TEXTURE DYNAMICS FOR NEUTRINOS*

W. Królikowski

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00–681 Warsaw, Poland

An ansatz for mass matrix was recently proposed for charged leptons, predicting (in its diagonal approximation) $m_\tau \simeq 1776.80$ MeV from the experimental values of m_e and m_μ, in agreement with $m_\tau^{\mathrm{exp}} = 1777.00^{+0.30}_{-0.27}$ MeV. Now it is applied to neutrinos. If the amplitude of neutrino oscillations $\nu_\mu \to \nu_\tau$ is $\sim 1/2$ and $|m_{\nu_\tau}^2 - m_{\nu_\mu}^2| \sim (0.0003$ to $0.01)$ eV^2, as seems to follow from atmospheric-neutrino experiments, this ansatz predicts $m_{\nu_e} \ll m_{\nu_\mu} \sim (0.2$ to $1) \times 10^{-2}$ eV and $m_{\nu_\tau} \sim (0.2$ to $1) \times 10^{-1}$ eV, and also the amplitude of neutrino oscillations $\nu_e \to \nu_\mu \sim 2^{+4}_{-2} \times 10^{-4}$ (in the vacuum). Such a very small amplitude for $\nu_e \to \nu_\mu$ is implied by the value of $m_\tau^{\mathrm{exp}} - 1776.80$ MeV used to determine the deviation of the diagonalizing matrix $\widehat{U}^{(e)}$ from $\widehat{1}$ in the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} = \widehat{U}^{(\nu)\dagger}\widehat{U}^{(e)}$. Here, $\widehat{U}^{(\nu)}$ by itself gives practically no oscillations $\nu_e \to \nu_\mu$, while it provides the large oscillations $\nu_\mu \to \nu_\tau$.

PACS numbers: 12.15. Ff, 12.90. +b

1. Introduction

First, let us say a few introductory words about two familiar notions of neutrino weak-interaction states and neutrino mass states.

Since, apparently, neutrinos display no electromagnetic nor strong interactions, experimental detectors select their weak-interaction states, what is in contrast to mass states selected by detectors in the case of charged leptons and hadrons (built up from quarks).

Thus, if the neutrino mass matrix $\widehat{M}^{(\nu)}$ and/or charged-lepton mass matrix $\widehat{M}^{(e)}$ are originally nondiagonal in the bases

$$\vec{\nu}^{(0)} = \begin{pmatrix} \nu_e^{(0)} \\ \nu_\mu^{(0)} \\ \nu_\tau^{(0)} \end{pmatrix} \tag{1}$$

and

$$\vec{e}^{(0)} = \begin{pmatrix} e^{-(0)} \\ \mu^{-(0)} \\ \tau^{-(0)} \end{pmatrix}, \tag{2}$$

respectively, the neutrino weak-interaction states

$$\vec{\nu} = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \tag{3}$$

are, *mutatis mutandis*, analogues of the Cabibbo–Kobayashi–Maskawa transforms

$$\vec{d'} = \begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} \tag{4}$$

of down-quark mass states

$$\vec{d}^{(m)} \equiv \vec{d} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}. \tag{5}$$

It is so, though $\vec{\nu}$ are experimentally observed states, in contrast to $\vec{d'} \neq \vec{d}$, where $\vec{d}^{(m)} \equiv \vec{d}$ describe experimentally observed states.

In fact, neutrino weak-interaction states are defined as

$$\vec{\nu} = \widehat{V}^{-1}\vec{\nu}^{(m)}, \quad \widehat{V} = \widehat{U}^{(\nu)-1}\widehat{U}^{(e)}, \tag{6}$$

where

$$\vec{\nu}^{(m)} = \widehat{U}^{(\nu)-1}\vec{\nu}^{(0)}, \quad \widehat{U}^{(\nu)-1}\widehat{M}^{(\nu)}\widehat{U}^{(\nu)} = \mathrm{diag}\left(m_{\nu_e}, m_{\nu_\mu}, m_{\nu_\tau}\right) \tag{7}$$

are neutrino mass states

$$\vec{\nu}^{(m)} = \begin{pmatrix} \nu_e^{(m)} \\ \nu_\mu^{(m)} \\ \nu_\tau^{(m)} \end{pmatrix}, \tag{8}$$

while

$$\vec{e}^{(m)} = \widehat{U}^{(e)-1}\vec{e}^{(0)}, \quad \widehat{U}^{(e)-1}\widehat{M}^{(e)}\widehat{U}^{(e)} = \mathrm{diag}\left(m_e, m_\mu, m_\tau\right) \tag{9}$$

represent charged-lepton mass states

$$\vec{e}^{(m)} \equiv \vec{e} = \begin{pmatrix} e^- \\ \mu^- \\ \tau^- \end{pmatrix}. \tag{10}$$

Here, $\vec{e}^{(m)} \equiv \vec{e}$ describe experimentally observed states, in contrast to $\vec{\nu}^{(m)} \neq \vec{\nu}$. It can be readily seen that the states $\vec{\nu}^{(m)}(t)$, as given in Eq. (7), are eigenstates of the neutrino mass operator

$$\int d^3\vec{r} \sum_{ij} \nu_i^{(0)\dagger}(x) M_{ij}^{(\nu)} \nu_j^{(0)}(x) = \int d^3\vec{r} \sum_i m_{\nu_i} \nu_i^{(m)\dagger}(x) \nu_i^{(m)}(x), \tag{11}$$

where $\widehat{M}^{(\nu)} = \left(M_{ij}^{(\nu)} \right)$. Similarly, the states $\vec{e}^{(m)}(t)$, as defined in Eq. (9), represent eigenstates of the charged-lepton mass operator. The unitary matrix $\widehat{V}$, introduced in Eq. (6), is obviously a lepton analogue of the Cabibbo–Kobayashi–Maskawa matrix, because the lepton charge-changing weak current has the form

$$\begin{aligned} \vec{\nu}^{(0)\dagger}(x)\beta\gamma^\mu(1-\gamma^5)\vec{e}^{(0)}(x) &= \vec{\nu}^{(m)\dagger}(x)\widehat{V}\beta\gamma^\mu(1-\gamma^5)\vec{e}^{(m)}(x) \\ &= \vec{\nu}^\dagger(x)\beta\gamma^\mu(1-\gamma^5)\vec{e}(x), \end{aligned} \tag{12}$$

where Eqs (7), (9) and (6) are used.

