

GEOMETRY ASSOCIATED WITH SELF-DUAL YANG–MILLS AND THE CHIRAL MODEL APPROACHES TO SELF-DUAL GRAVITY

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A geometric formulation of the Moyal deformation for the self-dual Yang–Mills theory and the Chiral Model approach to self-dual gravity is given. We find in Fedosov’s geometrical construction of deformation quantization the natural geometrical framework associated to the Moyal deformation of the six-dimensional version of the second heavenly equation and the Park–Husain heavenly equation. The Wess–Zumino–Witten-like Lagrangian of self-dual gravity is reexamined within this context.

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1. Introduction

The purpose of this paper is to describe some conjectures in the geometry of deformation quantization for Self-Dual Yang–Mills (SDYM) theory and the Chiral Model approach to Self-Dual Gravity (SDG). This relation was originally suggested by Strachan in [1]. He has developed a deformed differential commutative geometry and has applied it to describe, within this geometrical framework, the multidimensional integrable systems. Here we intend to consider the application of some *non-commutative* geometry (Fedosov’s geometry) to self-dual gravity.

The relation, for instance, between SDYM theory, Conformal Field Theory and Principal Chiral Model, all of them with gauge group $\text{SDiff}(\Sigma)$

(area-preserving diffeomorphism group of two-dimensional simply connected and symplectic manifold Σ), has been studied only at the algebraic level [2–10]. The standard approach consists in studying a classical field theory invariant under some symmetry group, for instance, $SU(N)$. In the case of full Yang–Mills theory its large- N limit ($N \rightarrow \infty$) is somewhat mysterious, however it is very necessary to understand it in the searching for new faces of integrability [11]. Drastic simplifications in some classical equations seem to confirm these speculations [12]. However, geometric and topological aspects of the correspondence SDYM and SDG remain to be clarified [13].

Integrable deformations of SDG, in particular the Moyal deformation of the first heavenly equation were studied by Strachan [14] and Takasaki [15]. Moyal deformation of the second heavenly equation was considered in [16,17]. The Weyl–Wigner–Moyal (WWM) formalism has been very useful in order to find (beginning from SDYM theory) a version of the ‘master system’ (compare with [2]) which leads to different versions of the heavenly equations in SDG [18,19]. The application of WWM-formalism to the Lagrangians and equations of motion of SDG for a $\text{sdiff}(\Sigma)$ -valued scalar field on \mathcal{M} , leads to the Moyal deformation of the six-dimensional version of the second heavenly equation ($\text{sdiff}(\Sigma)$ can be seen as the Lie algebra of $\text{SDiff}(\Sigma)$). This equation and its associated Lagrangian was found in [20]. The application of the WWM-formalism to the Principal Chiral Model approach to dynamical SDG was done in terms of a scalar field by us in Ref. [21]. This dynamics was enclosed in the P-H heavenly equation [21]. In [21] also some explicit solutions were constructed using an explicit Lie algebra representation of linear operators acting on some Hilbert space. As an intermediate step of constructing solutions in SDG using WWM-formalism one can represent the dynamics using a Moyal deformation of the P-H heavenly equation [21]. WWM-formalism also has been very useful to show that some results about harmonic maps [22] can be carried over to SDG [23]. As a consequence one can define the “Gravitational Uniton” which seems to be simultaneously a uniton *i.e.*, an appropriate solution of chiral equations [22] and a solution of the P-H heavenly equation describing the SDG metric. A WZW-like action for SDG can be constructed as well. “How can one to interpret geometrically this WZW action and the associated “Wess-Zumino” term”? was a question proposed in the last part of [23]. Here we intend to give a partial answer to this question. In this paper, in a spirit of Strachan [1], we describe some conjectures about the geometry associated to integrable deformations of SDG and its possible relation with the standard geometry of gauge theory. The paper is organized as follows: To be the most self-contained, in Section 2 we briefly review the geometrical construction of deformation quantization given by Fedosov [24]. In Section 3 we discuss the Moyal deformation of SDYM theory, the associated six-dimensional version of the second heavenly

equation [20] and P-H heavenly equation given in Ref. [21]. In Section 4 we describe within Fedosov's geometrical construction, the Moyal deformation of the SDYM equations via Yang's and Donaldson–Nair–Schiff equations. In particular, the Moyal deformation of the six-dimensional version of the second heavenly equation is considered. In this same section we discuss the geometry associated with the principal chiral model approach to SDG. The Moyal deformation of the P-H heavens is described as well. Section 5 is devoted to study the WZW-like Lagrangian found in [23] in the same geometrical framework. Finally in Section 6 we give our final remarks.

2. Fedosov's geometry of deformation quantization

Deformations of the Poisson Lie algebra structure on symplectic manifolds have been studied by many authors [25]. Recently some extensions of the WWM-formalism have been carried over to the phase space as represented by cotangent bundle [26]. More recently interesting connections with non-commutative geometry have been studied by Reuter in [27].

In this section we review some aspects of the geometry of deformation quantization given by Fedosov in Ref. [24]. The most important objects we consider here involve Weyl algebra bundle, differential forms and the trace formula defined for this algebra.

2.1. Weyl algebra bundle

Let (\mathcal{M}, ω) be a symplectic manifold of dimension $2n$ and ω the corresponding symplectic form on \mathcal{M} . The formal Weyl algebra \mathcal{W}_x associated with the tangent space at the point $x \in \mathcal{M}$, $T_x\mathcal{M}$, is the *associative* algebra over \mathbf{C} with a unit. An element of \mathcal{W}_x can be expressed by

$$a(y) = \sum_{2k+l \geq 0} \hbar^k a_{k,i_1 \dots i_l} y^{i_1} \dots y^{i_l}, \quad (2.1)$$

where \hbar is the deformation parameter, $y = (y^1, \dots, y^{2n}) \in T_x\mathcal{M}$ is a tangent vector and the coefficients $a_{k,i_1 \dots i_l}$ constitute the symmetric covariant tensor of degree l at $x \in \mathcal{M}$.

The product on \mathcal{W}_x which determines the associative algebra structure is defined by

$$\begin{aligned} a \bullet b &\equiv \exp\left(+\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y^i}\frac{\partial}{\partial z^j}\right)a(y, \hbar)b(z, \hbar)|_{z=y} \\ &= \sum_{k=0}^{\infty} \left(+\frac{i\hbar}{2}\right)^k \frac{1}{k!} \omega^{i_1 j_1} \dots \omega^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \dots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \dots \partial y^{j_k}}, \end{aligned} \quad (2.2)$$

for all $a, b \in \mathcal{W}_x$. Here ω^{ij} are the components of the tensor inverse to ω_{ij} at x . Of course the product “ \bullet ” is independent of the basis.

Having this one can define an algebra bundle structure taking the disjoint union of Weyl algebras for all points $x \in \mathcal{M}$ i.e. $\widetilde{\mathcal{W}} = \bigsqcup_{x \in \mathcal{M}} \mathcal{W}_x$. $\widetilde{\mathcal{W}}$ is the total space and the fiber is isomorphic to a Weyl algebra \mathcal{W}_x . Thus we have the Weyl algebra bundle structure

$$\widetilde{\mathcal{W}} \xrightarrow{\pi} \mathcal{M}, \quad \mathcal{W}_x \cong \pi^{-1}(\{x\}), \quad (2.3)$$

where π is the canonical projection.

