Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 3

BACK REACTION ON THE METRIC. BEYOND THE PAGE APPROXIMATION

J. Matyjasek

Institute of Physics, Maria Curie-Skłodowska University
Marii Curie-Skłodowskiej 1, 20–031 Lublin, Poland
e-mail: matyjase@tytan.umcs.lublin.pl
jurek@iris.umcs.lublin.pl

(Received October 2, 1997)

The back reaction of the quantized conformal massless scalar field in the Hartle–Hawking state upon the Schwarzschild geometry is considered perturbatively. Recently proposed approximation of the renormalized stress-energy tensor, which is known to properly reproduce exact numerical calculations, is used as the source term in the semi classical Einstein equations. The nature of the resulting spherically-symmetric and static metric is studied through the construction of the effective potential for null and timelike circular orbits and the analysis of the corrections to the trace anomaly. The modifications caused by the back reaction on the temperature and entropy are analysed.

PACS numbers: 04.62. +v, 04.70. Dy

1. Introduction

More than a decade ago the problem of the back reaction of the quantized field upon the spacetime geometry was successfully attacked by York [1]. In his approach he confined himself to the quantum fields in the Hartle–Hawking state. Solving the first order (linearized) semi-classical field equations in the static and spherically symmetric black hole spacetime, he was able to calculate quantum corrections to the metric, assuming that the source term is given by the Page approximation of the stress energy tensor [2]. The latter is known to be a reasonable approximation of the exact renormalized $\langle T^{\mu}_{\nu}\rangle$ of the massless, conformally invariant scalar field in the Hartle–Hawking state [3]. In this situation the black hole exists in a thermodynamical equilibrium with surrounding radiation, and the system is analytically tractable. Indeed, the solution of the semi-classical equations reduces in such a case to two elementary quadratures. The stress energy tensor in the thermal state is asymptotically constant and consequently the corrected metric is not asymptotically flat. It follows then that the back reaction calculations cannot be performed unless the system is confined in the finite volume and specific physically motivated boundary conditions are introduced.

Since that time many efforts have been devoted to this group of problems and in numerous articles various consequences of the back reaction have been analysed [4–12]. Most of them, however, use the methods that may be traced back to those propounded in Ref. [1]. A simplified two-dimensional analysis of the back reaction problem based on the Vaidya-type metrics has been carried out, for example, in [13–16] and in the references cited therein.

The most important ingredient of the back reaction calculations is the renormalized mean value of the stress energy tensor of the quantized field constructed for a wide class of metrics. Approximate analytic expressions describing $\langle T_{\nu}^{\mu} \rangle$ of the scalar, electromagnetic, and spinor fields in the Schwarzschild spacetime in the Hartle–Hawking, Unruh, and Boulware states have been constructed in [17–31]. Since the back reaction calculations in the conformal spinor case are based on the energy momentum tensor constructed within the framework of the Brown–Ottewill–Page approximation [22] which validity has not been verified by the exact numerical calculations, their status is, as yet, uncertain. It seems, however, that we have some reasons to believe that although the stress-energy tensor of the conformal spinor field may substantially differ from the exact one near the event horizon the approximate formula it yields that describes the entropy of the radiation may be right. The case of the spin 2 field has been treated in the interesting recent paper by Hochberg and Sushkov [8].

The renormalized stress-energy tensor violates both weak and strong energy conditions in the vicinity of the black hole event horizon that reflects the fact that it is surrounded by a cloud of negative radiation. On the other hand, provided the back reaction effects are taken into account, the thermodynamic entropy has been found to be positive and monotonically increasing with radius. In a recent publication [7], which is based on the approximation given by Anderson, Hiscock, and Samuel [24], the back reaction has been analysed for the scalar field with arbitrary coupling. The importance of this research lies in the fact that it is the first instance in which the general thermodynamic considerations place limits on the allowable values of the fundamental constant.

The general conclusion of such calculations is that "the back reaction, no matter how "small", is [...] always significant in describing thermal properties of the spacetime geometries and fields near black holes" [5]. Moreover, some of the consequences of the back reaction programme may be applied in constructing the renormalized $\langle T_{\nu}^{\mu}\rangle$ of quantized fields for which even the approximate expressions are not known [8].

Howard [3] in his numerical study showed that the stress tensor of the scalar field in Schwarzschild spacetime naturally splits into the part described by the Page approximation and the second term which is a traceless combination of certain mode sums. Modifications of the Page approximation are significant in the region $2M < r < 5M$; for example the maximal deviation of the tangential pressure from the exact value is approximately 48% at $r = 2.5M$, and 17% on the event horizon.

In a recent paper we constructed approximate analytical expressions describing the renormalized $\langle T_{\nu}^{\mu}\rangle$ of the massless, conformally invariant scalar field in the Hartle–Hawking state in the Schwarzschild spacetime which satisfy all regularity and consistency conditions [25]. Since the obtained analytical formulas closely follow the numerical calculations it would be interesting to reexamine the back reaction problem with the aid of the improved $\langle T_{\nu}^{\mu}\rangle$. In that paper we shall undertake this issue and solve the linearized semi-classical field equations. As the stress energy tensor asymptotically approaches the $\langle T_{\nu}^{\mu}\rangle$ of the gas of massless particles one is forced to confine the system that consists of the black hole and quantized radiation in the spatially bounded region and adopt physically motivated boundary conditions. After a discussion of the maximal radius of the spherical box we construct the corrected temperature and entropy. We shall show that the entropy, ΔS, of the conformally coupled scalar field may be generally written as

$$\Delta S = \sum_{n=-3}^{5} \alpha_n w^n + \beta \ln w, \tag{1}$$

where $w = 2M/r$, and α_n, β are numerical coefficients. The case of the arbitrary coupling considered by Anderson *et al.* [7] accounts for the following form of ΔS :

$$\Delta S = \sum_{n=-3}^{3} \tilde{\alpha}_n w^n + \beta \ln w + \left(\xi - \frac{1}{6}\right)\left(\sum_{n=0}^{3} \gamma_n w^n + \delta \ln w\right), \tag{2}$$

where ξ is a parameter that describes a type of the coupling, and $\tilde{\alpha}_n$, γ_n, and δ belong to the another set of the numerical coefficients. It should be noted that for the conformally invariant scalar field, $\langle T_{\nu}^{\mu}\rangle$ propunded by Anderson *et al.* [23, 24] reduces to the expression given by Page approximation. This is the reason why for $\xi = 1/6$ Eq. (2) differs from (1).