Note that the formula

$$\vec{\nu} = \widehat{U}^{(e)-1}\vec{\nu}^{(0)} \tag{13}$$

follows generally from Eqs (6) and (7). This implies in the case when $\widehat{M}^{(e)}$ is diagonal (*i.e.*, $\widehat{U}^{(e)} = \widehat{1}$ and so, $\vec{e} = \vec{e}^{(0)}$) that $\vec{\nu} = \vec{\nu}^{(0)}$. In this case, $\widehat{V} = \widehat{U}^{(\nu)-1}$ and thus $\vec{\nu} = \widehat{U}^{(\nu)}\vec{\nu}^{(m)}$, what means that $\vec{\nu} \neq \vec{\nu}^{(m)}$ if $\widehat{U}^{(\nu)} \neq \widehat{1}$. When, alternatively, $\widehat{M}^{(\nu)}$ is diagonal (*i.e.*, $\widehat{U}^{(\nu)} = \widehat{1}$ and so, $\vec{\nu}^{(m)} = \vec{\nu}^{(0)}$), then Eq. (13) shows that $\vec{\nu} = \widehat{U}^{(e)-1}\vec{\nu}^{(m)}$ giving $\vec{\nu} \neq \vec{\nu}^{(m)}$ if $\widehat{U}^{(e)} \neq \widehat{1}$.

As is well known, neutrino mixing *i.e.*, the mixing of neutrino mass states $\nu_i^{(m)}$ within neutrino weak-interaction states ν_i, expressed by the formula (6),

$$\nu_i = \sum_j \left(\widehat{V}^{-1} \right)_{ij} \nu_j^{(m)}, \tag{14}$$

implies neutrino oscillations (in time) between states ν_i. They occur if masses m_{ν_i} are not all degenerate and, of course, the mass matrices $\widehat{M}^{(\nu)}$ and/or $\widehat{M}^{(e)}$ are nondiagonal. In fact, since time-dependent weak-interaction neutrino states are

$$\nu_i(t) = e^{-iHt}\nu_i = \sum_j \left(\widehat{V}^{-1} \right)_{ij} \nu_j^{(m)} e^{-iE_j t}, \tag{15}$$

the probability of oscillations $\nu_i \to \nu_j$ (in the vacuum) is given by the formula

$$P(\nu_i \to \nu_j, t) = |\langle \nu_j | \nu_i(t) \rangle|^2 = \sum_{k\,l} V_{j\,l}^* V_{i\,l} V_{j\,k} V_{i\,k}^* \exp\left(i\frac{m_{\nu_l}^2 - m_{\nu_k}^2}{2|\vec{p}|} t \right) , \tag{16}$$

where the ultrarelativistic relation

$$E_l - E_k = \frac{m_{\nu_l}^2 - m_{\nu_k}^2}{2|\vec{p}|} \tag{17}$$

is used for neutrino mass states. In Eq. (16), usually $t/|\vec{p}| = L/E$, what is replaced by $4 \times 1.26693\, L/E$ if $m_{\nu_l}^2 - m_{\nu_k}^2$, L and E are measured in eV^2, km and GeV, respectively. Here, L is the source-detector distance.

Concluding this introductory Section, we can see that the masses m_{ν_e}, m_{ν_μ}, m_{ν_τ} of neutrino mass states $\nu_e^{(m)}$, $\nu_\mu^{(m)}$, $\nu_\tau^{(m)}$ as well as their mixing parameters [involved in the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} \equiv (V_{ij})$] can be determined, if neutrino and charged-lepton mass matrices $\widehat{M}^{(\nu)} \equiv \left(M_{ij}^{(\nu)} \right)$ and $\widehat{M}^{(e)} \equiv \left(M_{ij}^{(e)} \right)$ are given explicitly. Once the mixing of $\nu_e^{(m)}$, $\nu_\mu^{(m)}$, $\nu_\tau^{(m)}$ (in weak interactions) and the masses m_{ν_e}, m_{ν_μ}, m_{ν_τ} are known, the oscillations (in time) between the neutrino weak-interaction states ν_e, ν_μ, ν_τ can be evaluated.

The notation $\nu_e^{(m)}$, $\nu_\mu^{(m)}$, $\nu_\tau^{(m)}$ for neutrino mass states and m_{ν_e}, m_{ν_μ}, m_{ν_τ} for their masses, though consequent, may be sometimes confusing about its difference with ν_e, ν_μ, ν_τ being the neutrino weak-interaction states to which masses cannot be ascribed. Thus, in the case of mass states, the notation ν_0, ν_1, ν_2 and m_{ν_0}, m_{ν_1}, m_{ν_2} is, perhaps, more adequate. We hope, however, that the Reader will not be seriously confused by the former notation used consequently throughout this paper (notice that in the Particle Tables of Ref. [2] the neutrino masses are also denoted by m_{ν_e}, m_{ν_μ}, m_{ν_τ}).

In the next Section, an ansatz for the mass matrices $\widehat{M}^{(e)}$ and $\widehat{M}^{(\nu)}$ will be described and its consequences derived. This ansatz introduces a kind of "texture dynamics" for leptons.

2. A model for $\widehat{M}^{(e)}$ and $\widehat{M}^{(\nu)}$

Let us consider the following ansatz [1] for charged-lepton and neutrino mass matrices:

$$\widehat{M}^{(e,\nu)} = \widehat{\rho}\, \widehat{h}^{(e,\nu)}\, \widehat{\rho} \tag{18}$$

with

$$\begin{aligned}
\widehat{h}^{(e,\nu)} &= \mu^{(e,\nu)} \left[\widehat{N}^2 - (1 - \varepsilon^{(e,\nu)2})\widehat{N}^{-2} \right] \\
&\quad + \left(\alpha^{(e,\nu)}\widehat{1} + \beta^{(e,\nu)}\widehat{n} \right) \widehat{a}\mathrm{e}^{i\varphi^{(e,\nu)}} + \widehat{a}^\dagger \left(\alpha^{(e,\nu)}\widehat{1} + \beta^{(e,\nu)}\widehat{n} \right) \mathrm{e}^{-i\varphi^{(e,\nu)}}
\end{aligned} \tag{19}$$

where

$$\widehat{\rho} = \frac{1}{\sqrt{29}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix}, \quad \widehat{N} = \widehat{1} + 2\widehat{n} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad (20)$$

and

$$\widehat{n} = \widehat{a}^\dagger \widehat{a} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \widehat{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}. \qquad (21)$$

For charged leptons we will assume about the coupling constants $\alpha^{(e)}/\mu^{(e)}$ and $\beta^{(e)}/\mu^{(e)}$ that the second term in the matrix $\widehat{h}^{(e)}$ can be treated as a small perturbation of the first term. For neutrinos we will conjecture two alternative options: either (i) the coupling constants $\alpha^{(\nu)}/\mu^{(\nu)}$ and $\beta^{(\nu)}/\mu^{(\nu)}$ enable us to apply the perturbative treatment (similarly as for charged leptons) and, in addition, $\varepsilon^{(\nu)\,2} \simeq 0$, or ($ii$) $\alpha^{(\nu)}/\mu^{(\nu)}$ only is a perturbative parameter and, additionally, $\varepsilon^{(\nu)\,2} \simeq 0$.