Let $\mathcal{E}(\widetilde{\mathcal{W}})$ be the set of sections of $\widetilde{\mathcal{W}}$ which also has a Weyl algebra structure with unit. Denote by $a(x, y, \hbar)$ an element of $\mathcal{E}(\widetilde{\mathcal{W}})$; it can be written as follows

$$a(x, y, \hbar) = \sum_{2k+l \geq 0} \hbar^k a_{k, i_1 \dots i_l}(x) y^{i_1} \dots y^{i_l}, \quad (2.4)$$

where $y = (y^1, \dots, y^{2n}) \in T_x \mathcal{M}$ is a tangent vector, $a_{k, i_1 \dots i_l}$ are smooth functions on \mathcal{M} and $x \in \mathcal{M}$.

2.2. Differential forms

In Section 4 we shall define a field theory on space-time with the fields taking values in the Weyl algebra of sections $\mathcal{E}(\widetilde{\mathcal{W}})$ (instead of the usual Lie algebra). In order to do that we need the notion of $\widetilde{\mathcal{W}}$ -valued differential q -form on \mathcal{M} . A q -form can be written as

$$\begin{aligned} a &= \sum \hbar^k a_{k, j_1 \dots j_q}(x, y) dx^{j_1} \wedge \dots \wedge dx^{j_q} \\ &= \sum_{2k+p \geq 0} \hbar^k a_{k, i_1 \dots i_p, j_1 \dots j_q}(x) y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_q}, \end{aligned} \quad (2.5)$$

where $a_{k, j_1 \dots j_q}(x, y) = a_{k, i_1 \dots i_p, j_1 \dots j_q}(x) y^{i_1} \dots y^{i_p}$.

The set of differential forms constitutes (similarly as the usual ones) a *Grassmann–Cartan* algebra $\mathcal{C} = \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda) = \bigoplus_{q=0}^{2n} \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q)$. In this space the multiplication $\overset{\bullet}{\wedge}$ is defined by

$$a \overset{\bullet}{\wedge} b = a_{[j_1 \dots j_p} \bullet b_{l_1 \dots l_q]} dx^{j_1} \wedge \dots \wedge dx^{j_p} \wedge dx^{l_1} \wedge \dots \wedge dx^{l_q}, \quad (2.6)$$

for all $a = \sum_k \hbar^k a_{k, j_1 \dots j_p}(x, y) dx^{j_1} \wedge \dots \wedge dx^{j_p} \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^p)$ and $b = \sum_k \hbar^k b_{k, l_1 \dots l_q}(x, y) dx^{l_1} \wedge \dots \wedge dx^{l_q} \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q)$. $a \overset{\bullet}{\wedge} b$ is defined by the usual wedge product on \mathcal{M} and the product \bullet in the Weyl algebra.

A very useful concept is that of central forms. In order to define it first consider the commutator defined on the sections $\mathcal{E}(\widetilde{\mathcal{W}})$ i.e., for all $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q_1})$ and $b \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q_2})$ we define

$$[a, b] \equiv a \overset{\bullet}{\wedge} b - (-1)^{q_1 q_2} b \overset{\bullet}{\wedge} a. \quad (2.7)$$

Thus a form $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda)$ is said to be central, if for any $b \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda)$ the commutator (2.7) vanishes. The set of central forms is designed by $\mathcal{Z} \otimes \Lambda$. Here \mathcal{Z} coincides with the algebra of quantum observables¹. In order to be more precise \mathcal{Z} is a linear space whose elements are

$$a = a(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k a_k(x), \quad (2.8)$$

where $a_k(x) \in C^\infty(\mathcal{M})$.

Let $a(x, y, \hbar)$ be an element of $\mathcal{E}(\widetilde{\mathcal{W}})$, we define the symbol map $\sigma : \mathcal{E}(\widetilde{\mathcal{W}}) \rightarrow \mathcal{Z}$, $a(x, y, \hbar) \mapsto a(x, 0, \hbar)$, that is the map σ is the projection of $\mathcal{E}(\widetilde{\mathcal{W}})$ onto \mathcal{Z} .

2.3. Differential operators

One can define some important differential operators. The operator $\delta : \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q) \rightarrow \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q+1})$ defined by

$$\delta a \equiv dx^k \wedge \frac{\partial a}{\partial y^k} \quad (2.9)$$

and its dual operator $\delta^\bullet : \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q) \rightarrow \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q-1})$ defined by

$$\delta^\bullet a \equiv y^k \frac{\partial}{\partial x^k} \rfloor a. \quad (2.10)$$

for all $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q)$, where \rfloor stands for the contraction.

A useful definition is that of the differential operator δ^{-1} acting on a monomial $y^{i_1} y^{i_2} \dots y^{i_p} dx^{j_1} \wedge dx^{j_2} \wedge \dots \wedge dx^{j_q}$. δ^{-1} is defined as

$$\delta^{-1} := \frac{\delta^\bullet}{(p+q)}, \quad p+q > 0$$

and $\delta^{-1} = 0$, for $p+q = 0$.

¹ The algebra of quantum observables can be defined introducing an associative product operation $*$ on the vector space \mathcal{Z} of functions $a(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k a_k(x)$ and $b(x, \hbar) = \sum_{k=0}^{\infty} \hbar^k b_k(x)$, with $a_k(x), b_k(x) \in C^\infty(\mathcal{N})$ (\mathcal{N} is an ‘internal’ manifold). The product $*$ is defined by $a * b = c = \sum_{k=0}^{\infty} \hbar^k c_k(x)$ for all $a, b, c \in \mathcal{Z}$ satisfying the properties (i) — c_k are polynomials in a_k and b_k and their derivatives, (ii) — $c_0(x) = a_0(x)b_0(x)$, (iii) — $[a, b] \equiv a * b - b * a = +i\hbar\{a_0, b_0\}_P + \dots$.

The operators δ and δ^\bullet satisfy several properties very similar to those for the usual differential and co-differential; for instance, there exists an analogue of Hodge–de Rham decomposition [24].

2.3.1. Symplectic connection

Assume the existence of a torsion-free connection on \mathcal{M} which preserves its symplectic structure. This connection is known as *symplectic connection* ∂_i .

The definition of an operator which do not change the degree of the Weyl algebra and only changes the degree of differential forms is also possible. This operator is a connection defined in the bundle $\widetilde{\mathcal{W}}$ as $\partial : \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^q) \rightarrow \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q+1})$ and is defined in terms of the symplectic connection as follows

$$\partial a \equiv dx^i \wedge \partial_i a. \quad (2.11)$$

In Darboux local coordinates this connection is written as

$$\partial a = da + \frac{1}{i\hbar}[\Gamma, a], \quad (2.12)$$

where $\Gamma = \frac{1}{2}\Gamma_{ijk}y^i y^j dx^k$ is a local one-form with values in $\mathcal{E}(\widetilde{\mathcal{W}})$, Γ_{ijk} are the symplectic connection's coefficients, $d = dx^i \wedge \frac{\partial}{\partial x^i}$ and ∂_i is the covariant derivative on \mathcal{M} with respect to $\frac{\partial}{\partial x^i}$. The connection ∂ satisfies the following properties:

$$\partial(a \overset{\bullet}{\wedge} b) = \partial a \overset{\bullet}{\wedge} b + (-1)^{q_1} a \overset{\bullet}{\wedge} \partial b \quad (13a)$$

$$\partial(\phi \bullet a) = d\phi \overset{\bullet}{\wedge} b + (-1)^q \phi \bullet \partial a \quad (13b)$$

for all $\phi \in \mathcal{E}(\Lambda^q)$ and $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q_1})$, $b \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^{q_2})$.