As the spacetime geometry may be better understood analysing motion of test particles, we shall also construct and study the effective potential for both massive and massless particles moving in the modified black hole background. Finally, we briefly analyse corrections to the trace anomaly of the modified metric.

2. Stress-energy tensor

The stress energy tensor as proposed in Ref. [25] may be compactly written as

$$\langle T_{\mu}^{\nu}\rangle = \langle T_{\mu}^{\nu}\rangle^{\rm Page} + \Delta_{\nu}^{\mu}, \tag{3}$$

where

$$\begin{aligned}\langle T_{\nu}^{\mu}\rangle^{\rm Page} &= \frac{1}{90\pi^2(8M)^4}\left\{\frac{1-(\frac{2M}{r})^6(4-\frac{6M}{r})^2}{(1-\frac{2M}{r})^2}\,{\rm diag}[-3,1,1,1]_{\nu}^{\mu}\right.\\ &\quad\left. + 24\left(\frac{2M}{r}\right)^6 {\rm diag}[3,1,0,0]_{\nu}^{\mu}\right\},\end{aligned} \tag{4}$$

is the Page approximation of $\langle T_{\nu}^{\mu}\rangle$, and Δ_{μ}^{ν} is a traceless conserved tensor given by

$$\begin{aligned}8\pi^2\Delta_t^t = \frac{M^2}{r^6}\left(\frac{\beta}{240}+\frac{17}{6}\alpha_4\right) &- \frac{M^3}{r^7}\left(\frac{433}{27}\alpha_4+\frac{\beta}{120}+\frac{4}{405}A_8\right)\\ &+ \frac{M^4}{r^8}\left(\frac{11}{540}A_8+\frac{385}{18}\alpha_4\right),\end{aligned} \tag{5}$$

$$\begin{aligned}8\pi^2\Delta_r^r = \frac{M^2}{r^6}\left(\frac{\beta}{720}+\frac{17}{18}\alpha_4\right) &- \frac{M^3}{r^7}\left(\frac{121}{27}\alpha_4+\frac{\beta}{360}+\frac{A_8}{405}\right)\\ &+ \frac{M^4}{r^8}\left(\frac{35}{6}\alpha_4+\frac{A_8}{180}\right),\end{aligned} \tag{6}$$

and

$$\begin{aligned}8\pi^2\Delta_\theta^\theta = -\frac{M^2}{r^6}\left(\frac{17}{9}\alpha_4+\frac{\beta}{360}\right) &+ \frac{M^3}{r^7}\left(\frac{277}{27}\alpha_4+\frac{\beta}{180}+\frac{A_8}{162}\right)\\ &- \frac{M^4}{r^8}\left(\frac{245}{18}\alpha_4+\frac{7}{540}A_8\right).\end{aligned} \tag{7}$$

Parameters α_4 and A_8 may be easily determined from the knowledge of the horizon value of one of the components of the stress energy tensor and the field fluctuation $\langle\phi^2\rangle$. Simple considerations give $\alpha_4 = 0.171$, and $A_8 = -164$; in Ref. [25, 26] was also shown that taking $\beta = -80$, one obtains very good agreement with the exact numerical results.

Since β has been fixed by making use of the best-fit argument, it would be desirable to obtain constraints on this very parameter from the back reaction calculations. Unfortunately, as we shall show, the radial component of the metric perturbation does not depend on the parameter β and the analyses of the time component of perturbation cannot be exploited without additional information. We expect, however, that some informations may be gained from the thermodynamic considerations.

In order to incorporate into the stress energy tensor the term describing arbitrary coupling of the scalar field, the Page approximation should be supplemented with a term $(\xi - 1/6)D_{\nu}^{\mu}$, where D_{ν}^{μ}, according to analyses carried out in [23, 24] and [7], is given by

$$D_t^t = \frac{1}{128\pi^2 r^6}(r^2 + 4Mr - 20M^2)\,, \tag{8}$$

$$D_r^r = -\frac{1}{384M\pi^2 r^6}(2r - 3M)(12M^2 + 4Mr + r^2)\,, \tag{9}$$

and

$$D_\theta^\theta = D_\phi^\phi = \frac{1}{384M\pi^2 r^6}(r^3 + 4Mr^2 + 12M^2 r - 96M^3)\,. \tag{10}$$

3. The linearized back reaction equations

Since the stress-energy tensor is in the Hartle–Hawking state one expects that the spherically-symmetric quantum corrected metric is generally of the form

$$ds^2 = -U(r)dt^2 + V^{-1}(r)dr^2 + r^2 d\Omega^2, \tag{11}$$

where

$$U(r) = V(r)\mathrm{e}^{2\psi(r)} \tag{12}$$

and

$$V(r) = 1 - 2m(r)/r. \tag{13}$$

The semi-classical field equations may be solved perturbatively to first order in $\varepsilon = (M_{\mathrm{P}}/M)^2$, M_{P} is the Planck mass, assuming that to $O(\varepsilon)$ the functions e^{ψ} and $m(r)$ have the expansions

$$\mathrm{e}^{\psi} = 1 + \varepsilon\rho(r) \tag{14}$$

and

$$m(r) = M(1 + \varepsilon\mu(r)). \tag{15}$$

Indeed, substituting (11–13) into the semi-classical field equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle, \tag{16}$$

and retaining the $O(\varepsilon)$ terms, one obtains two independent linear equations governing $\mu(r)$ and $\rho(r)$

$$\frac{\varepsilon M}{4\pi r^2}\frac{d\mu}{dr} = -\langle T_t^t\rangle, \tag{17}$$

and

$$\frac{\varepsilon}{4\pi r}\frac{d\rho}{dr} = \left(1-\frac{2M}{r}\right)^{-1}\left(\langle T_r^r\rangle - \langle T_t^t\rangle\right). \tag{18}$$