Note from Eqs (21) that the "truncated" annihilation and creation matrices in the family space, $\widehat{a}$ and $\widehat{a}^\dagger$, satisfy the familiar commutation relations with $\widehat{n}$

$$[\widehat{a}, \widehat{n}] = \widehat{a}, \quad \left[\widehat{a}^\dagger, \widehat{n}\right] = -\widehat{a}^\dagger \qquad (22)$$

and, additionally, the "truncation" identities

$$\widehat{a}^3 = 0, \quad \widehat{a}^{\dagger\,3} = 0. \qquad (23)$$

Thus, $\widehat{n}|n\rangle = n|n\rangle$ as well as $\widehat{a}|n\rangle = \sqrt{n}|n-1\rangle$ and $\widehat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ ($n = 0, 1, 2$), but $\widehat{a}^\dagger|2\rangle = 0$ *i.e.*, $|3\rangle = 0$ (in addition to $\widehat{a}|0\rangle = 0$ *i.e.*, $|-1\rangle = 0$). Evidently, $n = 0, 1, 2$ plays the role of an index i in our three-dimensional matrix calculations.

For both labels e and ν the mass matrix (18) can be written explicitly in the form

$$\widehat{M} = \frac{1}{29} \begin{pmatrix} \mu\varepsilon^2 & 2\alpha e^{i\varphi} & 0 \\ 2\alpha e^{-i\varphi} & 4\mu(80 + \varepsilon^2)/9 & 8(\alpha + \beta)\sqrt{3}e^{i\varphi} \\ 0 & 8(\alpha + \beta)\sqrt{3}e^{-i\varphi} & 24\mu(624 + \varepsilon^2)/25 \end{pmatrix} \qquad (24)$$

(with obvious suppression of labels e and ν).

The unitary matrix $\widehat{U} \equiv (U_{ij})$, diagonalizing the mass matrix $\widehat{M} \equiv (M_{ij})$ according to the equation $\widehat{U}^{-1}\widehat{M}\widehat{U} = \mathrm{diag}(m_0, m_1, m_2)$, has the form

$$\widehat{U} = \begin{pmatrix} A_0 & -A_1\frac{M_{01}}{M_{00}-m_1} & A_2\frac{M_{22}-m_2}{M_{21}}\frac{M_{01}}{M_{00}-m_2} \\ -A_0\frac{M_{00}-m_0}{M_{01}} & A_1 & -A_2\frac{M_{22}-m_2}{M_{21}} \\ A_0\frac{M_{00}-m_0}{M_{01}}\frac{M_{21}}{M_{22}-m_0} & -A_1\frac{M_{21}}{M_{22}-m_1} & A_2 \end{pmatrix}, \qquad (25)$$

where

$$A_0 = \left\{ 1 + \frac{(M_{00} - m_0)^2}{|M_{01}|^2} \left[1 + \frac{|M_{12}|^2}{(M_{22} - m_0)^2} \right] \right\}^{-1/2},$$

$$A_1 = \left[1 + \frac{|M_{01}|^2}{(M_{00} - m_1)^2} + \frac{|M_{12}|^2}{(M_{22} - m_1)^2} \right]^{-1/2},$$

$$A_2 = \left\{ 1 + \frac{(M_{22} - m_2)^2}{|M_{12}|^2} \left[1 + \frac{|M_{01}|^2}{(M_{00} - m_2)^2} \right] \right\}^{-1/2}. \tag{26}$$

The elements of lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} \equiv (V_{ij}) = \widehat{U}^{(\nu)\dagger} \widehat{U}^{(e)}$ can be calculated from the formulae $V_{ij} = \sum_k U_{ki}^{(\nu)*} U_{kj}^{(e)}$ $(i, j = 0, 1, 2)$. Here, the secular equations $\det(\widehat{M} - \widehat{1}\, m_i) = 0$ $(i = 0, 1, 2)$ give

$$(M_{00} - m_i)(M_{11} - m_i)(M_{22} - m_i) = |M_{01}|^2 (M_{22} - m_i) + |M_{12}|^2 (M_{00} - m_i) \tag{27}$$

due to $M_{02} = 0 = M_{20}$. In particular, Eq. (27) implies that $M_{00} - m_0 = 0$ and $(M_{11} - m_i)(M_{22} - m_i) = |M_{12}|^2$ $(i = 1, 2)$ if $M_{01} = 0 = M_{10}$. Also, $(M_{00} - m_0)(M_{22} - m_0)^{-1} \to -|M_{01}|^2 |M_{12}|^{-2}$ if $\mu \to 0$.

3. Charged-lepton masses

Applying to the matrix $\widehat{M}$ given in Eq. (24) the lowest order perturbative calculation with respect to its off-diagonal elements, we obtain

$$m_0 = \frac{\mu}{29} \left[\varepsilon^2 - \frac{36}{320 - 5\varepsilon^2} \left(\frac{\alpha}{\mu} \right)^2 \right],$$

$$m_1 = \frac{\mu}{29} \left[\frac{4}{9} (80 + \varepsilon^2) + \frac{36}{320 - 5\varepsilon^2} \left(\frac{\alpha}{\mu} \right)^2 - \frac{10800}{31696 + 29\varepsilon^2} \left(\frac{\alpha + \beta}{\mu} \right)^2 \right],$$

$$m_2 = \frac{\mu}{29} \left[\frac{24}{25} (624 + \varepsilon^2) + \frac{10800}{31696 + 29\varepsilon^2} \left(\frac{\alpha + \beta}{\mu} \right)^2 \right]. \tag{28}$$

These formulae imply the following mass sum rule:

$$m_2 = \frac{6}{125} (351 m_1 - 136 m_0)$$

$$+ \frac{216}{3625} \left[\frac{105300}{31696 + 29\varepsilon^2} \left(\frac{\alpha + \beta}{\mu} \right)^2 - \frac{487}{320 - 5\varepsilon^2} \left(\frac{\alpha}{\mu} \right)^2 \right] \mu. \tag{29}$$

In the case of charged leptons, the mass formulae (28) with $m_e = m_0$, $m_\mu = m_1$, $m_\tau = m_2$ lead to

$$m_\tau = 1776.80 \text{ MeV} + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\mu^{(e)} + O\left\{\left[\frac{\alpha^{(e)} + \beta^{(e)}}{\mu^{(e)}}\right]^2\right\}\mu^{(e)},$$

$$\varepsilon^{(e)\,2} = 0.172329 + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right],$$

$$\mu^{(e)} = 85.9924 \text{ MeV} + O\left[\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right]\mu^{(e)} + O\left\{\left[\frac{\alpha^{(e)} + \beta^{(e)}}{\mu^{(e)}}\right]^2\right\}\mu^{(e)},$$

(30)

if the experimental values of m_e and m_μ [2] are used as an input. Thus, the sum rule (29) gives

$$m_\tau = \left[1776.80 + 9.20087\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right] \text{ MeV},$$

(31)

if we put for the sake of simplicity $\beta^{(e)} = 0$. With the experimental value $m_\tau = 1777.00^{+0.30}_{-0.27}$ MeV, Eq. (31) shows that

$$\left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2 = 0.022^{+0.033}_{-0.029}.$$

(32)

So, as yet, the value of $\alpha^{(e)}$ is consistent with zero (of course, from the viewpoint of our model, the acceptable lower error in Eq. (32) is -0.022).