Following Fedosov, we define a more general connection D in the Weyl bundle $\widetilde{\mathcal{W}}$ as follows

$$Da = \partial + \frac{1}{i\hbar}[\gamma, a], \quad (2.14)$$

where $\gamma \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda^1)$ is globally defined on \mathcal{M} . The *curvature* of the connection D is given by

$$\frac{1}{i\hbar}\Omega = \frac{1}{i\hbar}\left(R + \partial\gamma + \frac{1}{i\hbar}\gamma^2\right), \quad (2.15)$$

with the normalizing condition $\gamma_0 = 0$. Here R is defined by $R := \frac{1}{4}R_{ijkl}y^i y^j dx^k \wedge dx^l$ where R_{ijkl} is the curvature tensor of the symplectic connection. In [24] it was shown that for any section $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda)$ we have

$$D^2 a = \frac{1}{i\hbar}[\Omega, a]. \quad (2.16)$$

2.3.2. Abelian connection

One very important definition is that of the *Abelian connection*. A connection D is Abelian if for any section $a \in \mathcal{E}(\widetilde{\mathcal{W}} \otimes \Lambda)$

$$D^2 a = \frac{1}{i\hbar} [\Omega, a] = 0. \quad (2.17)$$

From Eq. (2.7) one immediately sees that the curvature of the Abelian connection, Ω , is central.

In Fedosov's paper the Abelian connection takes the form

$$D = -\delta + \partial + \frac{1}{i\hbar} [r, \cdot], \quad (2.18)$$

where ∂ is a fixed symplectic connection and $r \in \mathcal{E}(\widetilde{\mathcal{W}}_3 \otimes \Lambda^1)$ a globally defined one-form with the Weyl normalizing condition $r_0 = 0$. This connection has curvature

$$\Omega = -\frac{1}{2} \omega_{ij} dx^i \wedge dx^j + R - \delta r + \partial r + \frac{1}{i\hbar} r^2 \quad (2.19)$$

with

$$\delta r = R + \partial r + \frac{1}{i\hbar} r^2. \quad (2.20)$$

This last equation has a *unique* solution satisfying the condition

$$\delta^{-1} r = 0. \quad (2.21)$$

By iterative techniques one can finally construct r and therefore the Abelian connection D . Thus we have

$$r = \frac{1}{8} R_{ijkl} y^i y^j y^k dx^l + \frac{1}{20} \partial_m R_{ijkl} y^i y^j y^k y^m dx^l + \dots, \quad (2.22)$$

where ∂_m is a covariant derivative with respect to $\frac{\partial}{\partial x^m}$.

2.3.3. Algebra of Quantum Observables

Now consider the subalgebra $\mathcal{E}(\widetilde{\mathcal{W}}_D)$ of $\mathcal{E}(\widetilde{\mathcal{W}})$ consisting of *flat sections i.e.*

$$\mathcal{E}(\widetilde{\mathcal{W}}_D) = \{a \in \mathcal{E}(\widetilde{\mathcal{W}}) | Da = 0\}. \quad (2.23)$$

This subalgebra is called *the algebra of Quantum Observables*.

Now an important theorem is:

Theorem (Fedosov [24]). For any $a_0 \in \mathcal{Z}$ there exists a unique section $a \in \mathcal{E}(\widetilde{\mathcal{W}}_D)$ such that $\sigma(a) = a_0$.

As a direct consequence of this theorem we can construct a section $a \in \mathcal{E}(\widetilde{\mathcal{W}}_D)$ by its symbol $a_0 = \sigma(a)$

$$a = a_0 + \partial a_0 y^i + \frac{1}{2} \partial_i \partial_j a_0 y^i y^j + \frac{1}{6} \partial_i \partial_j \partial_k a_0 y^i y^j y^k - \frac{1}{24} R_{ijkl} \omega^{lm} \partial_m a_0 y^i y^j y^k + \dots \quad (2.24)$$

In the case when the phase space is flat $R_{ijkl} = 0$ the last equation reads

$$a = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} a_0) y^{i_1} y^{i_2} \dots y^{i_k}. \quad (2.25)$$

The last theorem states that there exists the bijective map

$$\sigma : \mathcal{E}(\widetilde{\mathcal{W}}_D) \rightarrow \mathcal{Z}. \quad (2.26)$$

Therefore there exists the inverse map $\sigma^{-1} : \mathcal{Z} \rightarrow \mathcal{E}(\widetilde{\mathcal{W}}_D)$. It is possible to use this bijective map to recover the Moyal product $*$ in \mathcal{Z} .

$$a_0 * b_0 = \sigma(\sigma^{-1}(a_0) \bullet \sigma^{-1}(b_0)). \quad (2.27)$$

2.4. A definition of trace on the Weyl algebra on \mathbf{R}^{2n}

In order to work with a variational principle which involves Moyal geometry we would like to get a definition of trace. In the case $\mathcal{M} = \mathbf{R}^{2n}$ with the standard symplectic structure

$$\omega = \sum_{j=1}^{2n} dp_j \wedge dq_j. \quad (2.28)$$

Here the Abelian connection D in $\widetilde{\mathcal{W}} \xrightarrow{\pi} \mathbf{R}^{2n}$ is $D = -\delta + d$.

In this case the product \star coincides with the usual Moyal product [8],

$$\begin{aligned} \sigma(a \bullet b) &= \exp\left(\frac{+i\hbar}{2} \omega^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j}\right) \sigma(a(x, \hbar)) \sigma(b(z, \hbar))|_{z=x} \\ &= \sum_{k=0}^{\infty} \left(+\frac{i\hbar}{2}\right)^k \frac{1}{k!} \omega^{i_1 j_1} \dots \omega^{i_k j_k} \frac{\partial^k a_0}{\partial x^{i_1} \dots \partial x^{j_1}}, \frac{\partial^k b_0}{\partial x^{j_1} \dots \partial x^{j_k}}, \end{aligned} \quad (2.29)$$

where $a_0 := \sigma(a)$ and $b_0 = \sigma(b)$.

The trace in the Weyl algebra $\mathcal{E}(\widetilde{\mathcal{W}}_D)$ over \mathbf{R}^{2n} is the linear functional on the ideal $\mathcal{E}(\mathcal{W}_D^{\text{Comp}})$ over $\mathcal{M} = \mathbf{R}^{2n}$ (which consists of the flat sections with compact support) given by

$$\text{tr } a = \int_{\mathbf{R}^{2n}} \sigma(a) \frac{\omega^n}{n!}, \quad (2.30)$$

where $\sigma(a)$ means the projection on the center $\sigma(a(x, y, \hbar)) := a(x, 0, \hbar)$.

