delta\varepsilon \ll \hbar/t$. Here $\Delta\varepsilon$ is the energy interval at which DOS is changed substantially, and t is the retardation time. The $\Delta\varepsilon$ is of order of $\hbar\omega_D$ or ε_F, for the DOS influenced by the electron-phonon interaction or not, respectively. We find the relation $v_F\Delta p \sim \hbar\omega_D$, where v_F is the Fermi velocity and Δp the radial uncertainty of the electron momentum [6]. This estimation is the result of the characteristic energy denominator in the element of the phonon induced interelectron interaction as well as the growth of DOS for phonons with ω as ω^2. From this estimation and from the Heisenberg relations, we find the range of such interelectron interaction, $\Delta r \sim v_F/\omega_D$. By dividing this quantity by the velocity of phonons v_ϕ we obtain the retardation time $t \sim v_F/v_\phi\omega_D$. By substituting this quantity into the strong inequality which must be fulfilled, we find that the opposite strong inequality holds, because $v_F/v_\phi \gg 1$. Hence, it is impossible for the electrons to feel the difference in the value of DOS. To some extent, the electrons average their DOS in the pairing domain, *(ii)*. As a result, it is impossible to consider the details of the band structure with phonon mediated interelectron attraction out of the Eliashberg equations, [23]. All the above considerations concern the case *(ii)*, they are not true for *(i)*, if the mechanism of the interelectron attraction is undetermined. Note that it is rather hopeless to introduce the retardation into the Hubbard mechanism, [24]. Even if PHA does not play an important role, then the jump of the specific heat differs substantially at constant p and ρ, in contrast to all classical low temperature systems, *cf.* Eqs (2) and (3). This is so, because the specific heat at constant

ρ and p is proportional to the second derivative of (2) and (3) with regard to the temperature, respectively. Moreover, $(d\ln p/d\ln\rho)$ is equal to 5/3 and 2 for the three- and two-dimensional parabolic bands. The role of the retardation in such phenomena as well as the importance of the difference between C_v and C_p have been emphasized for the first time in our paper [5]. The temperature dependence of $\Delta\mu$ is a subject of recent interest, because of the measurements of the work function [25, 26], particularly at $T <\approx T_c$. If the $O(\tau)$ behaviour of the variation of the work function [25] leads to the same behaviour of $\delta\mu$, then we deal with first order transition, because of (2) and (3). This is because it is hard to expect that the terms of $\delta\Omega$ can compensate for the $O(\tau)$ term in any model of superconductivity. On the other hand, $\delta\mu$ refers to a single electron.

Because of the long-range Coulomb forces, one should maintain the electroneutrality, *i.e.* one should introduce the chemical potential for the atom [27]. Note that the number of electrons sustained in the system at a constant voltage depends not only on its size but also on its form; the result in [27] is valid for the spherical specimen, [5].

The outline of our problem shows that the vast majority of our results can be qualitatively understood with the help of the principles of the physics of condensed matter as well as the knowledge of the BCS model. In fact, it is most difficult to explain why some of the BCS results coincide with ours. The next tree chapters of our paper serve as the workshop for this outline. In the second chapter the low-temperature properties will be discussed, in the third chapter — the subcritical properties. The last chapter will be devoted to the discussion of the compressibilty of the one-component system, expressed by $(\partial N/\partial\mu)_s$. This quantity jumps at T_c, as it should do for the first-order phase transition. Note that the jump of the compressibility of superconductors in a *dc* magnetic field at T_c, expressed by the jump of their volume at constant p, was observed almost half a century ago, [28]. An attempt was made to explain the difference between the elastic modulae of the superconducting and normal metallic crystals [29]. Unfortunately, this paper looks strange, because the difference in the superconducting and normal states is the linear function of Δ. This result allows for the suspicion that the gauge invariance of the theory was not fulfilled. Let us mention that measurements of this difference have been made starting from the early sixties, [30]. The most curious feature appearing in this part of the paper is the difference between the compressibility of the normal and the superconducting system even in the PHS case, which contradicts the widespread belief. This belief was accepted even under the sanction of the Fermi liquid theory, in the BCS and BWV cases as well, [12, 13, 31]. The coincidence of the compressibilities for the normal and superfluid system in these papers results from the exact coincidence of the static long-wavelength autocorrela-