Solving (17) and (18) with $\langle T_t^t\rangle$ and $\langle T_r^r\rangle$ given by (3–7), we obtain

$$\begin{aligned}
K\mu(r) &= 4\ln\left(\frac{r}{2M}\right) + \frac{M^5}{r^5}\left(\frac{24640}{3}a_4 + \frac{352}{45}A_8\right) \\
&\quad -\frac{M^4}{r^4}\left(\frac{69280}{9}a_4 + \frac{128}{27}A_8 + 4\beta\right) + \frac{M^3}{r^3}\left(88 + \frac{5440}{3}a_4 + \frac{8}{3}\beta\right) \\
&\quad -12\frac{M^2}{r^2} - 10\frac{M}{r} - \frac{22}{3} - \frac{20}{9}a_4 + \frac{7}{135}A_8 - \frac{\beta}{12} \\
&\quad +\frac{3}{2}\frac{r}{M} + \frac{r^2}{4M^2} + \frac{r^3}{24M^3} + C_0 \\
&= K\mu_0(r) + C_0\,,
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
K\rho(r) &= 4\ln\left(\frac{r}{2M}\right) - \frac{M^5}{r^5}\left(\frac{8960}{3}a_4 + \frac{128}{45}A_8\right) + \frac{M^4}{r^4}\left(\frac{2720}{3}a_4 + \frac{4}{3}\beta\right) \\
&\quad -\frac{112}{3}\frac{M^3}{r^3} - 20\frac{M^2}{r^2} - \frac{40}{3}\frac{M}{r} + 14 + \frac{110}{3}a_4 + \frac{4}{45}A_8 \\
&\quad -\frac{\beta}{12} + \frac{r}{M} + \frac{r^2}{12M^2} + k_0 \\
&= K\rho_0(r) + k_0,
\end{aligned} \tag{20}$$

where $K = 3840\pi$, and, C_0 and k_0 are integration constants. Note that the functions $\mu_0(r)$ and $\rho_0(r)$ are constructed in such a way that they vanish on the event horizon.

Generally, we observe that the differences between the results obtained with the aid of the Page approximation and these obtained with regard to the tensor Δ_ν^μ are expected to be important near the event horizon.

Solving Eqs. (17) and (18) with the source term given by

$$\langle T_\nu^\mu\rangle^{\text{Page}} + \left(\xi - \frac{1}{6}\right) D_\nu^\mu \tag{21}$$

one obtains the solution propounded by Anderson *et al.* [7].

$$\begin{aligned}
K\mu(r) &= 4\ln\left(\frac{r}{2M}\right) + 88\frac{M^3}{r^3} - 12\frac{M^2}{r^2} - 10\frac{M}{r} - \frac{22}{3} + \frac{3r}{2M} + \frac{r^2}{4M^2} + \frac{r^3}{24M^3} \\
&\quad -\left(\xi - \frac{1}{6}\right)\left(800\frac{M^3}{r^3} - 240\frac{M^2}{r^2} - 120\frac{M}{r} + 20\right) + C_0
\end{aligned} \tag{22}$$

and

$$K\rho(r) \;=\; 4\ln\left(\frac{r}{2M}\right)+\frac{112}{3}\frac{M^3}{r^3}-20\frac{M^2}{r^2}-\frac{40}{3}\frac{M}{r}+14+\frac{r}{M}+\frac{r^2}{12M^2}$$
$$+\left(\xi-\frac{1}{6}\right)\left(640\frac{M^3}{r^3}+240\frac{M^2}{r^2}+80\frac{M}{r}-180\right)+k_0\,. \qquad (23)$$

Inspection of equations (15) and (19) shows that the constant C_0 may be included into the mass term thus renormalizing the black hole (bare) mass. It is possible because a bare mass has no meaning in the $O(\varepsilon)$ calculations. Accordingly, in all our subsequent analyses we assume that the black hole mass has been renormalized, *i.e.* M stands for the dressed mass. Therefore, in order to complete the first order back reaction calculations, one has to determine only one integration constant-k_0, and the choice of k_0 should be dictated by the nature of the problem.

The modifications of the effective mass function of the conformally invariant theory caused by terms (5)–(7) are exhibited in Fig. 1. The effective mass of the radiation evaluated with the aid of the Page approximation is zero at the event horizon, approaches its minimal value at $r \sim 2.34M$, and passes through zero at $r \sim 2.8M$. As is seen in Fig. 1, one has similar behaviour of the effective mass function evaluated for the stress energy tensor given by (3). Specifically, it starts from zero at $r = 2M$, achieves its minimal value at $r = 2.44M$, and passes through zero at $r \sim 3.02M$. For illustrative purposes we presented also the effective mass function evaluated for the model of Anderson *et al.* for the minimal coupling ($\xi = 0$) and for $\xi = 1/3$.

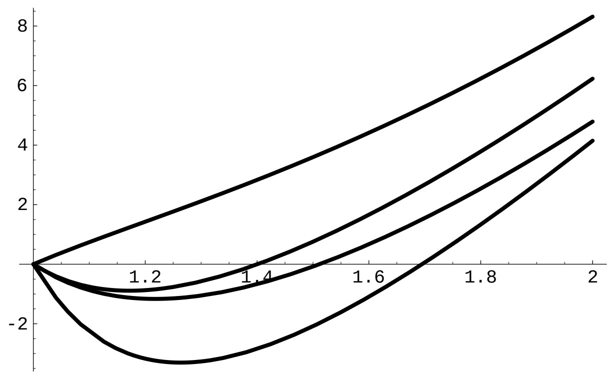

Fig. 1. The effective mass μ as a function of $y = r/2M$. Top to bottom the curves are for $\xi = 1/3$, $\xi = 1/6$ (Page approximation), $\xi = 1/6$ (our approximation), and $\xi = 0$. The effective mass for the nonconformal coupling is evaluated within the framework of the Anderson *et al.* model.

The solutions (19) and (20), and their asymptotic behaviour as $r \to \infty$, shows that it is necessary to enclose the system that consists of the black hole and the quantized scalar field in a cavity of a definite radius and to impose appropriate boundary conditions. Typically, one imposes either microcanonical or canonical boundary conditions. In the microcanonical ensemble one considers as a boundary an ideal massless and perfectly reflecting spherical wall of negligible mass and thickness, and the total effective energy is constant, whereas in the canonical ensemble the temperature is kept fixed at boundaries of the system by coupling the cavity wall with the external heat reservoir. To determine the constant k_0 we shall adopt the microcanonical boundary conditions.