This definition of the trace satisfies a series of useful properties

$$a) \quad \text{tr}(a \bullet b) = \text{tr}(b \bullet a), \quad (31a)$$

$$b) \quad \text{tr}(b) = \text{tr}(A_f b), \quad (31b)$$

for all the sections $a \in \mathcal{E}(\widetilde{\mathcal{W}}_D)$, $b \in \mathcal{E}(\widetilde{\mathcal{W}}_D^{\text{comp}})$. In last equation, A_f is an isomorphism $A_f : \mathcal{E}(\widetilde{\mathcal{W}}_D^{\text{comp}})(\mathcal{O}) \rightarrow \mathcal{E}(\widetilde{\mathcal{W}}_D^{\text{comp}})(f(\mathcal{O}))$, where f is a symplectic diffeomorphism.

On the other hand it is possible to construct a trace on the sections algebra $\mathcal{E}(\widetilde{\mathcal{W}}_D)$ of the Weyl bundle over an arbitrary symplectic manifolds \mathcal{M} . Although this definition satisfies the property (2.31), unfortunately it is too formal and we don't consider it here.

3. Moyal deformation of self-dual gravity

We now recall some results of Refs. [20,21]. We first review the chiral model approach to SDG and its Moyal deformation [21]. Then we consider the Yang–Mills approach to SDG [20].

3.1. The principal chiral model approach to self-dual gravity

We start with the principal chiral model approach to SDG à la Husain [9], Park [7] and Ward [6]. Husain has shown the equivalence between SDG and $\text{sdiff}(\Sigma)$ -valued principal chiral model. Since then some solutions of this model has been found in terms of harmonic maps [28].

In Ref. [21] we found that the Moyal deformation of Park–Husain (P-H) heavenly equation can be obtained from the operator algebra valued two-dimensional principal chiral model. To this end the WWM-formalism has been employed. We have finally reproduced the P-H heavenly equation by taking the limit $\hbar \rightarrow 0$, instead of $N \rightarrow \infty$.

3.1.1. The principal chiral model

The \mathcal{G} -valued principal chiral equations on a two-dimensional simply connected manifold Ω with local Cartesian coordinates $\{x, y\}$ read

$$\partial_x A_y - \partial_y A_x + [A_x, A_y] = 0, \quad (1a)$$

$$\partial_x A_x + \partial_y A_y = 0, \quad (1b)$$

where $A_\mu \in \mathcal{G} \otimes C^\infty(\Omega)$, $\mu \in \{x, y\}$, stand for the chiral potentials and \mathcal{G} is a Lie algebra of the Lie group \mathbf{G} .

One can proceed as follows. From (3.1a) it follows that A_μ , $\mu \in \{x, y\}$, is of the pure gauge form, *i.e.*, there exists a \mathbf{G} -valued function $g = g(x, y)$ such that

$$A_\mu = g^{-1} \partial_\mu g. \quad (3.2)$$

Substituting (3.2) into (3.1b) we get the principal chiral equations

$$\partial_\mu (g^{-1} \partial_\mu g) = 0. \quad (3.3)$$

(Summation over μ is assumed.)

Chiral equations are the dynamical equations for the fields $g : \Omega \rightarrow \mathbf{G}$ which under specific boundary conditions are called *harmonic maps* [22].

It is very easy to see that the Lagrangian for equations of motion (3.3) reads (we assume that \mathcal{G} is semisimple)

$$\begin{aligned} \mathcal{L}_{\text{Ch}} &= -c \text{Tr} \{ (g^{-1} \partial_\mu g) (g^{-1} \partial_\mu g) \} \\ &= c \text{Tr} \{ (\partial_\mu g) (\partial_\mu g^{-1}) \}, \end{aligned} \quad (3.4)$$

where $c > 0$ is a constant and ‘Tr’ is an invariant form on the Lie algebra \mathcal{G} .

Let $\hat{\mathbf{G}}$ be some Lie group of linear operators acting on the Hilbert space $L^2(\mathbb{R}^1)$ and let $\hat{\mathcal{G}}$ be the Lie algebra of $\hat{\mathbf{G}}$. Consider the $\hat{\mathbf{G}}$ principal chiral model. The principal chiral equations read now

$$\partial_x \hat{A}_y - \partial_y \hat{A}_x + [\hat{A}_x, \hat{A}_y] = 0, \quad (5a)$$

$$\partial_x \hat{A}_x + \partial_y \hat{A}_y = 0, \quad (5b)$$

where $\hat{A}_\mu = \hat{A}_\mu(x, y) \in \hat{\mathcal{G}} \otimes C^\infty(\Omega)$, $\mu \in \{x, y\}$.

From the constraint (3.5a) one infers that

$$\hat{A}_\mu = \hat{g}^{-1} \partial_\mu \hat{g}, \quad (3.6)$$

where $\hat{g} = \hat{g}(x, y)$ is some $\hat{\mathbf{G}}$ -valued function on Ω . Substituting (3.6) into (3.5b) we get the principal chiral equations

$$\partial_\mu (\hat{g}^{-1} \partial_\mu \hat{g}) = 0. \quad (3.7)$$

Within WWM-formalism we can transform the above equation into a new equation defined on the four manifold $\mathcal{K}^4 = \Omega \times \Sigma$, being $\Sigma \subset \mathbf{R}^2$.

The Weyl *correspondence* \mathcal{W}^{-1} leads from $\hat{g} = \hat{g}(x, y)$ to the function on \mathcal{K}^4 , $g = g(x, y, p, q; \hbar)$, *i.e.*, $g = \mathcal{W}^{-1}(\hat{g})$, according to the formula (compare with [17-21, 23])

$$g = g(x, y, p, q) := \mathcal{W}^{-1}(\hat{g}(x, y)) = \int_{-\infty}^{+\infty} \langle q - \frac{\xi}{2} | \hat{g} | q + \frac{\xi}{2} \rangle \exp(\frac{ip\xi}{\hbar}) d\xi \quad (3.8)$$

from Eq. (3.7) one can infer that the above function fulfills the following equation

$$\partial_\mu \left(g^{-1*} * \partial_\mu g \right) = 0, \quad (3.9)$$

where $*$ stands for the *Moyal $*$ -product* (see (3.24)) and g^{-1*} denotes the inverse of g in the sense of the Moyal $*$ -product *i.e.*,

$$g^{-1*} * g = g * g^{-1*} = 1. \quad (3.10)$$

Comparing (3.9) with (3.7) we can say that the function $g = \mathcal{W}^{-1}(\hat{g}(x, y))$ defines a *harmonic map* $g : \Omega \rightarrow \mathbf{G}_*$, being $\mathbf{G}_* := \mathcal{W}^{-1}(\hat{\mathbf{G}})^2$.

The Lagrangian associated to Eq. (3.9) can be written as [21]

$$\mathcal{L}_{SG}'^{(M)} := -\frac{\hbar^2}{2} (g^{-1*} * \partial_\mu g) * (g^{-1*} * \partial_\mu g), \quad (3.11)$$

or equivalently by the Lagrangian

$$\mathcal{L}_{SG}''^{(M)} = \frac{\hbar^2}{2} (\partial_\mu g) * (\partial_\mu g^{-1*}). \quad (3.12)$$

3.1.2. The Moyal deformation of Park–Husain heavenly equation

To proceed further, we need the Moyal deformation of P-H version of SDG.