tion function of the density in both cases. In turn, the last quantity should be given by $-(\partial\rho/\partial\mu)_T$ because in the static quasihomogeneous field u we deal with the heterogeneous equilibrium by the condition $\mu + u = \text{const}$. To determine the autocorrelation functions in refs [12, 13, 31] procedure of eliminating integrations far from the Fermi surface is applied, in the equation for vertex and the autocorrelation functions. It occurs in the particle-particle and the particle-hole channels. Both of them are important for the superconducting liquid. For normal liquid the last channel is important. In the particle-particle channel, the price for elimination is the introduction of proper irreducible amplitudes, being the functions of angle between quasiparticles on the Fermi surface. Its Legendre amplitudes are proportional to $(2l+1)[\ln(\zeta/C_l)]^{-1}$, with C_l representing some constants such that $C_l \ll \zeta$, where ζ is the cut-off energy. In the actual pairing channel, *i.e.* for $l = 0$ and 1 for BCS and BWV systems, respectively, $C_l = \Delta/2$, [12, 13] we suspect that the elimination of integrations far from the Fermi surface is accurate only up the terms proportional to large logarithms $\ln(2\zeta/\Delta)$. It seems as though this is the only possible escape from the paradox, because the theories [12, 13, 31] are the extensions of the Gorkov approach, [32] to the important quasiparticle interaction in the paring channel. Because of the character of this outline, our "workshop" will be rather free of physical comments, they will replaced by mathematical ones. The single exception will be the comment connected with the stability conditions in the concrete form, because of possible projection out of BCS and BWV systems. Consequently, in this outline, the questions connected with the phase transition for BWV system have been omitted. They can be found in our paper [5]. In our workshop, both motives *(i)* and *(ii)* will be paralleled, if possible.

2. The low-temperature properties

Let us establish the general equation for the chemical potential, expressing it by the functional equation depending on Δ and T as parameters. For the case *(i)* we do it for the DOS (4), for *(ii)* — for DOS in the form $2N(\varepsilon)g(\varepsilon-\varepsilon_F)$, $g(0) = 1$. In our final calculations we will put $g(x) = 1+ax$, for $|x| < \omega_D$. After relatively simple calculations, especially in the case *(i)*, we find

$$2\varepsilon_F - 1 + a\left(\varepsilon^2 - \frac{1}{2}\right) = -a\int_0^1 d\varepsilon_F \frac{\xi^2}{E_\xi}\tanh\left(\frac{E_\xi}{2T}\right) - 2T\ln\frac{\cosh(E_{1-\mu}/2T)}{\cosh(E_\mu/2T)}\,, \tag{6i}$$

$$\int_0^{\omega_D-\Delta\mu} d\xi g(\varepsilon_F+\xi+\Delta\mu)R(\xi,T) - \int_0^{\omega_D+\Delta\mu} d\xi g(\varepsilon_F-\xi+\Delta\mu)R(\xi,T)$$
$$= G(\varepsilon_F+\omega_D) + G(\varepsilon_F-\omega_D) - 2G(\varepsilon_F)\,, \qquad (6ii)$$

where $R(\xi,T) \equiv \frac{\xi}{E_\xi}\tanh\left(\frac{E_\xi}{2T}\right)$.

In Eq. (6*i*), $E_\mu = [\mu^2+\Delta^2]^{1/2}$, *etc.*, and the last term is equal to $(1+a\mu)(E_{1-\mu}-E_\mu)+O[\exp(-\varepsilon_F/T)]$. In Eq. (6*ii*) $\Delta\mu \equiv \mu-\varepsilon_F$ and $G(\varepsilon)$ is the primary function of $g(\varepsilon)$. The r.h.s. of this equation is the second difference of the function $G(\varepsilon)$ at $\varepsilon=\varepsilon_F$. Note that $\hbar$ is hereafter equal put to unity.

If $a=0$, then at $T \ll \varepsilon_F$ one finds

$$\mu = \frac{1}{2} + \left(\varepsilon_F - \frac{1}{2}\right)\left[1+\frac{\Delta^2}{\varepsilon_F(1-\varepsilon_F)}\right]^{1/2}$$
$$\approx \varepsilon_F + \frac{\Delta^2(2\varepsilon_F-1)}{4\varepsilon_F(1-\varepsilon_F)} + O(\Delta^4)\,. \qquad (7i)$$

Since $f(0,\Delta)=\Delta^2/4$ and $(\partial\ln\rho/\partial\mu)_n \approx 1/\varepsilon_F$ whereas $(\partial\ln p/\partial\mu)_n \approx 2/\varepsilon_F$, by virtue of Eqs (2) and (3), the superconductivity is stable for $\varepsilon<2/3$ and $3/4$, at constant V and p, respectively at low temperatures (L.T.), $T \ll \Delta(0)$. In the subcritical temperatures this stability occurs for $\varepsilon<1/2$. In the case *(ii)* for $g=1$, in the pairing area, we find $\mu=\varepsilon_F+O(\exp(-\omega_D/T))$. PHS can be broken if the pairing area is nonsymmetric with respect to ε_F but this is meaningless for the phonon mediated interelectron attraction.