Outside the spherical wall the metric is strictly Schwarzschildian

$$ds^2 = -\left(1 - \frac{2m(R)}{r}\right) dt^2 + \left(1 - \frac{2m(R)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{24}$$

and is described by a total mass

$$m(R) = M(1 + \varepsilon\mu_0(R)), \tag{25}$$

where R is a radius of the cavity. Inside one has the corrected metric described by (11) with (19) and (20). Requirement of the continuity of the inner and outer metric on the reflecting wall results in the condition

$$k_0 = -K\rho(R), \tag{26}$$

and therefore in the region $2M < r < R$ one has

$$ds^2 = -\left(1 - \frac{2m(r)}{r}\right)(1 + 2\varepsilon\tilde{\rho}(r))\, dt^2 + \left(1 - \frac{2m(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{27}$$

where $\tilde{\rho}(r) = \rho(r) - \rho(R)$.

The radius itself may not be arbitrarily great because the system consisting of the black hole and the quantized radiation would collapse producing a larger black hole. On the other hand, it should not be too small because the surface term of the full stress-energy tensor is ignored in the calculations. Moreover, modifications of the exterior Schwarzschild metric caused by the semi-classical effects in the Boulware state are also ignored. It is a standard procedure, and for discussion of this issue see York's paper [1]. As is well known the semi-classical theory has only limited range of validity and in general context it should be replaced by unknown as yet quantum gravity. Therefore, in order to remain within the range of applicability of the quantum field theory in curved background and be able to treat the back reaction programme perturbatively, one should take $M > M_{\mathrm{P}}$.

A rough estimate of the radius may be obtained from the hoop conjecture, that in its original version states that a horizon forms if matter with a total mass M gets compactified in a region whose circumference, $\mathcal{L}$, in all directions satisfies the inequality $\mathcal{L} < 4\pi M$ [32]. Since the total mass of the system is given by (25) one concludes, by the hoop conjecture, that the maximal radius of the cavity, R, should satisfy

$$R > 2M + 2\varepsilon M \mu_0(R). \tag{28}$$

The maximal radius of the cavity may be determined imposing reasonable requirement that the changes of the metric caused by the quantized field remain small unless $r < R$, and hence a natural condition for tractability of the back reaction calculations is

$$\varepsilon |h^{\mu}_{\nu}| = \varepsilon_0 < 1, \tag{29}$$

where ε_0 is a dimensionless parameter. The fractional corrections to the metric, h^{μ}_{ν}, may be obtained from

$$g_{\mu\nu} = \tilde{g}_{\mu\nu}(\delta^{\mu}_{\nu} + h^{\mu}_{\nu}), \tag{30}$$

where $\tilde{g}_{\mu\nu}$ is the uncorrected Schwarzschild metric. The perturbations of the metric are explicitly given by

$$h^{r}_{r} = 2M \frac{\mu(r)}{r - 2M} \tag{31}$$

and

$$h^{t}_{t} = -2[\rho(R) - \rho(r)] - h^{r}_{r}. \tag{32}$$

Simple calculations show that the consequences of the hoop conjecture and these of equations (29) and (31) are equivalent, and h^{r}_{r} evaluated on the event horizon does not depend on the parameter β. On the other hand, the condition (29) applied to the metric perturbation h^{t}_{t} is more restrictive, and the maximal radius of the cavity may be obtained from the analysis of its behaviour on the event horizon. Results of the computations of $y_{\max} = R_{\max}/2M$ as a function of ε for several values of ε_0 are displayed in Fig. 2. The horizontal line refers to the simplified case $\varepsilon = \varepsilon_0$.

Note, that although the results presented in Fig. 2. do satisfy the condition (29), some of them cannot be used as the maximal radii. The reason is that the calculations have been carried out with the aid of the stress-tensor (3), which is constructed for infinite spacetime. Clearly, in the presence of the boundary some of the radial modes should be ignored. In this connection, an useful condition is, that the radius of the cavity should be greater

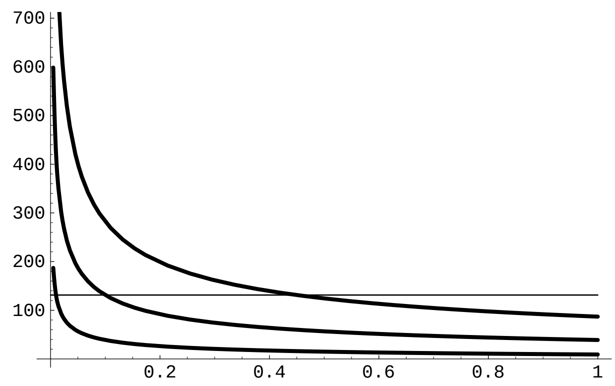

Fig. 2. The maximal radius of the spherical box $y_{\rm max} = R_{\rm max}/2M$ as a function of ε. Top to bottom, the curves are for $\varepsilon_0 = 0.45, 0.1, 0.01$. The horizontal line refers to a simplified case of $\varepsilon = \varepsilon_0$.

than the wavelength of the least-damped quasinormal mode of the lowest angular momentum, which is approximately $42M$. Such a mode is, in turn, associated with the longest characteristic wavelength of Hawking radiation.

More detailed information concerning the maximal radii may be presented thanks to the observation that $y_{\rm max}$ depends only on the ratio of $\varepsilon_0/\varepsilon$. Figure 3 shows $y_{\rm max}$ as a function of $\varepsilon_0/\varepsilon$.

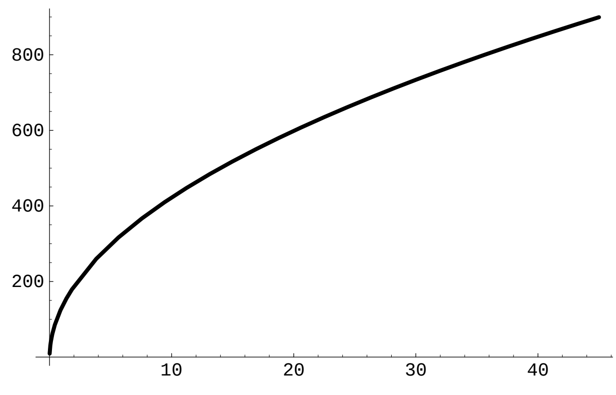

Fig. 3. The maximal radius of the spherical box $y_{\rm max} = R_{\rm max}/2M$ as a function of $\varepsilon_0/\varepsilon$.