We start with Eq. (3.1b). This equation says that there exists a scalar function $\theta = \theta(x, y) \in \mathcal{G} \otimes C^\infty(\Omega)$ such that

$$A_x = -\partial_y \theta, \quad \text{and} \quad A_y = \partial_x \theta. \quad (3.13)$$

Inserting Eqs. (3.13) into (3.1a) one gets the principal chiral equations to read

$$\partial_x^2 \theta + \partial_y^2 \theta + [\partial_x \theta, \partial_y \theta] = 0. \quad (3.14)$$

Under the assumption that the algebra \mathcal{G} is semisimple one can construct a Lagrangian leading to (3.14) as follows

² In a sense the Moyal bracket algebra can be considered to be an infinite dimensional matrix Lie algebra. Especially interesting is the case when the group \mathbf{G}_* is a subgroup of \mathbf{U}_* , where $\mathbf{U}_* := \{f = f(p, q) \in C^\infty(\mathbf{R}^2); f * \bar{f} = \bar{f} * f = 1\}$; (the bar stands for the complex conjugation.). It means that $\hat{\mathbf{G}} = \mathcal{W}(\mathbf{G}_*)$ is a subgroup of the group $\hat{\mathbf{U}}$ of unitary operators acting on $L^2(\mathbf{R}^1)$. Now one quickly finds that if $g = g(x, y, p, q)$ is an \mathbf{U}_* -valued function, then $g^{-1*} * \partial_\mu g = \bar{g} * \partial_\mu g$ is *pure imaginary*.

$$\mathcal{L}_{\text{Ch}} := c' \text{Tr} \left\{ \frac{1}{3} \theta [\partial_x \theta, \partial_y \theta] - \frac{1}{2} ((\partial_x \theta)^2 + (\partial_y \theta)^2) \right\}, \quad (3.15)$$

where $c' > 0$ is a constant and ‘Tr’ is defined as (3.4).

Similarly from (3.5b) it follows that there exists the operator-valued scalar function $\hat{\theta} = \hat{\theta}(x, y) \in \hat{\mathcal{G}} \otimes C^\infty(\Omega)$ such that

$$\hat{A}_x = -\partial_y \hat{\theta}, \quad \text{and} \quad \hat{A}_y = \partial_x \hat{\theta}. \quad (3.16)$$

The analog of Eq. (3.14) is

$$\partial_x^2 \hat{\theta} + \partial_y^2 \hat{\theta} + [\partial_x \hat{\theta}, \partial_y \hat{\theta}] = 0. \quad (3.17)$$

Now, it is convenient to define a new operator-valued function $\hat{\Theta} = \hat{\Theta}(x, y) \in \hat{\mathcal{G}} \otimes C^\infty(\Omega)$ by

$$\hat{\Theta} := i\hbar \hat{\theta}. \quad (3.18)$$

Thus, by (3.17), $\hat{\Theta}$ satisfies the following equation

$$\partial_x^2 \hat{\Theta} + \partial_y^2 \hat{\Theta} + \frac{1}{i\hbar} [\partial_x \hat{\Theta}, \partial_y \hat{\Theta}] = 0. \quad (3.19)$$

Then we put

$$\Theta = \Theta(x, y, p, q, \hbar) := \mathcal{W}^{-1}(\hat{\Theta}(x, y)) = \int_{-\infty}^{+\infty} \langle q - \frac{\xi}{2} | \hat{\Theta} | q + \frac{\xi}{2} \rangle \exp\left(\frac{ip\xi}{\hbar}\right) d\xi. \quad (3.20)$$

It is clear that Θ satisfies the *Moyal deformation of the P-H heavenly equation*

$$\partial_x^2 \Theta + \partial_y^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_M = 0, \quad (3.21)$$

where the bracket $\{\cdot, \cdot\}_M$ denotes the *Moyal bracket i.e.*,

$$\{f_1, f_2\}_M := \frac{1}{i\hbar} (f_1 * f_2 - f_2 * f_1) = f_1 \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) f_2, \quad (3.22)$$

$$\overleftrightarrow{\mathcal{P}} := \overleftarrow{\frac{\partial}{\partial q}} \overrightarrow{\frac{\partial}{\partial p}} - \overleftarrow{\frac{\partial}{\partial p}} \overrightarrow{\frac{\partial}{\partial q}}; \quad f_1 = f_1(x, y, p, q), \quad f_2 = f_2(x, y, p, q). \quad (3.23)$$

The *Moyal *-product* is defined by

$$f_1 * f_2 := f_1 \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}\right) f_2. \quad (3.24)$$

If the functions f_1 and f_2 are independent of \hbar , then

$$\lim_{\hbar \rightarrow 0} f_1 * f_2 = f_1 f_2, \quad \lim_{\hbar \rightarrow 0} \{f_1, f_2\}_M = \{f_1, f_2\}_P := f_1 \overset{\leftrightarrow}{\mathcal{P}} f_2, \quad (3.25)$$

where $\{\cdot, \cdot\}_P$ denotes the Poisson bracket.

As it has been shown in [21] the Lagrangian leading to Eq. (3.21) reads

$$\mathcal{L}_{SG}^{(M)} = -\frac{1}{3}\Theta * \{\partial_x \Theta, \partial_y \Theta\}_M + \frac{1}{2}((\partial_x \Theta) * (\partial_x \Theta) + (\partial_y \Theta) * (\partial_y \Theta)) . \quad (3.26)$$

3.1.3. The Park–Husain heavenly equation

Assume now that the function Θ is analytic in \hbar , i.e., [14]

$$\Theta = \sum_{n=0}^{\infty} \hbar^n \Theta_n, \quad (3.27)$$

where $\Theta_n = \Theta_n(x, y, p, q)$, $n = 0, 1, \dots$, are independent of \hbar . If Θ is a solution of (3.21), then by (3.27) one concludes that the function Θ_0 satisfies *P-H heavenly equation* [21]

$$\partial_x^2 \Theta_0 + \partial_y^2 \Theta_0 + \{\partial_x \Theta_0, \partial_y \Theta_0\}_P = 0. \quad (3.28)$$

Moreover, the Lagrangian \mathcal{L}_{SG} leading to Eq. (3.28) can be quickly found to read

$$\mathcal{L}_{SG} = \lim_{\hbar \rightarrow 0} \mathcal{L}_{SG}^{(M)} = -\frac{1}{3}\Theta_0 \{\partial_x \Theta_0, \partial_y \Theta_0\}_P + \frac{1}{2}((\partial_x \Theta_0)^2 + (\partial_y \Theta_0)^2) \quad (3.29)$$

(compare with the Lagrangian well known in SDG [29]).

Therefore, *self-dual gravity appears to be the $\hbar \rightarrow 0$ limit of the principal chiral model for the Moyal bracket algebra*, or equivalently, one can interpret self-dual gravity to be the *principal chiral model for the Poisson bracket algebra* [6,7,9].

If one is interested in searching for solutions of P-H heavenly equation (3.28), one must take an explicit Lie algebra homomorphism $\Psi : \mathcal{G} \rightarrow \hat{\mathcal{G}}$

$$\hat{\Theta} = \hat{\Theta}(x, y) = i\hbar \theta_a(x, y) \hat{X}_a, \quad (3.30)$$

where $\hat{X}_a := \Psi(\tau_a)$ satisfies Eq. (3.19). Therefore, the function Θ defined by (3.20)

$$\begin{aligned} \Theta = \Theta(x, y, p, q) &= i\hbar \theta_a(x, y) X_a(p, q) \\ X_a(p, q) &:= \mathcal{W}^{-1}(\hat{X}_a) \end{aligned} \quad (3.31)$$

fulfills the Moyal deformation of P-H heavenly equation (3.21).