For the normal system, we have [4]:

$$\mu_n = \varepsilon_F - \frac{a(\pi T)^2}{6(1+a\varepsilon_F)} + O(T^4), \qquad (8i)$$

whereas in the case *(ii)* one should put $\varepsilon_F=0$ into the denominator. Let us discuss the L.T solutions for $\delta\mu$. First, let us rewrite the formula (6*ii*) for DOS (4) restricting ourselves to the terms $O(\delta\mu)$, *i.e.* $O(\Delta^2)$. We have

$$\left(1-\frac{\Delta^2}{2\omega_D^2}+\frac{a\Delta\mu}{2}\right)\Delta\mu = a\left[\omega_D^2 - \int_0^{\omega_D} d\xi\xi^2\frac{\tanh(E_\xi/2T)}{E_\xi}\right] + O(\Delta^6), \quad (9ii)$$

where $\Delta\mu \equiv \mu_s-\varepsilon_F$. Note that this formula is valid if $T \ll \omega_D$. From this point, throughout the paper, $\Delta\mu$ or $\delta\mu$ without their arguments will denote only $\Delta\mu(T,\rho)$ or $\delta\mu(T,\rho)$. Under formulated restrictions, we get at L.T.:

$$\delta\mu = -\Delta^2 \times \left\{ \frac{1-2\varepsilon_F}{\varepsilon_F(1-\varepsilon_F)} - \frac{a}{1-\varepsilon_F} + a\ln\frac{4\varepsilon_F(1-\varepsilon_F)}{\Delta^2} + 8a(T/\Delta)K_1(\Delta/T) \right\} \times \frac{4}{1+a\varepsilon_F} + O[\exp(-2\Delta/T)] \qquad (10i)$$

$$\Delta\mu = \Delta\mu(0) - 2aT\Delta K_1(\Delta/T) + O[\exp(-2\Delta/T)] \qquad (10ii)$$

where $\Delta\mu(0) = -\Delta^2 a\left[\ln(2\omega_D/\Delta) - 1/2 + O(\Delta^2)\right]$. Here, K_1 is the usual *Mc Donald* function. In the calculations leading to Eqs (10) usual procedures of the asymptotic expansions of the exponential integrals have been used, *cf. e.g.* [33].

Our gap equations can be rewritten in the following form:

$$(1+a\mu)\int_{-\mu}^{1-\mu} d\xi \tanh\frac{E_\xi/2T}{E_\xi} = 2/\kappa + a(E_\mu - E_{1-\mu}), \qquad (i),$$

$$(1+a\Delta\mu)\int_{-\omega_D-\Delta\mu}^{\omega_D-\Delta\mu} d\xi \tanh\frac{E_\xi/2T}{E_\xi} = 2/\kappa + a(E_{\omega_D+\Delta\mu} - E_{\omega_D-\Delta\mu}), \quad (ii), \qquad (11)$$

Note that the terms $O[\exp(-\varepsilon_F/T)]$, *(i)*, and $O[\exp(-\omega_D/T)]$, *(ii)*, have been neglected. Let us solve Eqs (10), $(11i)$ at L.T. For $T = 0$, restricting ourselves to the main exponent, we get

$$2/\kappa = (1+a\mu)\ln[4\mu(1-\mu)/\Delta^2] + a(E_{1-\mu} - E_\mu),$$

and hence, taking into account that
$\mu = \varepsilon_F + O(\Delta^2)$, we find:

$$\Delta(0) = 2\sqrt{\varepsilon_F(1-\varepsilon_F)}\exp\left\{-\left[\tfrac{1}{\kappa} - a\left(\tfrac{1}{2} - \varepsilon_F\right)\right](1+a\varepsilon_F)^{-1}\right\}. \qquad (12i)$$

Substituting Eq. $(11i)$ into $(9i)$ one finds $\delta\mu(0)$ expressed with the parameters of the problem. We have

$$\delta\mu(0) = -\frac{\Delta^2(0)}{4(1+a\varepsilon_F)}\left\{\frac{1-2\varepsilon_F}{\varepsilon_F(1-\varepsilon_F)} - \frac{a}{1-\varepsilon_F} + \frac{a[1/\kappa - a(1-2\varepsilon_F)]}{1+a\varepsilon_F}\right\}. \qquad (13i)$$

From Eqs $(11i)$ and $(13i)$ we get

$$\delta\mu = 2\delta\mu(0)\frac{d\Delta}{\Delta(0)} - 2aT\Delta(0)K_1\left(\frac{\Delta(0)}{T}\right), \qquad (14i)$$

where $d\Delta \equiv \Delta(T) - \Delta(0)$, *etc.* In order to find $d\Delta$ let us subtract from the formula $(11i)$ the same formula for $\Delta = \Delta(T)$ but with T put equal to zero in the argument of the hyperbolic tangent. Restricting ourselves to the linear approximation with respect to $d\Delta$ and $d\mu$, using $(12i)$ we find:

$$\begin{aligned} & d\left\{(1+a\mu)\ln\frac{4\mu(1-\mu)}{\Delta^2} + a(E_{1-\mu} - E_\mu)\right\} \\ & = -2\frac{d\Delta}{\Delta(0)} + ad\mu\frac{2/\kappa - a(1-2\varepsilon_F)}{1+a\varepsilon_F} \\ & = 4(1+a\varepsilon_F)K_0\left(\frac{\Delta(0)}{T}\right) + O[\exp(-2\Delta/T)]\,. \end{aligned} \tag{15i}$$