We can see that our model for $\widehat{M}^{(e)}$, even in the zero-order perturbative calculation, predicts excellently the mass m_τ [1].

4. Neutrino masses (the first option)

In the case of neutrinos consistent with our first option ($\alpha^{(\nu)}/\mu^{(\nu)}$ and $\beta^{(\nu)}/\mu^{(\nu)}$ are perturbative parameters and $\varepsilon^{(\nu)\,2} \simeq 0$), the mass formulae (28) take the form

$$m_{\nu_e} = \frac{\mu^{(\nu)}}{29}\varepsilon^{(\nu)\,2} - \frac{9}{2320}\left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2\mu^{(\nu)} \simeq 0,$$

$$m_{\nu_\mu} \simeq \frac{320}{261}\mu^{(\nu)},$$

$$m_{\nu_\tau} \simeq \frac{14976}{725}\mu^{(\nu)},$$

(33)

if $\left(\alpha^{(\nu)}/\mu^{(\nu)}\right)^2 < 1$ and $\left(\beta^{(\nu)}/\mu^{(\nu)}\right)^2 < 1$. Here, the possible minus sign at m_{ν_e} can be changed (if considered from the phenomenological point of view) into the plus sign since only $m_{\nu_e}^2$ is relevant relativistically (*cf.* the Dirac equation). Note from Eqs (33) that

$$\frac{m_{\nu_\tau}}{m_{\nu_\mu}} \simeq \frac{2106}{125} = 16.8480\,, \tag{34}$$

thus this ratio is practically equal to

$$\frac{m_\tau}{m_\mu} \simeq \frac{1777.00}{105.658} = 16.8184\,. \tag{35}$$

The recent Super-Kamiokande experiments for atmospheric neutrinos [3] seem to show that

$$|m_{\nu_\tau}^2 - m_{\nu_\mu}^2| \sim (0.0003 \text{ to } 0.01) \text{ eV}^2\,, \tag{36}$$

with the value 0.005 eV^2 being preferable (if mixing of $\nu_\mu^{(m)}$ and $\nu_\tau^{(m)}$ is maximal). In this case, Eqs (33) give

$$\mu^{(\nu)} \simeq \frac{\left(m_{\nu_\tau}^2 - m_{\nu_\mu}^2\right)^{1/2}}{20.6201} \sim (0.0008 \text{ to } 0.005) \text{ eV}\,. \tag{37}$$

Then, from Eqs (33) we predict

$$m_{\nu_e} \sim (0.3 \text{ to } 2) \times 10^{-5} \left[9\varepsilon^{(\nu)\,2} - \left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2\right] \text{ eV} \simeq 0\,,$$
$$m_{\nu_\mu} \sim (1 \text{ to } 6) \times 10^{-3} \text{ eV}\,,$$
$$m_{\nu_\tau} \sim (0.2 \text{ to } 1) \times 10^{-1} \text{ eV}\,. \tag{38}$$

Hence,

$$m_{\nu_\mu}^2 - m_{\nu_e}^2 \simeq m_{\nu_\mu}^2 \sim (0.1 \text{ to } 4) \times 10^{-5} \text{ eV}\,. \tag{39}$$

Here, the sign "$\sim$" means approximate equality deduced with the use of bounds (36).

If we put tentatively

$$\left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2 \simeq \left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\,, \tag{40}$$

where the rhs is estimated as in Eq. (32), then from the first of Eqs (38) we obtain

$$|m_{\nu_e}| \sim (0.7^{+1}_{-0.9} \text{ to } 4^{+6}_{-5}) \times 10^{-7} \text{ eV} \tag{41}$$

if $\varepsilon^{(\nu)\,2} = 0$ for the sake of simplicity (of course, from the viewpoint of our model, the realistic lower errors in Eq. (41) are –0.7 and –4, rerspectively). In such a case, from Eqs (33)

$$\frac{|m_{\nu_e}|}{m_{\nu_\mu}} \simeq 7^{+10}_{-9} \times 10^{-5} \quad , \tag{42}$$

thus this ratio is much smaller than

$$\frac{m_e}{m_\mu} = \frac{0.510999}{105.658} = 4.83635 \times 10^{-3} \tag{43}$$

(obviously, the realistic lower error in Eq. (42) is –7).

We can see from Eqs (38) that $m_{\nu_e} + m_{\nu_\mu} + m_{\nu_\tau} \simeq m_{\nu_\tau} \sim (0.02 \text{ to } 0.1)$ eV, so (if our model for $\widehat{M}^{(\nu)}$ works) neutrinos cannot be candidates for hot dark matter, because such a possibility requires several eV for the neutrino mass sum [4].

5. Neutrino oscillations (the first option)

In order to calculate elements of the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} \equiv (V_{ij}) = \widehat{U}^{(\nu)\,\dagger}\widehat{U}^{(e)}$ we use the formulae $V_{ij} = \sum_k U^{(\nu)\,*}_{ki} U^{(e)}_{kj}$ $(i,\,j = 0,\,1,\,2)$, where the unitary matrices $\widehat{U}^{(\nu,e)}$ are given by Eq. (25) in cooperation with Eq. (24) (here, the labels ν and e are made explicit). In $\widehat{M}^{(\nu)}$ we put $\varepsilon^{(\nu)\,2} = 0$, while in $\widehat{M}^{(e)}$ we have approximately $\varepsilon^{(e)\,2} \simeq 0$. Then, in the lowest (linear) order in $\alpha^{(\nu)}/\mu^{(\nu)}$, $\beta^{(\nu)}/\mu^{(\nu)}$ and $\alpha^{(e)}/\mu^{(e)}$, $\beta^{(e)}/\mu^{(e)}$ we obtain

$$
\begin{aligned}
V_{01} &= -\frac{\sqrt{4}}{29}\left(\frac{\alpha^{(\nu)}}{m_{\nu_\mu}}e^{i\varphi^{(\nu)}} - \frac{\alpha^{(e)}}{m_\mu}e^{i\varphi^{(e)}}\right) = -V_{10}^*, \\
V_{12} &= -\frac{\sqrt{192}}{29}\left(\frac{\alpha^{(\nu)}+\beta^{(\nu)}}{m_{\nu_\tau}}e^{i\varphi^{(\nu)}} - \frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau}e^{i\varphi^{(e)}}\right) = -V_{21}^*, \\
V_{02} &= 0 = V_{20}, \\
V_{00} &= V_{11} = V_{22} = 1.
\end{aligned}
\tag{44}
$$