Consequently, if Θ is of the form (3.31) then Θ_0 satisfies *P-H heavenly equation*, (3.28). Moreover, if the Lie group $\hat{\mathbf{G}}$ defined by the Lie algebra $\hat{\mathcal{G}}$ appears to be a subgroup of the group $\hat{\mathbf{U}}$ of unitary operators in $L^2(\mathbf{R}^1)$ then the functions Θ and Θ_0 are real.

The procedure described here, which leads to the construction of the solutions to P-H heavenly equation is somewhat speculative. The main problem is to find representations for interesting Lie algebras and show how it works in practice. (For $\mathfrak{su}(2)$ see [21]).

3.2. SDYM theory approach to self-dual gravity

We will now describe the $\mathfrak{su}(N)$ SDYM equations in the flat 4-dimensional real, simply connected flat manifold $X \subset \mathbf{R}^4$ with local coordinates $(x, y, \tilde{x}, \tilde{y})$ chosen in such a way that the metric takes the form $dS^2 = 2(dx \otimes_s d\tilde{x} + dy \otimes_s d\tilde{y})$.

Then the $\mathfrak{su}(N)$ SDYM equations read

$$F_{xy} = 0, \quad F_{\tilde{x}\tilde{y}} = 0, \quad F_{x\tilde{x}} + F_{y\tilde{y}} = 0, \quad (3.32)$$

where, as usually, $F_{\mu\nu} \in \mathfrak{su}(N) \otimes C^\infty(X)$, $\mu, \nu \in \{x, y, \tilde{x}, \tilde{y}\}$, stands for the Yang–Mills field tensor.

In terms of the Yang–Mills potentials $A_\mu \in \mathfrak{su}(N) \otimes C^\infty(X)$ the SDYM equations can be rewritten in the gauge $A_x = A_y = 0$ as

$$\partial_{\tilde{x}} A_{\tilde{y}} + \partial_{\tilde{y}} A_{\tilde{x}} + [A_{\tilde{x}}, A_{\tilde{y}}] = 0, \quad (33a)$$

$$\partial_x A_{\tilde{x}} + \partial_y A_{\tilde{y}} = 0. \quad (33b)$$

Now assume that the potentials A_μ are now the anti-hermitian operator-valued functions on $X \subset \mathbf{R}^4$ acting in a Hilbert space $\mathcal{H} = L^2(\mathbf{R}^1)$. In this case Eqs. (3.32) give

$$\partial_{\tilde{x}} \hat{A}_{\tilde{y}} - \partial_{\tilde{y}} \hat{A}_{\tilde{x}} + [\hat{A}_{\tilde{x}}, \hat{A}_{\tilde{y}}] = 0, \quad (34a)$$

$$\partial_x \hat{A}_{\tilde{x}} + \partial_y \hat{A}_{\tilde{y}} = 0, \quad (34b)$$

where

$$\hat{A}_{\tilde{x}}^\dagger = -\hat{A}_{\tilde{x}}, \quad \hat{A}_{\tilde{y}}^\dagger = -\hat{A}_{\tilde{y}}.$$

3.2.1. Moyal deformation of the six dimensional version of the second heavenly equation

Eq. (3.34b) implies that

$$\hat{A}_x = -\partial_y \hat{\theta}, \quad \hat{A}_y = \partial_x \hat{\theta}, \quad (3.35)$$

under the condition

$$\hat{\theta} = \hat{\theta}(x, y, \tilde{x}, \tilde{y}) = -\hat{\theta}^\dagger.$$

It is easy to see from Eq. (3.34a) and (3.35) that

$$\partial_x \partial_{\tilde{x}} \hat{\Theta} + \partial_y \partial_{\tilde{y}} \hat{\Theta} + \frac{1}{i\hbar} [\partial_x \hat{\Theta}, \partial_y \hat{\Theta}] = 0. \quad (3.36)$$

where $\hat{\Theta} := i\hbar\theta$.

The above equation can be derived as equation of motion from the Lagrangian [20]

$$\mathcal{L}^{(q)} = \text{Tr} \left\{ 2\pi\hbar \left(-\frac{1}{3i\hbar} \hat{\Theta} [\partial_x \hat{\Theta}, \partial_y \hat{\Theta}] + \frac{1}{2} [(\partial_x \hat{\Theta})(\partial_{\tilde{x}} \hat{\Theta}) + (\partial_y \hat{\Theta})(\partial_{\tilde{y}} \hat{\Theta})] \right) \right\}, \quad (3.37)$$

where ‘Tr’ is the standard trace of a linear operator in the orthonormal base $\{|\psi >_j\}_{j \in \mathbf{N}}$ of the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^1)$.

Using the WWM-formalism it can be shown [20] that the above Lagrangian can be transformed into a new Lagrangian defined on the six-dimensional manifold $\mathcal{K}^6 = X \times \Sigma$, $\Sigma = \mathbf{R}^2$. Let $\{p, q\}$ be the local coordinates on Σ . This new Lagrangian reads

$$\mathcal{L}^{(M)} = -\frac{1}{3} \Theta * \{\partial_x \Theta, \partial_y \Theta\}_M + \frac{1}{2} [(\partial_x \Theta) * (\partial_{\tilde{x}} \Theta) + (\partial_y \Theta) * (\partial_{\tilde{y}} \Theta)]. \quad (3.38)$$

Furthermore the *Weyl correspondence* \mathcal{W}^{-1} leads from $\hat{\Theta} = \hat{\Theta}(x, y, \tilde{x}, \tilde{y})$ to the function $\Theta = \Theta(x, y, \tilde{x}, \tilde{y}, p, q, \hbar)$, $\Theta = \mathcal{W}^{-1}(\hat{\Theta})$, defined on $X \times \Sigma$ (here $\Sigma = \mathbf{R}^2$) according to the formula

$$\Theta = \Theta(x, y, \tilde{x}, \tilde{y}, p, q, \hbar) := \int_{-\infty}^{+\infty} \exp\left(\frac{ip\xi}{\hbar}\right) < q - \frac{\xi}{2} | \hat{\Theta} | q + \frac{\xi}{2} > d\xi. \quad (3.39)$$

In Ref. [20] it has been also shown that $\Theta \in C^\infty(X \times \mathbf{R}^2)$ satisfies the *Moyal deformation of the six-dimensional version of the heavenly equation*

$$\partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + \{\partial_x \Theta, \partial_y \Theta\}_M = 0, \quad (3.40)$$

which is, of course, the equation of motion of the Lagrangian (3.38).