Substituting $(14i)$ into $(15i)$ and omitting small terms in the coefficients at the exponentially small terms one finds:

$$\begin{aligned} \frac{d\Delta}{\Delta(0)} \;=\; & -2K_0\left(\frac{\Delta(0)}{T}\right) - a^2T\Delta(0)K_1\left(\frac{\Delta(0)}{T}\right)\frac{2/\kappa - a(1-2\varepsilon_F)}{1+a\varepsilon_F} \\ & + O[\exp(-2\Delta/T)]\,. \end{aligned} \tag{16i}$$

Performing analogous, even simpler, calculations for the case *(ii)* one finds:

$$\Delta\mu(0) = \delta\mu(0) = -\Delta^2(0)a\left(\frac{1}{\kappa} - \frac{1}{2} + O(\Delta^2)\right), \tag{13ii}$$

$$\frac{d\Delta}{\Delta(0)} = -2K_0\left(\frac{\Delta(0)}{T}\right) - a^2T\Delta(0)\left(\frac{2}{\kappa} - 1\right)K_1\left(\frac{\Delta(0)}{T}\right). \tag{16ii}$$

The term with the *Mc Donald* function K_0 reproduces the BCS result.

The function $f(T,\Delta)$, for the simple models of DOS, (4), coincides with the BCS result [1, 6]. Hence, at constant μ, the difference of the entropy is given by:

$$-\left(\frac{\partial\delta\Omega}{\partial T}\right)_V = \left(\frac{\partial N}{\partial\mu}\right)_n \frac{d}{dT}\, f(T,\Delta).$$

On the other hand, $\delta\mu = O(\Delta^2)$ as well as $d\mu$ unless we deal with PHS. For PHA $-(\partial\delta\Omega/\partial T)_V$ does not coincides with difference of entropies at constant V or p. The l.h.s. of Ineqs (2) and (3) are the differences of thermodynamic potentials, per particle, proper at constant V and p, respectively. Hence, the difference of the entropies per particle at constant V and p are given by, [4]:

$$\begin{aligned} \delta S_V \;&=\; -(\partial\delta\mu/\partial T) + \delta S_{\rm BCS}\,, \\ \delta S_p \;&=\; -(\partial\delta\mu/\partial T) + (d\ln p/d\ln\rho)_n\delta S_{\rm BCS}\,. \end{aligned} \tag{17}$$

It was taken into account that the BCS $\delta\Omega$ under the T– differentiation must yield $-\delta S_{\mathrm{BCS}}$. It is clear that Eqs (17) are fulfilled also for quantities dS, $d\mu$ and dS_{BCS}. The common term in both Eqs (17) in the d–version leads to the common result in the models *(i)* and *(ii)* because of the expression for $d\mu$. Applying the asymptotics of *Mc Donald* functions, [33], one finds in L.T.

$$-T\left(\frac{\partial^2 d\mu}{\partial T^2}\right)_\rho = \sqrt{2\pi}a\Delta^{5/2}(0)T^{-3/2}\exp(-\Delta(0)/T)[1+O(T)]. \qquad (18)$$

This is the L.T. specific heat caused by PHA. The analogous BCS result for the model *(i)* is

$$C_{\mathrm{BCS}} = 2\sqrt{2\pi}\Delta^{5/2}(0)T^{-3/2}\exp(-\Delta(0)/T)\frac{1+x}{\varepsilon_F(2+x)}, \qquad (19i)$$

where $x \equiv a\varepsilon_F$. As we see from (18), its order of magnitude coincides with the order of (19). To write the expression for the model *(ii)* one should divide the conventional value, per unit volume, by the particle density. The double logarithmic derivative in the second of Eqs (17) can be determined only in the model *(i)* because DOS in the model *(ii)* is still arbitrary at $|\varepsilon - \varepsilon_F| > \omega_D$. We have:

$$\frac{d\ln p}{d\varepsilon_F} = \frac{3(2+x)}{\varepsilon_F(3+x)}, \qquad \frac{d\ln\rho}{d\varepsilon_F} = \frac{2(1+x)}{\varepsilon_F(2+x)}, \qquad x \equiv a\varepsilon_F. \qquad (20i)$$

It is easy to see that $(d\ln p/d\ln\rho)$ is always bigger than unity in this model. Hence, at constant p, it is easier to be a superconductor than at constant V.