Inserting the matrix elements (44) into Eq. (16), we get in the lowest (quadratic) order in $\alpha^{(\nu)}/\mu^{(\nu)}$, $\beta^{(\nu)}/\mu^{(\nu)}$ and $\alpha^{(e)}/\mu^{(e)}$, $\beta^{(e)}/\mu^{(e)}$ the following

neutrino-oscillation probabilities (in the vacuum):

$$P\left(\nu_e \to \nu_\mu, t\right) = \frac{16}{841}\left[\left(\frac{\alpha^{(\nu)}}{m_{\nu_\mu}}\right)^2 + \left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\right.$$

$$\left. -2\frac{\alpha^{(\nu)}}{m_{\nu_\mu}}\frac{\alpha^{(e)}}{m_\mu}\cos\left(\varphi^{(\nu)} - \varphi^{(e)}\right)\right]$$

$$\times \sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right),$$

$$P\left(\nu_\mu \to \nu_\tau, t\right) = \frac{768}{841}\left[\left(\frac{\alpha^{(\nu)}+\beta^{(\nu)}}{m_{\nu_\tau}}\right)^2 + \left(\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau}\right)^2\right.$$

$$\left. -2\frac{\alpha^{(\nu)}+\beta^{(\nu)}}{m_{\nu_\tau}}\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau}\cos\left(\varphi^{(\nu)}-\varphi^{(e)}\right)\right]$$

$$\times \sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right),$$

$$P\left(\nu_e \to \nu_\tau, t\right) = 0. \tag{45}$$

If we make use of Eqs (28), neglecting there the terms $O\left[\left(\alpha^{(e)}/\mu^{(e)}\right)^2\right]$ and also $O\left\{\left[\left(\alpha^{(e)} + \beta^{(e)}\right)/\mu^{(e)}\right]^2\right\}$ (what leads to the relations $m_\mu \simeq (320/261)\mu^{(e)}$ and $m_\tau \simeq (14976/725)\mu^{(e)}$ up to terms proportional to $\varepsilon^{(e)\,2} \simeq 0$), we can conclude from Eqs (45) that

$$P\left(\nu_e \to \nu_\mu, t\right) = 0.0126\left[\left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2 + \left(\frac{\alpha^{(e)}}{\mu^{(e)}}\right)^2\right.$$

$$\left. -2\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\frac{\alpha^{(e)}}{\mu^{(e)}}\cos\left(\varphi^{(\nu)} - \varphi^{(e)}\right)\right]$$

$$\times \sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right),$$

$$P\left(\nu_\mu \to \nu_\tau, t\right) = 0.00214\left[\left(\frac{\alpha^{(\nu)} + \beta^{(\nu)}}{\mu^{(\nu)}}\right)^2 + \left(\frac{\alpha^{(e)} + \beta^{(e)}}{\mu^{(e)}}\right)^2\right.$$

$$\left. -2\frac{\alpha^{(\nu)} + \beta^{(\nu)}}{\mu^{(\nu)}}\frac{\alpha^{(e)} + \beta^{(e)}}{\mu^{(e)}}\cos\left(\varphi^{(\nu)} - \varphi^{(e)}\right)\right]$$

$$\times \sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right). \tag{46}$$

Here, the factors $[\ \ ] < 1$, so the order of amplitude of $P\,(\nu_\mu \to \nu_\tau, t)$ is smaller than $O(10^{-3})$.

We can see that this result, valid in the case of our first option, appears to be inconsistent with the experiments for atmospheric neutrinos [3–5] which seem to indicate that the order of amplitude of $P\,(\nu_\mu \to \nu_\tau, t)$ is $O(1)$.

6. Neutrino masses (the second option)

In the case of neutrinos consistent with the second option (where $\alpha^{(\nu)}/\mu^{(\nu)}$ only is a perturbative parameter and $\varepsilon^{(\nu)\,2} \simeq 0$), the second term in the matrix $\widehat{h}^{(\nu)}$ given in Eq. (19) cannot be treated as a small perturbation of the first term.

When $\alpha^{(\nu)}/\mu^{(\nu)} = 0$, the neutrino mass matrix (24) takes the unperturbed form

$$\widehat{M}^{(\nu)} = \frac{1}{29}\begin{pmatrix} \mu^{(\nu)}\varepsilon^{(\nu)\,2} & 0 & 0 \\[2mm] 0 & \frac{4\mu^{(\nu)}(80+\varepsilon^{(\nu)\,2})}{9} & 8\beta^{(\nu)}\sqrt{3}\mathrm{e}^{i\varphi} \\[2mm] 0 & 8\beta^{(\nu)}\sqrt{3}\mathrm{e}^{-i\varphi} & \frac{24\mu^{(\nu)}(624+\varepsilon^{(\nu)\,2})}{25} \end{pmatrix}. \qquad (47)$$

Evidently, its eigenvalues can be found exactly, reading

$$m_0 \;=\; M_{00}^{(\nu)} = \frac{\mu^{(\nu)}}{29}\varepsilon^{(\nu)\,2}\,,$$

$$m_{1,2} \;=\; \frac{M_{11}^{(\nu)} + M_{22}^{(\nu)}}{2} \mp \left[\left(\frac{M_{11}^{(\nu)} - M_{22}^{(\nu)}}{2}\right)^2 + |M_{12}^{(\nu)}|^2\right]^{1/2}$$

$$\;=\; \left[10.9 \mp 0.478\frac{\beta^{(\nu)}}{\mu^{(\nu)}}\sqrt{1 + \left(20.3\frac{\mu^{(\nu)}}{\beta^{(\nu)}}\right)^2}\right]\mu^{(\nu)}\,, \qquad (48)$$

if $\varepsilon^{(\nu)\,2} < 0.1$. These eigenvalues give three unperturbed neutrino masses

$$m_{\nu_e} = m_0\,, \quad m_{\nu_\mu} = m_1\,, \quad m_{\nu_\tau} = m_2 \qquad (49)$$

if, by convention, we ascribe the minus sign in Eq. (48) to m_{ν_μ}. Note that in the limit of $\mu^{(\nu)} \to 0$ Eqs (48) give $m_0 \to 0$ and $m_{1,2} \to \mp|M_{12}^{(\nu)}| = \mp 0.478\beta^{(\nu)}$.