Taking the limit $\hbar \rightarrow 0$ in the Lagrangian (3.38) one obtains

$$\mathcal{L}_\infty = -\frac{1}{3} \Theta \{\partial_x \Theta, \partial_y \Theta\}_P + \frac{1}{2} [(\partial_x \Theta)(\partial_{\tilde{x}} \Theta) + (\partial_y \Theta)(\partial_{\tilde{y}} \Theta)], \quad (3.41)$$

which has as Euler-Lagrange equation precisely the *six-dimensional version of the second heavenly equation*³

$$\partial_x \partial_{\tilde{x}} \Theta + \partial_y \partial_{\tilde{y}} \Theta + \{\Theta_x, \Theta_y\}_P = 0, \quad (3.42)$$

where $\Theta = \Theta(x, y, \tilde{x}, \tilde{y}, p, q)$ is a smooth function on the six-dimensional manifold \mathcal{K}^6 . As it has been demonstrated in Ref. [20], the different (but equivalent) versions of the heavenly equation (for instance, first and second heavenly equations, Grant's equation [30], P-H equation and the evolution form of the second heavenly equation [31]) are symmetry reductions of Eq. (3.42).

4. A Geometry associated with the Moyal deformation of self-dual gravity

The purpose of this section is to reformulate some results of [20,21,23], in terms of non-commutative geometry developed by Fedosov [24] and described in Sec. 2.

4.1. Geometry of deformation quantization associated with the principal chiral model approach to self-dual gravity

4.1.1. Equations

Let us now work out the principal chiral model (described in Section (3.1)) in geometrical terms. First of all note that Eqs. (3.1a,b) can be normally written as

$$F = dA + A \wedge A = 0, \quad (4.1a)$$

$$d \star A = 0, \quad (4.1b)$$

where \star is the standard Hodge operator and $A \in \mathcal{E}(\mathcal{G} \otimes \Lambda^1)$ is the connection one form.

The corresponding Eqs. (3.2) and (3.3) are

$$A = g^{-1} dg, \quad (4.2a)$$

$$d \star (g^{-1} dg) = 0. \quad (4.2b)$$

The first equation (4.1a) is the condition of *flat connection* and the second one is the equation of motion.

³ The six-dimensional version of the second heavenly equation can be obtained from the $\mathfrak{su}(N)$ -SDYM equations by taking the large- N limit, $N \rightarrow \infty$.

In coordinates $(x, y) \in \Omega$

$$A = A_x dx + A_y dy, \quad (4.3)$$

with

$$A_\mu(x, y) = \sum_{a=0}^{\dim G} A_\mu^a(x, y) \tau_a \in \mathcal{G} \otimes C^\infty(\Omega), \quad \mu = x, y. \quad (4.4)$$

Now we generalize this gauge connection from \mathcal{G} -valued connection one-form (4.3) to the corresponding $\widetilde{\mathcal{W}}_D$ -valued connection one-form

$$\tilde{A} = \tilde{A}_x dx + \tilde{A}_y dy, \quad (4.5)$$

with

$$\begin{aligned} \tilde{A}_\mu = \tilde{A}_\mu(x, y, p, q; \hbar) = & a_\mu + \partial_i a_\mu y^i + \frac{1}{6} \partial_i \partial_j \partial_k a_\mu y^i y^j y^k \\ & - \frac{1}{24} R_{ijkl} \omega^{lm} \partial_m a_\mu y^i y^j y^k + \dots, \end{aligned} \quad (4.6)$$

for the case of non-flat phase-space. While that for the flat case we have

$$\tilde{A}_\mu(z, \bar{z}) = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \partial_{i_2} \dots \partial_{i_k} a_\mu) y^{i_1} y^{i_2} \dots y^{i_k}. \quad (4.7)$$

In the above formulas $a_\mu = a_\mu(x, y, p, q; \hbar); (x^1, x^2) \equiv (q, p) \in \mathcal{M}$.

The mentioned correspondence also implies that Eqs. (4.1a,b) have a counterpart in terms of Fedosov's geometry

$$\tilde{F} = d\tilde{A} + \tilde{A} \overset{\bullet}{\wedge} \tilde{A} = 0, \quad (4.8a)$$

$$d \star \tilde{A} = 0, \quad (4.8b)$$

where $\tilde{A} \in \mathcal{E}(\widetilde{\mathcal{W}}_D \otimes \Lambda^1)$ and $\tilde{F} \in \mathcal{E}(\widetilde{\mathcal{W}}_D \otimes \Lambda^2)$. Equations (4.2a,b) are written now as

$$\tilde{A} = g^{-1} \overset{\bullet}{\bullet} dg, \quad (4.9a)$$

$$d \star (g^{-1} \overset{\bullet}{\bullet} dg). \quad (4.9b)$$

4.1.2. Lagrangian

Now we will show that Eqs. (4.8a,b) can be obtained from a variational principle from a Lagrangian of the standard principal chiral model.

First we recall that the action which gives Eqs. (4.1a,b) reads

$$S = \int_{\Omega} \mathcal{L},$$

where

$$\mathcal{L} = \frac{1}{2} \text{Tr}(g^{-1} dg \wedge \star g^{-1} dg), \quad (4.10)$$

where $g : \Omega \rightarrow \mathbf{G}$ and d is the exterior differential on Ω *i.e.* $d = dx\partial_x + dy\partial_y$ and Tr is an invariant form on the Lie algebra of \mathbf{G} , $\text{Lie}(\mathbf{G}) = \mathcal{G}$. Here we have assumed that \mathbf{G} is semisimple.

The above action can be generalized to Fedosov's geometry as follows

$$S^{\bullet} = \int_{\Omega} \mathcal{L}^{\bullet},$$

where

$$\begin{aligned} \mathcal{L}^{\bullet} &= -\frac{\hbar^2}{2} \text{tr}(\tilde{A} \overset{\bullet}{\wedge} \star \tilde{A}) \\ &= -\frac{\hbar^2}{2} \text{tr}(g^{-\dot{1}} \bullet dg \overset{\bullet}{\wedge} \star g^{-\dot{1}} \bullet dg), \end{aligned} \quad (4.11)$$

where 'tr' is the Fedosov's trace and $g : \Omega \rightarrow \mathbf{G}_{\bullet}$ is the generalized gravitational uniton (see [23]).

In the case of flat phase-space $R_{ijkl} = 0$, the trace can be expressed by (2.30)

$$\begin{aligned} \mathcal{L}^{\bullet} &= -\frac{\hbar^2}{2} \text{tr}(\tilde{A} \overset{\bullet}{\wedge} \star \tilde{A}) = -\frac{\hbar^2}{2} \int_{\mathbf{R}^2} \sigma(\tilde{A} \overset{\bullet}{\wedge} \star \tilde{A}) dp \wedge dq, \\ &= -\frac{\hbar^2}{2} \int_{\mathbf{R}^2} \sigma(g^{-\dot{1}} \bullet dg \overset{\bullet}{\wedge} \star g^{-\dot{1}} \bullet dg) dp \wedge dq. \end{aligned} \quad (4.12)$$

This corresponds to (3.11).

4.2. Moyal deformation of SDYM theory

4.2.1. Moyal deformation of Yang and Donaldson–Nair–Schiff equations

First of all, we will apply our method described in [20,21,23] to the well known Yang and Donaldson–Nair–Schiff equations. We will find that these equations admit Moyal deformations via Fedosov’s geometry.