4. Temperature and entropy

As is well known the equilibrium temperature of static and self gravitating system , $T_{\rm loc}(r)$, is given by the Tolman formula

$$T_{\rm loc}(r) = T|g_{tt}|^{-1/2}. \tag{33}$$

When the back reaction effects are taken into account the black hole temperature, T, is no longer of the form $T = 1/(8\pi M)$. The $O(\varepsilon)$ black hole temperature may be constructed by means of the general formula

$$T = \frac{\kappa}{2\pi}, \tag{34}$$

where κ is the black hole surface gravity. What is the meaning of T in the back reaction calculations with a microcanonical boundary conditions? Simple considerations indicate that it is the corrected temperature at spatial infinity.

The surface gravity may be easily calculated form

$$\kappa^2 = -\tfrac{1}{2} t_{\mu;\nu} t^{\mu;\nu}{}_{|r=2M}\,, \tag{35}$$

where t^μ is a timelike Killing vector field. Retaining the terms linear in ε one has

$$\kappa = \frac{1}{4M}\left[1 + \frac{\varepsilon}{K}\left(k_0 + 12 + \frac{350}{9}a_4 + \frac{A_8}{27}\right)\right]. \tag{36}$$

Note that the result does not depend on the parameter β. Moreover,

$$T_{\rm loc}(r) = \frac{1}{8\pi M}\left[1 + \frac{\varepsilon}{K}\left(12 - \rho_0(r) + \frac{350}{9}a_4 + \frac{A_8}{27}\right)\right]\left(1 - \frac{2m(r)}{r}\right)^{-1/2} \tag{37}$$

is independent of k_0 as have been observed earlier by York [1].

The black hole temperature may be also found from the examination of the complexified metric (11) obtained from the Wick rotation $t \to -it$. Indeed, with a periodicity of the time coordinate, β_H, suitably adjusted to remove the conical singularities of the line element one has

$$\beta_H = \frac{1}{T} = 4\pi \lim_{r\to 2M}\left(U(r)V^{-1}(r)\right)^{1/2}\left(\frac{dU(r)}{dr}\right)^{-1}. \tag{38}$$

It is a straightforward task to show that to $O(\varepsilon)$ the black hole temperature obtained from (38) coincides with (33). Euclideanized version of the line element allows to identify the inverse of the local temperature, for fixed r, θ, and ϕ, with the periodically identified proper length of the t

$$\frac{1}{T_{\rm loc}(r)} = \int\limits_0^{\beta_H} U(r)^{1/2} d\tau. \tag{39}$$

The main features of the entropy of the quantized field have been extensively investigated in numerous papers; a general expression, the proof of positivity, and monotonic increase with r is firmly established for conformally invariant massless scalar (Page approximation [2]), vector (Jensen and Ottewill approximation [19]), and to certain extent spinor field [22]. Unfortunately, there are no numerical calculations of the $\langle T_\nu^\mu \rangle$ of the quantized conformal spinor and therefore validity of the Brown–Ottewill–Page approximation in this context is unknown.

Detailed discussion and the method for constructing entropy has been presented in Ref. [5]. Simple and compact expression describing ΔS has been recently derived by Zaslavskii [33]

$$\Delta S = 32M\pi^2 \int\limits_{2M}^{r} dr' r'^2 \left[\langle T_r^r \rangle \; - \; \langle T_t^t \rangle - \langle T_\mu^\mu \rangle \ln \left(\frac{r}{r'} \right) \right], \tag{40}$$

where the arbitrary integration constant has been fixed by the demand of vanishing of ΔS at the event horizon of the black hole. When applied to regular stress energy tensor (40) exhibits some general features: the radial derivative of the entropy vanishes at the event horizon, and S is a positive function of r monotonically increasing with radius. It should be noted that this features, as have been observed by Hochberg, Kephart, and York, do not hold if one ignores the back reaction.

Now we shall analyse the modifications of the ΔS caused by the Δ_ν^μ piece. Substituting (3) with (5–7) into (40), after simple integration, one finds

$$\begin{aligned} \Delta S = {} & \frac{1}{1080} w^{-3} + \frac{1}{360} w^{-2} + \frac{1}{120} w^{-1} - \frac{1}{540} + \frac{a_4}{54} + \frac{A_8}{10800} \; - \; \frac{\beta}{8640} - \frac{w}{72} - \frac{w^2}{120} \\ & + \left(\frac{13}{1080} + \frac{17}{54} a_4 + \frac{\beta}{2160} \right) w^3 - \left(\frac{13}{18} a_4 + \frac{A_8}{2160} + \frac{\beta}{2880} \right) w^4 \; + \; \left(\frac{7}{18} a_4 + \frac{A_8}{2700} \right) w^5 \\ & + \; \frac{1}{90} \ln w \, . \end{aligned} \tag{41}$$

In order to recover the result obtained in Ref. [7], *i.e.* when an arbitrary coupling is taken into account, the right hand side of the above equation (with β, a_4, and A_8 set to zero) should be supplemented with the following term

$$\left(\xi - \frac{1}{6}\right)\left(-\frac{1}{8} + \frac{w}{12} + \frac{w^2}{8} - \frac{w^3}{12} - \frac{1}{12}\ln w\right). \tag{42}$$

Simple calculation show that the first derivative of the entropy with respect to the radial coordinate vanishes, as expected, at the event horizon whereas the second derivative there equals

$$\frac{\partial^2 \Delta S}{\partial r^2} = \frac{1}{M^2}\left(\frac{1}{45} + \frac{a_4}{4} + \frac{A_8}{2160} - \frac{\beta}{2880}\right). \tag{43}$$

Putting in the above equation β, a_4 and A_8 equal to zero and subtracting

$$\frac{1}{24M^2}\left(\xi - \frac{1}{6}\right) \tag{44}$$

one obtains the analogous formula to that from Ref. [7].

One of the most interesting and important consequences of the back reaction calculations is the possibility to determine the inequalities that are to be satisfied by unknown parameters describing either the renormalized stress tensor or the coupling constants. Indeed, the positivity of the second derivative of the entropy at the event horizon yields definite constraint whereas the reasonable smallness of the perturbations of the quantum corrected metric may yield, in certain cases, another one. It should be noted however that the second requirement is much less conclusive and in the case at hand it would involve some sort of prescience. Indeed, taking β as a free parameter, from (43) one has $\beta < -31.5$, and further determination of the parameter β would require knowledge of the maximal radius of the spherical wall.