From Eqs (48) we can evaluate the difference of mass squared :

$$m_2^2 - m_1^2 = 2|M_{12}^{(\nu)}| \left(M_{11}^{(\nu)} + M_{22}^{(\nu)} \right) \left[1 + \left(\frac{M_{11}^{(\nu)} - M_{22}^{(\nu)}}{2|M_{12}^{(\nu)}|} \right)^2 \right]^{1/2}$$

$$= 20.9 \, \beta^{(\nu)} \mu^{(\nu)} \sqrt{1 + \left(20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}} \right)^2} \,, \tag{50}$$

if $\varepsilon^{(\nu)\,2} < 0.1$. Note also from Eq. (48) that

$$\frac{M_{11}^{(\nu)} - m_1}{M_{12}^{(\nu)}} = X e^{-i\varphi} \,, \frac{M_{22}^{(\nu)} - m_2}{M_{21}^{(\nu)}} = -X e^{i\varphi} \,, \tag{51}$$

where

$$X = \frac{M_{11}^{(\nu)} - M_{22}^{(\nu)}}{2|M_{12}^{(\nu)}|} + \left[1 + \left(\frac{M_{11}^{(\nu)} - M_{22}^{(\nu)}}{2|M_{12}^{(\nu)}|} \right)^2 \right]^{1/2}$$

$$= -20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}} + \left[1 + \left(20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}} \right)^2 \right]^{1/2} \,, \tag{52}$$

if $\varepsilon^{(\nu)\,2} < 0.1$. Here, $(M_{22}^{(\nu)} - m_i)M_{21}^{(\nu)\,-1} = M_{12}^{(\nu)}(M_{11}^{(\nu)} - m_i)^{-1}$ $(i = 1, 2)$, as it follows from the secular equations $\det(\widehat{M}^{(\nu)} - \widehat{1}m_i) = 0$ with $M_{02}^{(\nu)} = 0 = M_{20}^{(\nu)}$ and $M_{01}^{(\nu)} = 0 = M_{10}^{(\nu)}$. Also $M_{11}^{(\nu)} + M_{22}^{(\nu)} = m_1 + m_2$ and $M_{11}^{(\nu)} M_{22}^{(\nu)} - |M_{12}^{(\nu)}|^2 = m_1 m_2$. Note that Eqs (48) and (52) give $m_{1,2} = M_{11,22}^{(\nu)} \mp |M_{12}^{(\nu)}|X$.

Now, let us assume that the neutrino mass matrix (47) is perturbed by the matrix

$$\delta\widehat{M}^{(\nu)} = \begin{pmatrix} 0 & \delta M_{01}^{(\nu)} & 0 \\ \delta M_{10}^{(\nu)} & 0 & \delta M_{12}^{(\nu)} \\ 0 & \delta M_{21}^{(\nu)} & 0 \end{pmatrix}$$

$$= \frac{\alpha^{(\nu)}}{29} \begin{pmatrix} 0 & 2e^{i\varphi^{(\nu)}} & 0 \\ 2e^{-i\varphi^{(\nu)}} & 0 & 8\sqrt{3}e^{i\varphi^{(\nu)}} \\ 0 & 8\sqrt{3}e^{-i\varphi^{(\nu)}} & 0 \end{pmatrix}. \tag{53}$$

Remember that in the unperturbed mass matrix (47) $M_{01}^{(\nu)} = 0 = M_{10}^{(\nu)}$, while $M_{12}^{(\nu)} = (8\beta^{(\nu)}\sqrt{3}/29) \exp(i\varphi^{(\nu)}) = M_{21}^{(\nu)\,*}$. Then, the secular equations $\det[\widehat{M}^{(\nu)} + \delta\widehat{M}^{(\nu)} - \widehat{1}(m_i + \delta m_i)] = 0$ give in the lowest (linear or

quadratic) perturbative order in $\alpha^{(\nu)}/\mu^{(\nu)}$ the following neutrino mass corrections:

$$\delta m_0 = -\frac{|\delta M_{01}^{(\nu)}|^2 M_{22}^{(\nu)}}{m_1 m_2},$$

$$\delta m_{1,2} = \mp \frac{|\delta M_{12}^{(\nu)}||M_{12}^{(\nu)}|}{m_2 - m_1}, \tag{54}$$

where $|\delta M_{01}^{(\nu)}|^2 = 0.00476\alpha^{(\nu)\,2}$, $|\delta M_{12}^{(\nu)}| = 0.478\alpha^{(\nu)}$, $M_{22}^{(\nu)} = 20.7\mu^{(\nu)}$ and $|M_{12}^{(\nu)}| = 0.478\beta^{(\nu)}$. Here, we neglect all terms proportional to $\varepsilon^{(\nu)\,2}$ (this is correct for $\varepsilon^{(\nu)\,2} < 0.1$). From Eqs (54) it follows that $\delta m_0 + \delta m_1 + \delta m_2 = 0$, as it should be because of $\operatorname{tr}\delta\widehat{M}^{(\nu)} = 0$. Note that in the limit of $\mu^{(\nu)} \to 0$ Eqs (54) give $\delta m_0 \to 0$ and $\delta m_{1,2} \to \mp|\delta M_{12}^{(\nu)}|/2$. Thus, in this limit $(m_1 + \delta m_1)^2 = (m_2 + \delta m_2)^2$ as well as $m_1^2 = m_2^2$.

7. Neutrino oscillations (the second option)

The unitary matrix (25), diagonalizing the unperturbed neutrino mass matrix (47) according to the equation $\widehat{U}^{(\nu)\,-1}\widehat{M}^{(\nu)}\widehat{U}^{(\nu)} = \operatorname{diag}(m_0\,,\,m_1\,,\,m_2)$, can be written as

$$\widehat{U}^{(\nu)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A_1^{(\nu)} & A_2^{(\nu)}Xe^{i\varphi^{(\nu)}} \\ 0 & -A_1^{(\nu)}Xe^{-i\varphi^{(\nu)}} & A_2^{(\nu)} \end{pmatrix}, \tag{55}$$

where

$$A_1^{(\nu)} = \left(1 + X^2\right)^{-1/2} = A_2^{(\nu)}. \tag{56}$$

Here, X is given as in Eq. (51). Note that in the limit of $\mu^{(\nu)} \to 0$ Eqs (52) and (56) give $X \to 1$ and $A_1^{(\nu)} = A_2^{(\nu)} \to 1/\sqrt{2}$.