Let us discuss L.T. stability conditions for the superconductivity. Let us start from the model *(i)*. Because $f(0,\Delta) = \Delta^2/4$, [1, 6], using Ineqs (2,3) and Eqs (10), (20i) we find that the stability conditions at constant p and V are the third order inequalities with respect to $x = a\varepsilon_F$. Restricting ourselves to small κ and $x = O(1)$, we find that the superconductivity is stable if

$$\begin{aligned} a\varepsilon_F(1-\varepsilon_F) &> \kappa(3\varepsilon_F - 2), \\ a\varepsilon_F(1-\varepsilon_F) &> \kappa(4\varepsilon_F - 3), \end{aligned} \qquad (21i)$$

at constant V and p, respectively. Hence, if r.h.s. of inequalities (21i) are positive, the superconductivity is stable at suitably big positive a. Note that the conditions (21i) are valid unless ε_F or $1-\varepsilon_F$ are much less than unity. For the model *(i)*, at L.T., the stability condition of the superconductivity is, by virtue of the formulae (2), (3) and (12i):

$$-2a\left(\frac{2}{\kappa} - 1\right) < \left(\frac{d\ln\rho}{d\varepsilon_F}\right)_n \quad \text{or} \quad \left(\frac{d\ln p}{d\varepsilon_F}\right)_n, \qquad (21ii)$$

at constant V and p, respectively.

The appearance of the term $O(1/\kappa)$ in the formulae of this section is a testimony of the nonanalytical character of the formulae for $\delta\mu$. Let us remark also another nonanalyticity in the model *(i)*. Namely:

$$\lim_{\varepsilon_F \to 0} \delta\mu(0) = -\exp(-2/\kappa + a),$$

by virtue of the formulae (12), (13i). Hence, the chemical potentials in the phases "s" and "n" differ even if the particles are absent. It is easy to see that it does not lead to any troubles with the thermodynamic functions. Unfortunately, it is impossible to calculate $\lim_{\varepsilon_F \to 0} \delta\mu(T)$ at L.T. because our expansion formulae are valid at $T \ll \Delta$.

3. The subcritical properties

Let us discuss the subcritical (S.C.) thermodynamic functions, starting from $\delta\mu$. In this case, the crucial role plays the series expansion with respect to Δ^2 of the integral appearing in Eqs (6i), (9ii). We have:

$$\int_0^b d\xi \frac{\xi^2}{E_\xi} \tanh(E_\xi/2T) = \sum_{n=0} \frac{\Delta^{2n}}{(2n)!!(2T)^{2n-2}} \int_0^{b/2T} du u^2 \left(\frac{1}{u}\frac{d}{du}\right)^n \left(\frac{\tanh u}{u}\right).$$

By means of elementary integrations and the weak-coupling BCS integrals one finds:

$$\begin{aligned} \int_0^b d\xi \xi^2 \tanh(E_\xi/2T)/E_\xi \;&=\; \frac{1}{2}b^2 - \frac{1}{6}\pi^2 T^2 + \frac{\Delta^2}{2}\left[1 - \ln\left(\frac{2\gamma b}{\pi T}\right)\right] \\ &\quad + \frac{7\zeta(3)\Delta^4}{32\pi^2 T_c^2} + O(\Delta^6)\,, \end{aligned} \tag{22}$$

where $\gamma = e^C$ and C is the Euler constant. The logarithmic term in Eq. (22) can be replaced by $\ln(2\gamma b/\pi T_c) + \tau + O(\tau^2)$, where $\tau \equiv 1 - T/T_c$.

As it was remarked, the S.C. solution for Δ, expressed by T_c, preserves its form characteristic for $a = 0$, with the accuracy up to the small terms in the coefficients. Hence, from the expansion in Ref. [22]:

$$\begin{aligned} \Delta^2 \;&=\; \frac{8\pi^2 T_c^2}{7\zeta(3)}\tau(1 - q\tau) + O(\tau^3)\,, \\ q \;&=\; \frac{3}{2} - \frac{93\zeta(5)}{98\zeta^2(3)} \approx 0.8190\,. \end{aligned} \tag{23}$$

Substituting the formula (22) into Eqs (6*i*) and (9*ii*), exploiting next Eq. (12*i*) together with the relation between T_c and $\Delta(0)$, as well as analogous formulae for the model *(ii)* one finds:

$$\delta\mu = -r\Delta^2 - \frac{2\pi^2 T_c^2 a\tau^2}{7(1+a\varepsilon_F)\zeta(3)} + O(\tau^3)\,, \tag{24}$$

where

$$\begin{aligned} r &= \frac{1-2\varepsilon_F}{4\varepsilon_F(1-\varepsilon_F)} + \frac{a}{2(1+a\varepsilon_F)^2}\left[\frac{1}{\kappa} - 1 - a\left(\frac{1}{2}-\varepsilon_F\right)\right], \quad (i)\,, \\ r &= \frac{a}{2}\left(\frac{1}{\kappa}-1\right), \quad (ii)\,. \end{aligned} \tag{25}$$

Note that ε_F should be put equal to zero in the denominator of Eq. (24), for the model *(ii)*. Moreover, the small terms $O[(aT_c/\kappa)^2]$ have been omitted in Eqs (24), (25). On the other hand, the second term in the r.h.s. of Eq. (24) is the result of the main term of Eq. (23).