5. The effective potential

Further informations concerning the nature of the modified black hole geometry may be gained by studying the motion of the test particles. Now, we shall analyse the effective potential for circular orbits of both massless and massive particles.

Because of staticity and spherical-symmetry of the quantum-corrected metric the first integral of the test particle's geodesic equation in the equatorial plane $\theta = \pi/2$, may be easily constructed

$$E^2 = -g_{tt}g_{rr}\dot{r}^2 - g_{tt}\left(\tilde{\kappa} + \frac{L^2}{r^2}\right), \tag{45}$$

where $E = -g_{tt}\dot{t}$, is particle's total energy, $L = g_{\phi\phi}\dot{\phi}$, is its orbital angular momentum, and $\tilde{\kappa} = 0, 1$ for the null and timelike geodesics, respectively;

the overdot denotes differentiation with respect to an affine parameter (null curves) or the proper time (timelike orbits). One may therefore identify

$$V^2 = \left(1 - \frac{2m(r)}{r}\right)\left[1 + 2\varepsilon\left(\rho(r) - \rho(R)\right)\right]\left(\tilde{\kappa} + \frac{L^2}{r^2}\right) \tag{46}$$

with the effective potential for circular orbits. Retaining in the effective potential $O(\varepsilon)$ terms one has

$$V^2(r) = \left(\tilde{\kappa} + \frac{L^2}{r^2}\right)\left(1 - \frac{2M}{r}\right)\left(1 + \varepsilon h_t^t\right). \tag{47}$$

Since h_t^t depends on the magnitude of the box so does the effective potential. In the further calculations we assume, for simplicity, that the radius of the spherical wall for fixed ε and ε_0 is a maximal one, in a sense that it is obtained from (29) and (32) evaluated on the event horizon. With this choice of R the fractional effective potential is a function of r and the ratio $\varepsilon_0/\varepsilon$.

First, consider the case of the null geodesics. Inspection of (47) shows that the only local extremum of V^2 is a maximum corresponding to the unstable circular photon orbit and that the maximum of the effective potential for a given black hole mass decreases with increase of ε_0. It is interesting to note that for a fixed ε_0 the difference

$$\mathit{\Delta} = \frac{4M^2}{L^2}\left[V^2(\varepsilon, w) \; - \; V^2(\varepsilon', w)\right] \tag{48}$$

may be written as

$$\mathit{\Delta} = (\varepsilon - \varepsilon')F(w), \tag{49}$$

where

$$F(w) = w^2(1 - w)\left[2\rho(w) - h_r^r(w) + h_r^r(1)\right], \tag{50}$$

and $w = 2M/r$. Analysis of $F(w)$ shows that it is a positive function of w nowhere exceeding 1.5×10^{-4}. This explains why for a given ε_0 the effective potential is practically independent of ε. This does not contradict the observation of Hochberg *et al.* (Ref. [6]), that V^2 depends on the ε since they considered simplified case of $\varepsilon = \varepsilon_0$. Typical results of calculations for $\varepsilon = 0.1$ and $\varepsilon_0 = 0.02$, 0.1, and 0.45 are presented in Fig. 4.

For a given ε the magnitude of V^2 decreases with the incerase of ε_0 or equivalently with the increase of the radius of the spherical wall. It should be noted that the results for $\varepsilon_0 > 0.5$ are unreliable in the first order calculations.

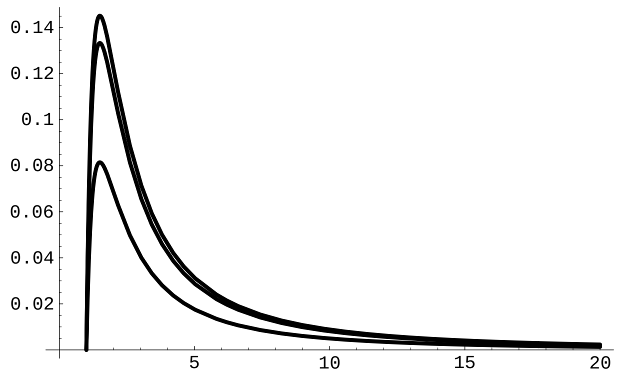

Fig. 4. This graph shows $\frac{4M^2}{L^2}V^2$ of the massless particle as a function of $y = r/(2M)$ for $\varepsilon = 0.1$. Top to bottom the curves are for $\varepsilon_0 = 0.02$, $\varepsilon_0 = 0.1$, and $\varepsilon_0 = 0.45$.

As in the classical Schwarzschild geometry the case of massive particles is more complicated. The shape of the effective potential for timelike circular geodesics and consequently the existence of local extrema of V^2 critically depends on the value of L. From (47) with $\tilde{\kappa} = 1$ one concludes that there are no stable orbits for $L < L_{\rm crit}$ whereas for $L > L_{\rm crit}$ in the interval $2M < r < R$ the potential V^2 has both the local minimum and maximum. In Fig. 5 we illustrated the run of the potential for some typical cases.

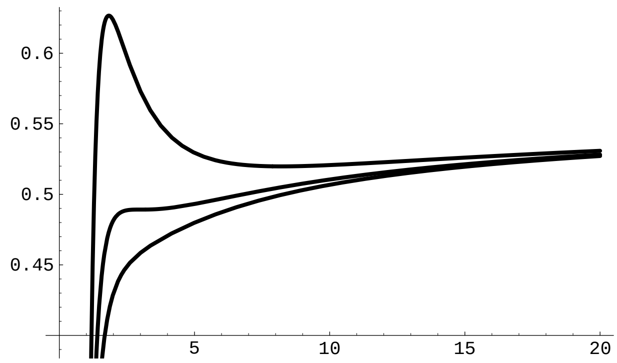

Fig. 5. The effective potential V^2 of a massive particle as a function of $r/2M$ for $\varepsilon = 0.1$ and $\varepsilon_0 = 0.45$. Top to bottom the curves are for $l = 3$, $l = l_{\rm crit} = 3.466$, and $l = 4.5$. At the point of inflexion the last stable circular orbit occur.

Although for various values of ε qualitative features of V^2 constructed for a fixed ε_0 remain unchanged there are important quantative differences, especially as $r \to \tilde{R}$, where $\tilde{R}$ denotes the smaller value of $R_{\rm max}$.