Assuming tentatively that $\alpha^{(e)}$ and $\beta^{(e)}$, which are experimentally consistent with zero [*cf.* Eq. (32)], are really zero *i.e.*, $\widehat{U}^{(e)} = \widehat{1}$, we have

$$\widehat{V} = \widehat{U}^{(\nu)\dagger} = \left(U_{ji}^{(\nu)*}\right) \tag{57}$$

with $\widehat{U}^{(\nu)} = \left(U_{ij}^{(\nu)}\right)$. Then, Eqs (16) and (55) give the following unperturbed neutrino oscillation probabilities (in the vacuum):

$$P\left(\nu_e \to \nu_\mu, t\right) = 0 = P\left(\nu_e \to \nu_\tau, t\right),$$

$$P\left(\nu_\mu \to \nu_\tau, t\right) = 4\frac{X^2}{(1 + X^2)^2}\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right). \tag{58}$$

Note from Eq. (52) that the oscillation amplitude $4X^2(1 + X^2)^{-2} \to 1$ in the limit of $\mu^{(\nu)}/\beta^{(\nu)} \to 0$ as then $X \to 1$. The atmospheric neutrino experiments seem to indicate that this oscillation amplitude is of the order $O(1)$, perhaps $\sim 1/2$ [4]. So, taking $4X^2(1 + X^2)^{-2} \sim 1/2$ as an input, we estimate $X \sim (3 - 2\sqrt{2})^{1/2} = \sqrt{2} - 1$, what through Eq. (52) implies that

$$20.3 \frac{\mu^{(\nu)}}{\beta^{(\nu)}} \sim 1 \tag{59}$$

or $\beta^{(\nu)}/\mu^{(\nu)} \sim 20.3$ and $\mu^{(\nu)}/\beta^{(\nu)} \sim 0.05$.

Now, assuming as another input the Super-Kamiokande bound (36), we obtain from Eqs (50) and (59)

$$29.6\mu^{(\nu)}\beta^{(\nu)} \sim (0.0003 \ \text{to} \ 0.01)\,\text{eV}^2 \,. \tag{60}$$

Of course, this relation excludes $\mu^{(\nu)} = 0$, what would give $m_1^2 = m_2^2$ as well as $(m_1 + \delta m_1)^2 = (m_2 + \delta m_2)^2$. Making use of Eqs (59) and (60), we estimate

$$\mu^{(\nu)} \sim (0.71 \ \text{to} \ 4.1) \times 10^{-3}\text{eV} \tag{61}$$

and

$$\beta^{(\nu)} \sim (1.4 \ \text{to} \ 8.3) \times 10^{-2}\text{eV} \,. \tag{62}$$

Finally, using the estimates (59) and (61), we can calculate from Eqs (48) the unperturbed neutrino masses

$$
\begin{aligned}
m_0 &= \frac{\mu^{(\nu)}}{29}\varepsilon^{(\nu)\,2} \sim (0.24 \ \text{to} \ 1.4) \times 10^{-4}\varepsilon^{(\nu)\,2}\text{eV} \ll |m_1| \,, \\
m_1 &\sim -2.82\mu^{(\nu)} \sim -(0.20 \ \text{to} \ 1.2) \times 10^{-2}\text{eV} \,, \\
m_2 &\sim 24.6\mu^{(\nu)} \sim (0.17 \ \text{to} \ 1.0) \times 10^{-1}\text{eV} \,,
\end{aligned} \tag{63}
$$

if $\varepsilon^{(\nu)\,2} < 0.1$. The minus sign at m_1 is irrelevant (*cf.* the Dirac equation) and so, can be changed (if considered from the phenomenological point of view) into the plus sign.

Similarly, from Eqs (54) we can evaluate the neutrino mass corrections in terms of $\alpha^{(\nu)}/\mu^{(\nu)}$:

$$
\begin{aligned}
\delta m_0 &\sim \ 0.0014 \left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2 \mu^{(\nu)} \sim (1.0 \ \text{to} \ 5.9) \times 10^{-6} \left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2 \text{eV} \,, \\
\delta m_{1,2} &\sim \mp 0.17\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\mu^{(\nu)} \sim \mp (1.2 \ \text{to} \ 6.9) \times 10^{-4} \frac{\alpha^{(\nu)}}{\mu^{(\nu)}} \ \text{eV} \,,
\end{aligned} \tag{64}
$$

if $\varepsilon^{(\nu)\,2} < 0.1$. Thus, $\delta m_0/m_0 \sim 4.2 \times 10^{-2} \left(1/\varepsilon^{(\nu)\,2}\right) \left(\alpha^{(\nu)}/\mu^{(\nu)}\right)^2$, $\delta m_1/m_1 \sim 6.0 \times 10^{-2} \alpha^{(\nu)}/\mu^{(\nu)}$ and $\delta m_2/m_2 \sim 6.9 \times 10^{-3} \alpha^{(\nu)}/\mu^{(\nu)}$, what implies that on our accuracy level we get $m_i + \delta m_i \sim m_i$ for $i = 1,\,2$.

We can see that the unperturbed result (58) for $P\left(\nu_\mu \to \nu_\tau, t\right)$, valid in the case of our second option, is consistent with the experiments for atmospheric neutrinos [3–5], which suggest a large neutrino-oscillation amplitude of the order $O(1)$. However, in the case of our second option, the vanishing $P\left(\nu_e \to \nu_\mu, t\right)$ and $P\left(\nu_e \to \nu_\tau, t\right)$ raise a problem for solar neutrinos. Of course, the perturbed neutrino mass matrix $\widehat{M}^{(\nu)} + \delta\widehat{M}^{(\nu)}$, as described by Eqs (47) and (53), induces a perturbation $\delta\widehat{U}^{(\nu)}$ for the diagonalizing unitary matrix $\widehat{U}^{(\nu)}$ given in Eq. (55), and so, a perturbation $\delta\widehat{V}$ for the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} = \widehat{U}^{(\nu)\,\dagger}$. Obviously, when $\widehat{V} \to \widehat{V} + \delta\widehat{V}$ in consequence of $\widehat{M}^{(\nu)} \to \widehat{M}^{(\nu)} + \delta\widehat{M}^{(\nu)}$, then

$$
\begin{aligned}
P\left(\nu_e \to \nu_\mu, t\right) &= 0 \to \delta P\left(\nu_e \to \nu_\mu, t\right), \\
P\left(\nu_e \to \nu_\tau, t\right) &= 0 \to \delta P\left(\nu_e \to \nu_\tau, t\right), \\
P\left(\nu_\mu \to \nu_\tau, t\right) &\to P\left(\nu_\mu \to \nu_\tau, t\right) + \delta P\left(\nu_\mu \to \nu_\tau, t\right).
\end{aligned}
\tag{65}
$$

If the realistic $\alpha^{(e)}$ and/or $\beta^{(e)}$ are not zero *i.e.*, $\widehat{U}^{(e)} \neq \widehat{1}$, then Eqs (58) get also other corrections which will be discussed in detail in the next Section. The perturbed $\widehat{V} + \delta\widehat{V}$, strengthened by the mechanism of neutrino oscillations in the Sun matter [6, 4, 5], might help with the problem of solar neutrinos, practically not perturbing the oscillations (in the vacuum) of atmospheric neutrinos.