Some years ago C.N. Yang found that $\mathfrak{su}(2)$ -SDYM equations (in a particular gauge), always are related to a principal chiral model on a four-dimensional flat submanifold X of \mathbf{R}^4 [30] (see also [20]). Thus, SDYM equations (3.2a,b) can be written in appropriate real coordinates $\{x, y, \tilde{x}, \tilde{y}\}$ as follows

$$\partial_x(g^{-1}\partial_{\tilde{x}}g) + \partial_y(g^{-1}\partial_{\tilde{y}}g) = 0. \quad (4.13)$$

Equation (4.13) is called *Yang equation*.

Proceeding in a similar way as in the above section, we find the equation

$$\partial_x(g^{-1} \bullet \partial_{\tilde{x}}g) + \partial_y(g^{-1} \bullet \partial_{\tilde{y}}g) = 0, \quad (4.14)$$

where $g = g(x, y, \tilde{x}, \tilde{y}, p, q; \hbar) \in \mathcal{E}(\widetilde{\mathcal{W}}_D)$. This equation will be called *the Moyal deformation of Yang’s equation*.

It is very interesting to note that for Kähler and hyper-Kähler manifolds, Yang equation admits the very natural extension [31,32]

$$\omega \wedge \partial(g^{-1}\tilde{\partial}g) = 0, \quad (4.15)$$

where $\partial := dx\partial_x + dy\partial_y$, $\tilde{\partial} := d\tilde{x}\partial_{\tilde{x}} + d\tilde{y}\partial_{\tilde{y}}$ and ω is here the Kähler two-form on the spacetime manifold \mathbf{R}^4 where the theory is defined. This equation is well known as the *Donaldson–Nair–Schiff (DNS) equation*⁴.

The DNS equation can be derived from the so called DNS-action [31,32]

$$S_{\text{DNS}}[g] = \frac{1}{2} \int_X \omega \wedge \text{Tr}(g^{-1}\partial g \wedge g^{-1}\tilde{\partial}g) - \frac{1}{3} \int_{\tilde{X}} \omega \wedge \text{Tr}(g^{-1}dg)^3, \quad (4.16)$$

where $\tilde{X} = X \times I$ and ‘Tr’ is an invariant form on $\mathfrak{su}(2)$. The above action is of the WZW type and it can be obtained by dimensional reduction from the Kähler-Chern-Simons theory [32].

Applying now, our method described in Section 2 we find the *Moyal deformation of DNS*

$$\omega \wedge \partial(g^{-1} \bullet \tilde{\partial}g) = 0. \quad (4.17)$$

⁴ The DNS equation represents a coupling between the SDYM fields and the SDG. The natural framework to englobe this equation seems to be $N = 2$ heterotic string theory [33]. Connections to 4d analogues of WZW theory and Conformal Field Theories, can be found at [34].

After a simple calculations we find that the Lagrangian from which we can derive Eq. (4.19) is

$$S_{\text{DNS}}^{(\text{M})}[g] = -\frac{\hbar^2}{2} \int_X \omega \wedge \text{tr}(g^{-1} \bullet \partial g \wedge g^{-1} \bullet \bar{\partial} g) + \frac{\hbar^2}{3} \int_{\tilde{X}} \omega \wedge \text{tr}(g^{-1} \bullet dg)^3. \quad (4.18)$$

4.3. Geometry of WZW-like action of self-dual gravity

Now we study the geometry and topology associated to the WZW-like action obtained in Ref. [23]. We first study the case where the basic field θ is a Lie algebra \mathcal{G} -valued function on the spacetime manifold Ω ($\dim \Omega = 2$). The action reads

$$S(\theta) = -\frac{\alpha}{2} \int_{\Omega} \text{Tr}[d\theta \wedge d\theta] + \frac{\alpha}{3} \int_B \text{Tr}(d\theta \wedge d\theta \wedge d\theta), \quad (4.19)$$

where ‘Tr’ is an invariant form on the Lie algebra \mathcal{G} of G and $\theta \in \Lambda^0(S^2) \otimes \mathcal{G}$ and Ω is the boundary of B . The field θ can be seen as $\theta : \Omega \rightarrow \mathcal{G}$.

Taking $\Omega = S^2$ one can extend this map from S^2 to the 3-manifold $\tilde{\Omega} = \Omega \times I$ with $\partial\tilde{\Omega} = S^2$. This is due to the fact that $\pi_2(\mathcal{G}) = 0$.

The maps $\bar{\theta} : \tilde{\Omega} \rightarrow \mathcal{G}$ are classified by $\pi_3(\mathcal{G})$. The triviality of this group implies that the constant α must take \mathbf{R} -values *i.e.* $\alpha \in \mathbf{R}$. Thus we choose $\alpha = 1$. Therefore the WZ-like term is not in essential topological! Thus one can define globally an invariant 3-form ρ

$$\rho = \frac{1}{3} \text{Tr}(d\theta \wedge d\theta \wedge d\theta). \quad (4.20)$$

The form ρ can be written globally as an exact form $\rho = d\lambda$, where λ is a two-form on \mathcal{G} . Thus this term does not contribute to the classical equations of motion.

Action (4.21) can be generalized to Fedosov’s geometry as follows

$$S^\bullet(\Phi) = +\frac{1}{2} \int_{S^2} \text{tr}[d\Theta \wedge \bullet d\Theta] - \frac{1}{3i\hbar} \int_{\tilde{\Omega}} \text{tr}(d\bar{\Theta} \wedge \bullet d\bar{\Theta} \wedge \bullet d\bar{\Theta}), \quad (4.21)$$

where now $\Theta \in \mathcal{E}(\widetilde{\mathcal{W}}_D) \otimes C^\infty(S^2)$ and $\bar{\Theta} \in \mathcal{E}(\widetilde{\mathcal{W}}_D) \otimes C^\infty(\tilde{\Omega})$.

Moyal deformation of WZW-like action for SDG can be put in the form

$$S(\Phi) = +\frac{1}{2} \int_{S^2} \text{tr}[d\Theta \wedge \bullet d\Theta] + \Gamma_{WZ}, \quad (4.22)$$

where $\Gamma_{WZ} \cong -\frac{1}{i\hbar} \int_{\tilde{\Omega}} \bar{\Theta}^* \rho$.

5. Final remarks

In this paper we have reformulated different aspects of integrable deformation of SDG in terms of a Fedosov's geometry of deformation quantization. We find that this non-commutative geometry, appears to be a natural language to describe the SDYM and Chiral Model approaches to SDG.

Some further questions remain to be overcome. For instance, it would be very interesting to investigate the behavior of heavenly hierarchies of conserved quantities [35,36], within Fedosov's geometry and some other alternative geometries of deformation quantization. Some applications to hyperheavenly equations would be of capital importance. In connection with quantum groups and non-commutative geometry might be interesting to consider both, simultaneously, the Moyal deformation and q -deformation. Some results of [27,37,38] might serve as beginning point.

On the other hand, the strong connection between SDYM theory and SDG with $N = 2$ heterotic strings [33] might indicate that the results obtained in this paper for SDYM (via Yang's equation), concerning its Moyal deformation (and the geometrical interpretation) can be extended to $N = 2$ heterotic strings. It is very possible that we can find something like an integrable Moyal deformation of $N = 2$ heterotic strings.

The surprising relation between $N = 2$ Heterotic Strings with M and F theories [39,40,41] and the application of the former in the searching for geometrical structures of M and F theories, one would hope that some new geometrical descriptions of SDYM theory and SDG will be of some importance to give more deep insight into string theory dualities.

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