In order to complete the thermodynamical potentials per particle (2) and (3) let us write the subcritical f–function. Exploiting Eq. (23) and the results of Refs [1, 6] one finds: $f(T,\Delta) = 2(\pi T_c\tau)^2/7\zeta(3)$. Differentiating Eqs (2) and (3) with respect to T at $T = T_c$ we find the latent heat. It is equal to:

$$\delta Q \equiv Q_s - Q_n = -T_c\left(\frac{\partial\delta\mu}{\partial T}\right)_{T=T_c} = -\frac{8\pi^2 T_c^2 r}{7\zeta(3)}, \tag{26}$$

where r is expressed by Eqs (25). The latent heats at constant V and p coincide. They are negative, by virtue of the subcritical stability condition for the superconductivity. From thermodynamic formulae [4] the jump of the specific heat per particle is given by the second T-derivative of Eqs (2) and (3) multiplied by $-T_c$, without the δ-function component determining the latent heat. Hence, if we normalize the result to the BCS jump we find

$$\frac{\Delta C}{\Delta C_{\rm BCS}} = -\left(\frac{\partial\ln\rho}{\partial\mu}\right)_n^{-1}\left[4qr - \frac{a}{1+a\varepsilon_F}\right] + \left\{\begin{array}{c} 1 \\ \left(\frac{\partial\ln p}{\partial\ln\rho}\right)_n \end{array}\right\}, \tag{27}$$

where the upper line in the curly bracket corresponds to constant V whereas the lower one – to constant p. For the model *(ii)* one should put $\varepsilon_F = 0$ in the denominator in the square bracket. To obtain Eq. (27) out of Eqs (2), (3), (23), (24) one should take into account the form of $\Delta C_{\rm BCS}$ per particle which is $\frac{4\pi^2}{7\zeta(3)}\left(\frac{\partial\ln\rho}{\partial\mu}\right)_n$, [1,6]. In the case *(i)* the derivative $\left(\frac{\partial\ln p}{\partial\ln\rho}\right)_n = \frac{3(2+x)^2}{2(3+x)}$. For $x = 0$, corresponding to the parabolic band for $d = 2$, this double logarithmic derivative is equal to 2, whereas such band at $d = 3$ gives $\frac{5}{3}$.

The first term in the r.h.s. of Eq. (27) is negative, by virtue of the subcritical stability condition, $\delta\mu < 0$, and the solution of the gap (23) where $q > 0$. Because a is $O(1/\omega_D)$ or $O(1/\varepsilon_F)$ if DOS is influenced by the electron-phonon interaction or is not, respectively, at the weak coupling, when $r \ll 1$, the first negative term can easily overpass the second one making $\Delta C/\Delta C_{\rm BCS}$ negative. Here the estimation $\left(\frac{d\ln\rho}{d\mu}\right)_n = O(1/\varepsilon_F)$ has been used. Moreover, because $\Delta C_{\rm BCS}/C_n = O(1)$ the total specific heat may very easy to attain the negative value.

For the system *(ii)* the subcritical stability of the superconducting state is expressed in the extremely simple form: $a > 0$. For the system *(i)* it leads to quadratic inequality with respect to the variable x. Restricting ourselves to small κ and $x = O(1)$ one finds from (25i):

$$(2\varepsilon_F - 1)\kappa < a\varepsilon_F(1 - \varepsilon_F). \tag{28i}$$

On the other hand, a should be smaller than some positive quantity $O(1/\kappa)$, which is irrelevant at $x = O(1)$. It is easy to see that such a region of the parameter a always appears that the superconductivity is stable at L.T. but is ceasing to be stable below T_c. For example, in the system *(i)*, if $a < 0$ but $|a| \ll 1/T_c$ then the temperature of the jump of the order parameter can be expressed as follows:

$$T_j = T_c + 2aT_c\frac{1/\kappa - 1}{q},$$

where q is equal to $\frac{d\ln p}{d\varepsilon_F}$ or $\frac{d\ln\rho}{d\varepsilon_F}$ at constant p and V, respectively. Analogously, we can determine T_j for the model *(i)* but the formula is much more complicated and we omit it here.

The inequalities (21), (28i) and analogous inequalities for the model *(ii)* determining the stability of the superconducting state are very easy to interpret. Namely, the distribution function of particles (1) consists of the body for $\varepsilon < \mu$ and the tail. The body continuously passes into the tail at $\varepsilon \sim \mu$ and the tail is thick only if $|\varepsilon - \mu| = O(\Delta)$. If, in the model *(i)*, ε_F is too close to 1 then it is difficult to find the place for particles and, hence, the possible tactics of the electrons consists in the growth of μ. Similar problem does not appear for the normal system, where the tail is of the thermal character and can be neglected for $T \ll \varepsilon_F$. Similar arguments can be also used at $a < 0$. Note that the validity of this argument is not restricted to the BCS system. In usual models, out of the model of the Luttinger liquid [34], *cf.* for $d \geq 2$ [35], the occupation numbers are $\theta(\varepsilon_p - \varepsilon_F)$, with θ being the Heaviside step function. In the superconducting state this sharp step should be smeared out, *i.e.* the tail of the particle distribution function should appear. Hence, the vulnerability of the superconducting state by ε_F too

close to the top of the band or diminishing derivative of DOS in the region of the thick tail of the particle distribution function.