Numerical calculations show that $l_{\rm crit} = L_{\rm crit}/M$ weakly depends on the ε_0 and ε always remaining close to its Schwarzschild analogue $2\sqrt{3}$. Specifically, for fixed ε, $l_{\rm crit}$ increases with the increase of ε_0 and, for fixed ε_0 it increases with ε. Although the $l_{\rm crit}$ remains close to $2\sqrt{3}$, both $V^2_{\rm max}$ and $V^2_{\rm min}$ strongly depend on the back reaction. The minima and maxima of the effective potential as a function of the reduced angular momentum $l = L/M$ are displayed in Fig. 6. The cusp in Fig. 6 represents the inflexion point of the effective potential for $l = l_{\rm crit}$ visible in Fig. 5.

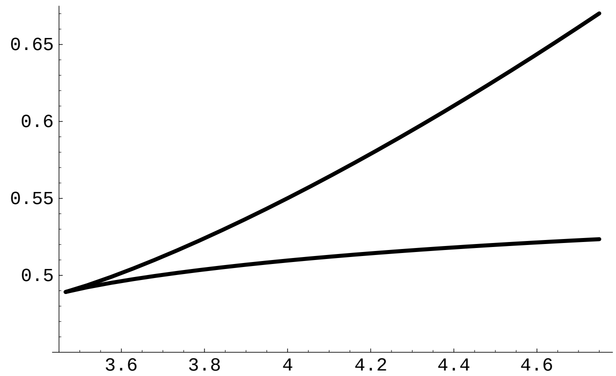

Fig. 6. This graph shows local extrema of the effective potential V^2 as a function of the reduced angular momentum l.

6. Trace anomaly of the corrected space

Corrections to the trace anomaly caused by the back reaction are given by a purely geometric terms constructed from the $O(\varepsilon)$ quantum-corrected metric. In general, for a conformally invariant masless scalar field the trace anomaly is given by

$$\langle T^{\mu}_{\mu}\rangle = aH + bG + c\,\Box R, \tag{51}$$

where H is a square of the Weyl tensor

$$H = C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau} = R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} - 2R^{\mu\nu}R_{\mu\nu} + \tfrac{1}{3}R^2\,, \tag{52}$$

and

$$G =^{*} R^{\mu\nu\sigma\tau}\,{}^{*}R_{\mu\nu\sigma\tau} = R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} - 4R^{\mu\nu}R_{\mu\nu} + R^2, \tag{53}$$

and, $a = 2/(3840\pi^2)$, $b = -1/3\,a$, $c = 2/3\,a$. It could be easily shown that for the metric at hand both $R^{\mu\nu}R_{\mu\nu}$ and R^2 are $O(\varepsilon^2)$, and in the first order calculations $H = G$. Remaining $O(\varepsilon^2)-$ terms in the trace anomaly are given by

$$R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau} = C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau} = \frac{48M^2}{r^6} + \frac{4\varepsilon}{3r^6}\left[72M^2\mu(r) - 48M^2 r\frac{d\mu}{dr}\right.$$
$$\left. - \left(60M^2 r - 12Mr^2\right)\frac{d\rho}{dr} + 12M^2r^2\frac{d^2\mu}{dr^2} + \left(24M^2r^2 - 12Mr^2\right)\frac{d^2\rho}{dr^2}\right] \qquad (54)$$

and

$$\Box R = \frac{2\varepsilon}{r^5}\left[\left(4Mr - 16M^2\right)\frac{d\mu}{dr} + \left(6Mr - 8M^2\right)\frac{d\rho}{dr} + \left(10M^2r - 4Mr^2\right)\frac{d^2\mu}{dr^2}\right.$$
$$+ \left(2M^2r - 6Mr^2\right)\frac{d^2\rho}{dr^2} - \left(2M^2r^2 - 2Mr^3\right)\frac{d^3\mu}{dr^3} + \left(2M^2r^2 + 7Mr^3 - 4r^4\right)\frac{d^3\rho}{dr^3}$$
$$\left. + \left(Mr^4 - 2M^2r^3\right)\frac{d^4\mu}{dr^4} + \left(4Mr^4 - 4M^2r^3 - r^5\right)\frac{d^4\rho}{dr^4}\right]. \qquad (55)$$

Making use of (19) and (20) one obtains

$$C^{\mu\nu\sigma\tau}C_{\mu\nu\sigma\tau} = \frac{48M^2}{r^6} + \left[\frac{2576a_4M^7}{3r^{11}} + \frac{184A_8M^7}{225r^{11}} - \frac{6164a_4M^6}{9r^{10}} - \frac{56A_8M^6}{135r^{10}}\right.$$
$$- \frac{11\beta M^6}{30r^{10}} + \frac{74M^5}{15r^9} + \frac{136a_4M^5}{r^9} + \frac{\beta M^5}{5r^9} - \frac{M^4}{2r^8} - \frac{M^3}{3r^7} - \frac{13M^2}{60r^6}$$
$$\left. - \frac{a_4M^2}{18r^6} + \frac{7A_8M^2}{5400r^6} - \frac{\beta M^2}{480r^6} + \frac{M}{40r^5} + \frac{1}{480r^4} + \frac{M^2}{10r^6}\log\left(\frac{r}{2M}\right)\right]\frac{\varepsilon}{\pi} \qquad (56)$$

and

$$\Box R = \left(\frac{48M^5}{5r^9} - \frac{4M^4}{r^8}\right)\frac{\varepsilon}{\pi}. \qquad (57)$$

Note that $\Box R$ is independent of the parameters a_4, A_8, and β and therefore equal to its equivalent evaluated within the framework of the Page approximation.

Instead of considering $\langle T_\mu^\mu\rangle$ itself, we shall analyse the fractional trace anomaly, Δ, defined as

$$\Delta = \frac{\langle T_\mu^\mu\rangle - \langle \tilde{T}_\mu^\mu\rangle}{\langle \tilde{T}_\mu^\mu\rangle}, \qquad (58)$$

where in $\langle \tilde{T}_\mu^\mu\rangle = M^2/(60\pi^2 r^6)$ one recognizes the pure conformal anomaly of the scalar field in the Schwarzschild spacetime. The results of the calculations of Δ are presented in Fig. 7.