3. The compressibility of superconducting systems

Here we are going to determine $\left(\frac{d\rho}{d\mu}\right)_{V,T}$ in both variants *(i)* and *(ii)*. This quantity is connected with the isothermal compressibility, α, as follows: $\alpha = \frac{1}{\rho^2}\left(\frac{d\rho}{d\mu}\right)_{V,T}$, [4]. For normal systems at our DOS we have:

$$\left(\frac{\partial\rho}{\partial\mu}\right)_{V,T} = \begin{cases} 2N(0)\left[1 + a\varepsilon_F - \frac{a^2\pi^2T^2}{6(1+a\varepsilon_F)}\right], & (i) \\ 2N(\varepsilon_F)\left[1 - \frac{a^2\pi^2T^2}{6}\right], & (ii). \end{cases} \tag{29}$$

cf. [4]. The derivative $\left(\frac{\partial\rho}{\partial\mu}\right)_{V,T}$ in the superconducting system is

$$\left(\frac{\partial\rho}{\partial\mu}\right)_{T} = \left(\frac{\partial\rho}{\partial\mu}\right)_{T,\Delta} + \left(\frac{\partial\rho}{\partial\Delta^2}\right)_{\mu,T}\left(\frac{\partial\Delta^2}{\partial\mu}\right)_{T}, \tag{30}$$

where the last derivative is calculated from the gap equation, $f(\Delta,\mu,T) = 0$ by means of the relation $df = 0$ at $dT = 0$. From Eqs (11) we find:

$$\begin{aligned}\left(\frac{\partial\Delta^2}{\partial\mu}\right)_{T} &= -2\left\{\frac{1}{E_\mu} - \frac{1+a}{E_{1-\mu}} + \frac{a}{1+a\mu}\left[\frac{2}{\kappa} + a(E_{1-\mu} - E_\mu)\right]\right\} \\ &\times \left\{\int_{-\mu}^{1-\mu}\frac{d\xi}{E_\xi}(1 + a\mu + a\xi)\frac{d}{dE_\xi}\left[\frac{\tanh(E_\xi/2T)}{E_\xi}\right]\right\}^{-1}, \quad (31i)\end{aligned}$$

$$\begin{aligned}&\left(\frac{\partial\Delta^2}{\partial\mu}\right)_{T} \\ &= -2\left\{\frac{1 - a\omega_D}{E_{\omega_D+\Delta\mu}} - \frac{1 + a\omega_D}{E_{\omega_D-\Delta\mu}} + \frac{a}{1+a\mu}\left[\frac{2}{\kappa} + a(E_{\omega_D-\Delta\mu} - E_{\omega_D+\Delta\mu})\right]\right\} \\ &\times \left\{\int_{-\omega_D-\Delta\mu}^{\omega_D-\Delta\mu}\frac{d\xi}{E_\xi}(1 + a\Delta\mu + a\xi)\frac{d}{dE_\xi}\left[\frac{\tanh(E_\xi/2T)}{E_\xi}\right]\right\}^{-1}. \quad (31ii)\end{aligned}$$

In these calculations the single non-elementary integral in the numerator coincides with that in Eqs (11) and can be eliminated from (31) by means of (11). The numerator in Eqs (31) we will denote in the future as $-2M$.

The quantity $\left(\frac{\partial\rho}{\partial\Delta^2}\right)$ can be found by taking into account $\frac{\partial}{\partial\Delta^2} = \frac{1}{2\xi}\frac{\partial}{\partial\xi}$ in the action on $f(E_\xi)$, integrating by parts and eliminating, once more, the same non-elementary integral via Eqs (11). Hence we get

$$\left(\frac{\partial\rho}{\partial\Delta^2}\right)_{T,\mu} = N(0)\frac{M}{2}, \tag{32i}$$

and for the model *(ii)* $N(0)$ is replaced by $N(\varepsilon_F)$. This coincidence is the result of the variational principle for BCS systems [1, 6] and the commutativity of partial derivatives. Accuracy $O(1)$ in the term M leads to the replacement $\mu = \varepsilon_F$ and $E_\xi = |\xi|$ in both models *(i), (ii)*.

The terms $\left(\frac{\partial\rho}{\partial\mu}\right)_{\Delta,T}$ can be found in an elementary way. Omitting, as previously, the exponentially small terms we get

$$\begin{aligned}\frac{1}{N(0)}\left(\frac{\partial\rho}{\partial\mu}\right)_{\Delta,T} &= \frac{\mu}{E_\mu} + \frac{(1+a)(1-\mu)}{E_{1-\mu}} + a(E_\mu - E_{1-\mu}) \\ &= 2(1+a\mu) - \frac{\Delta^2}{2}\left[1 - (2+a)\varepsilon_F + (1+4a)\varepsilon_F^2 - 2a\varepsilon_F^3\right] \\ &\times\frac{1}{\varepsilon_F^2(1-\varepsilon_F)^2} + O(\Delta^4)\,,\end{aligned} \tag{33i}$$

$$\begin{aligned}\frac{1}{N(\varepsilon_F)}\left(\frac{\partial\rho}{\partial\mu}\right)_{\Delta,T} &= \frac{(1-a\omega_D)(\omega_D+\Delta\mu)}{E_{\omega_D+\Delta\mu}} + \frac{(1+a\omega_D)(\omega_D-\Delta\mu)}{E_{\omega_D-\Delta\mu}} \\ &+a(E_{\omega_D+\Delta\mu} - E_{\omega_D-\Delta\mu}) = 2(1+a\Delta\mu) - \left(\frac{\Delta}{\omega_D}\right)^2 + O(\Delta^4)\,.\end{aligned} \tag{33ii}$$

Unfortunately, the denominator in Eqs (31) cannot be calculated in an universal way valid for low and subcritical temperatures. For L.T. we have

$$\int_{-c}^{b}\frac{d\xi}{E_\xi}(1+ay+a\xi)\frac{d}{dE_\xi}\left[\frac{\tanh(E_\xi/2T)}{E_\xi}\right] = -\frac{2(1+ay)}{\Delta^2}, \tag{33}$$

provided that $b, c \gg 1$. For the model *(i)* $y = \mu \to \varepsilon_F$ whereas for *(ii)* $y = \Delta\mu \to 0$. Analogously, in the subcritical region the integral above is equal to

$$\frac{1+ay}{2T_c^2}\int_0^\infty\frac{du}{u}\frac{d}{du}\left(\frac{\tanh u}{u}\right) + O(1)$$

According to the well known formula [1, 6] this integral is equal to

$$-\frac{7\zeta(3)(1+ay)}{2\pi^2T_c^2}\,, \tag{34}$$

and the same prescriptions for the choice y hold. Collecting together Eqs (30) and (31)–(33) we find the L.T. compressibility of the Fermi gas under pairing. We have:

$$\frac{1}{N(0)}\left(\frac{\partial\rho}{\partial\mu}\right)_T = 2(1+a\mu)$$
$$-\frac{\Delta^2}{2}\left[1-(2+a)\varepsilon_F+(1+4a)\varepsilon_F^2-2\varepsilon_F^3 a\right]\frac{1}{\varepsilon_F^2(1-\varepsilon_F)^2}$$
$$+\frac{\Delta^2}{2(1+a\varepsilon_F)}\left\{\frac{1}{\varepsilon_F}-\frac{1+a}{1-\varepsilon_F}+\frac{a}{1+a\varepsilon_F}\left[\frac{2}{\kappa}+a(1-2\varepsilon_F)\right]\right\}^2, \quad (35i)$$

$$\tfrac{1}{N(\varepsilon_F)}\left(\tfrac{\partial\rho}{\partial\mu}\right)_T = 2(1+a\Delta\mu)-\tfrac{\Delta^2}{\omega_D^2}+2a^2\Delta^2\left(\tfrac{1}{\kappa}-1\right)^2. \quad (35ii)$$

Analogously, in the S.C. region we find:

$$\frac{1}{N(0)}\left(\frac{\partial\rho}{\partial\mu}\right)_T = 2(1+a\mu)$$
$$-\frac{\Delta^2}{2}\left[1-(2+a)\varepsilon_F+(1+4a)\varepsilon_F^2-2\varepsilon_F^3 a\right]\frac{1}{\varepsilon_F^2(1-\varepsilon_F)^2}$$
$$+\frac{2\pi^2T_c^2}{7\zeta(3)(1+a\varepsilon_F)}\left\{\frac{1}{\varepsilon_F}-\frac{1+a}{1-\varepsilon_F}+\frac{a}{1+a\varepsilon_F}\left[\frac{2}{\kappa}+a(1-2\varepsilon_F)\right]\right\}^2, \quad (36i)$$

$$\tfrac{1}{N(\varepsilon_F)}\left(\tfrac{\partial\rho}{\partial\mu}\right)_T = 2(1+a\Delta\mu)-\tfrac{\Delta^2}{\omega_D^2}+\tfrac{8a^2\pi^2T_c^2}{7\zeta(3)}\left(\tfrac{1}{\kappa}-1\right)^2. \quad (36ii)$$

Let us add that the terms omitted in Eqs (35) are $O(\Delta^4)$ whereas in Eqs (36) — $O(T_c^4)$. Moreover, all partial derivatives are taken at constant V.

Because of the jump of the electron compressibility at T_c the Debye radius and the effective ion-ion interaction is changing by a jump at T_c. Hence, the instability of the superconductivity can be excluded in the metal as a whole, because of lattice contribution to the proper thermodynamic potential.

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