Inserting (56) and (57) in (51) and constructing Δ one readily finds that in the limit of large r the leading behaviour of Δ is proportional to r^2. More precisely, for the corrected metric determined by the Page approximation

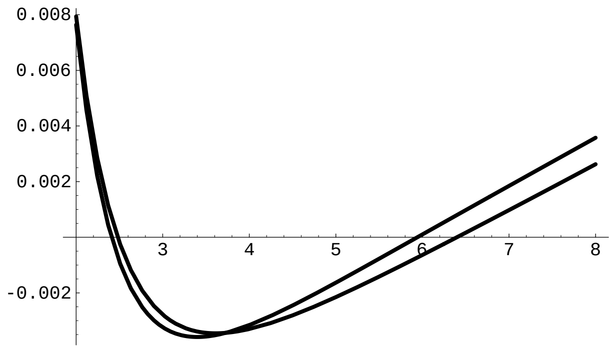

Fig. 7. This graph shows the rescaled fractional trace anomaly of the scalar field in the corrected spacetime as a function of r/M. $\pi\Delta/\varepsilon$ evaluated for the solution of the back reaction equations with the source term given by the improved approximation of the stress energy tensor is the line shifted to the right with respect to the one which has been determined with the aid of the Page approximation

$\pi\,\varDelta/\varepsilon$ is zero at $r = 2.41M$, approaches its minimal value at $r = 3.4M$, and passes through zero at $r = 5.94M$. Similarly for the improved approximation $\pi\,\varDelta/\varepsilon$ is zero at $r = 2.48M$, has the miniumum at $r = 3.61M$, and is zero at $r = 6.4M$. The fractional trace anomaly on the event horizon is

$$\varDelta = \frac{\varepsilon}{\pi}\left(\frac{11}{1440} + \frac{35a_4}{1728} + \frac{A_8}{51840}\right). \tag{59}$$

In the absence of $\varDelta^{\mu}_{\nu}$ — term (59) reduces to the analogous result presented in [1].

7. Concluding remarks

Let us now summarize and add a few remarks to the results presented in this paper. We have investigated the first order back reaction of the quantized scalar field upon spherically-symmetric static black hole by solving linearized semi-classical Einstein field equations. Adopted source term is given by recently proposed approximation of the stress- energy tensor, which reproduces exact $\langle T^{\mu}_{\nu}\rangle$ in the Hartle-Hawking state in the Schwarzschild spacetime to a high accuracy. Our analyses extend earlier works based on the Page approximation. As expected, modifications of the metric and the physical characteristics caused by the $\varDelta^{\mu}_{\nu}$ are significant near the event horizon. More important modifications are produced by the stress energy tensor of the scalar field with arbitrary coupling.

Our calculations suffer from one important drawback, they do not incorporate quantum fluctuations of the metric, *i.e.*, the graviton contribution (unknown as yet) to the stress-energy tensor has been ignored. Some light on the problem has been recently shed by Hochberg and Sushkov. It should be emphasised once again that we are dealt with the linear theory. Of course, it would be desirable to extend present analysis of the back reaction problem to higher order. This, however, would require incorporation of the quadratic curvature terms to the field equations, and, moreover, may be analitically intractable.

REFERENCES

[1] J.W. York, *Phys. Rev.* **D31**, 775 (1985).

[2] D.N. Page, *Phys. Rev.* **D25**, 1499 (1982).

[3] K.W. Howard, *Phys. Rev.* **D30**, 2532 (1984).

[4] D. Hochberg, T.W. Kephart, *Phys. Rev.* **D47**, 1465 (1993).

[5] D. Hochberg, T.W. Kephart, J.W. York, *Phys. Rev.* **D48**, 479 (1993).

[6] D. Hochberg, T.W. Kephart, J.W. York, *Phys. Rev.* **D49**, 5257 (1994).

[7] P.R. Anderson, W.A. Hiscock, J. Whitesell, J.W. York, *Phys. Rev.* **D50**, 6427 (1994).

[8] D. Hochberg, S.V. Sushkov, *Phys. Rev.* **D53**, 7094 (1996).

[9] C.F.P. Lee, *Phys. Rev.* **D47**, 2260 (1993).

[10] W.A. Hiscock, *Phys. Rev.* **D37**, 2142 (1988).

[11] R. Brout, S. Massar, R. Parentani, P. Spindel, *Phys. Rep.* **260**, 329 (1995).

[12] N. Sanchez, C.O. Lousto, *Phys. Lett.* **212B**, 411 (1988).

[13] J.W. York, in *Quantum Theory of Gravity*, edited by S. M. Christensen Hilger, Bristol 1984.

[14] R. Balbinot, *Phys. Rev.* **D33**, 1611 (1986).

[15] R. Mallett, *Phys. Rev.* **D34**, 1916 (1986).

[16] J. Matyjasek, *Acta. Phys. Pol.* **B21**, 851 (1990).

[17] T. Elster, *Phys. Lett.* **94A**, 205 (1983).

[18] B.P. Jensen, J.G. McLaughlin, A.C. Ottewill, *Phys. Rev.* **D43**, 4142 (1991).

[19] B.P. Jensen, A.C. Ottewill, *Phys. Rev.* **D39**, 1130 (1989).

[20] V. P. Frolov, A.I. Zel'nikov, *Phys. Rev.* **D35**, 3031 (1987).

[21] M. R. Brown, A.C. Ottewill, *Phys. Rev.* **D31**, 2514 (1985).

[22] M. R. Brown, A.C. Ottewill, D.N. Page, *Phys. Rev.* **D33**, 2840 (1986).

[23] P. R. Anderson, W.A. Hiscock, D.A. Samuel, *Phys. Rev. Lett.* **70**, 1739 (1993).

[24] P.R. Anderson, W.A. Hiscock, D.A. Samuel, *Phys. Rev.* **D51**, 4337 (1995).

[25] J. Matyjasek, *Phys. Rev.* **D53**, 794 (1996).

[26] J. Matyjasek, IF UMCS Report 1995, p.33.

[27] J. Matyjasek, *Phys. Rev.* **D55**, 809 (1997).

[28] J. Matyjasek, *Classical Quantum Gravity* **14**, L15 (1997).

[29] M. Visser, *Phys. Rev.* **D56**, 936 (1997).

[30] I.D. Novikov, V.P. Frolov, *Physics of Black Holes* , Nauka, Moscow 1986 (in Russian).

[31] J. Matyjasek, *Acta. Phys. Pol.* **B27**, 2005 (1996).

[32] Ch.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation*, Freeman, San Francisco 1973.

[33] O. B. Zaslavskii, *Phys. Lett.* **181A**, 105 (1993).