The perturbation $\delta\widehat{V} = \left(\delta\widehat{U}^{(\nu)}\right)^\dagger$ of the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} = \widehat{U}^{(\nu)\,\dagger}$ (in the case of $\widehat{U}^{(e)} = \widehat{1}$) can be calculated from Eq. (25) applied to the whole mass matrix $\widehat{M}^{(\nu)} + \delta\widehat{M}^{(\nu)}$ given by Eqs (47) and (53). Then, in this equation, $M_{01}^{(\nu)} \to \delta M_{01}^{(\nu)}$ and $M_{12}^{(\nu)} \to M_{12}^{(\nu)} + \delta M_{12}^{(\nu)}$, and so, $A_i^{(\nu)} \to A_i^{(\nu)} + \delta A_i^{(\nu)}$ as well as $m_i \to m_i + \delta m_i$ $(i = 0, 1, 2)$. Here, $\delta A_i^{(\nu)}$ are of the second order in $\alpha^{(\nu)}/\mu^{(\nu)}$, while δm_i are negligible. But, due to Eq. (54), the elements

$$
\begin{aligned}
U_{10}^{(\nu)} \to \delta U_{10}^{(\nu)} &= -A_0^{(\nu)} \frac{M_{00}^{(\nu)} - m_0 - \delta m_0}{\delta M_{01}^{(\nu)}} = A_0^{(\nu)} \frac{\delta m_0}{\delta M_{01}^{(\nu)}} \\
&= -A_0^{(\nu)} \frac{|\delta M_{01}^{(\nu)}|\, M_{22}^{(\nu)}}{m_1 m_2}
\end{aligned}
\tag{66}
$$

and $U_{20}^{(\nu)} \to \delta U_{20}^{(\nu)}$ are of the first order in $\alpha^{(\nu)}/\mu^{(\nu)}$. Remember that $A_0^{(\nu)} = 1$ and $A_1^{(\nu)} = A_2^{(\nu)} = (1 + X^2)^{-1/2}$. In this way, after some calculations, we

obtain in the lowest (linear) perturbative order in $\alpha^{(\nu)}/\mu^{(\nu)}$

$$\delta\widehat{U}^{(\nu)} = \begin{pmatrix} 0 & A_1^{(\nu)}\frac{\delta M_{01}^{(\nu)}}{m_1} & A_2^{(\nu)}\frac{\delta M_{01}^{(\nu)} M_{12}^{(\nu)}}{m_2(M_{22}^{(\nu)}-m_1)} \\ -\frac{\delta M_{10}^{(\nu)} M_{22}^{(\nu)}}{m_1 m_2} & 0 & A_2^{(\nu)}\frac{\delta M_{12}^{(\nu)}}{M_{22}^{(\nu)}-m_1} \\ \frac{\delta M_{10}^{(\nu)} M_{21}^{(\nu)}}{m_1 m_2} & -A_1^{(\nu)}\frac{\delta M_{21}^{(\nu)}}{M_{22}^{(\nu)}-m_1} & 0 \end{pmatrix}, \quad (67)$$

where

$$\delta M_{01}^{(\nu)} = (2\alpha^{(\nu)}/29)\exp(i\varphi^{(\nu)}) = \delta M_{10}^{(\nu)\,*}$$

and

$$\delta M_{12}^{(\nu)} = (8\alpha^{(\nu)}\sqrt{3}/29)\exp(i\varphi^{(\nu)}) = \delta M_{21}^{(\nu)\,*},$$

while $\widehat{U}^{(\nu)}$ is given as in Eq. (55) with $X\exp(i\varphi^{(\nu)}) = -\left(M_{22}^{(\nu)}-m_2\right)/M_{21}^{(\nu)}$. Here, all terms proportional to $\varepsilon^{(\nu)\,2} \simeq 0$ are neglected (it is correct already for $\varepsilon^{(\nu)\,2} < 0.1$).

The corrections $\delta P\left(\nu_e \to \nu_\mu, t\right)$, $\delta P\left(\nu_e \to \nu_\tau, t\right)$ and $\delta P\left(\nu_\mu \to \nu_\tau, t\right)$ to the neutrino oscillation probabilities (in the vacuum) can be evaluated from Eqs (16) applied to the whole Cabibbo–Kobayashi–Maskawa matrix $\widehat{V} + \delta\widehat{V} = \widehat{U}^{(\nu)\,\dagger} + \left(\delta\widehat{U}^{(\nu)}\right)^\dagger$ (in the case of $\widehat{U}^{(e)} = \widehat{1}$) given by Eqs (55) and (67). Then, after some calculations, we get in the lowest (linear or quadratic) perturbative order in $\alpha^{(\nu)}/\mu^{(\nu)}$ the following formulae:

$$\begin{aligned} \delta P\left(\nu_e \to \nu_\mu, t\right) = \;& 4|\delta V_{10}|^2 \left[\frac{M_{22}^{(\nu)}}{m_2}\sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \right. \\ & + \frac{|M_{12}^{(\nu)}|}{m_2}X\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \\ & \left. - \frac{|M_{12}^{(\nu)}|\,M_{22}^{(\nu)}}{m_2^2}X\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right) \right] \quad (68) \end{aligned}$$

for the oscillations $\nu_e \to \nu_\mu$,

$$\begin{aligned} \delta P\left(\nu_e \to \nu_\tau, t\right) = \;& 4|\delta V_{10}|^2\frac{|M_{12}^{(\nu)}|}{m_2}X\left[-\frac{|m_1|\,M_{22}^{(\nu)}}{m_2|M_{12}^{(\nu)}|}X\sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \right. \\ & + \frac{|m_1|}{m_2}\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \\ & \left. + \frac{M_{22}^{(\nu)}}{m_2}\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right) \right] \quad (69) \end{aligned}$$

for the oscillations $\nu_e \to \nu_\tau$, and

$$\delta P\left(\nu_\mu \to \nu_\tau, t\right) = 8|\delta V_{12}| \frac{X}{(1+X^2)^{3/2}} \sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|} t\right)$$

$$+4|\delta V_{10}|^2 \frac{|m_1|}{m_2} \frac{X^2}{1+X^2}\left[\sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|} t\right)\right.$$

$$\left. - \sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_e}^2}{4|\vec{p}|} t\right)\right] \tag{70}$$

for the oscillations $\nu_\mu \to \nu_\tau$. These corrections are to be added to the unperturbed values (58). Here, the following numbers are involved:

$$|\delta V_{10}|^2 = \frac{|\delta M_{10}^{(\nu)}|^2}{m_1^2(1+X^2)} = \frac{4}{841(1+X^2)}\left(\frac{\alpha^{(\nu)}}{m_1}\right)^2$$

$$\sim 5.25 \times 10^{-4}\left(\frac{\alpha^{(\nu)}}{\mu^{(\nu)}}\right)^2,$$

$$|\delta V_{12}| = \frac{|\delta M_{12}^{(\nu)}|}{(M_{22}^{(\nu)} - m_1)\sqrt{1+X^2}} = \frac{8\sqrt{3}}{29\sqrt{1+X^2}}\frac{\alpha^{(\nu)}}{M_{22}^{(\nu)} - m_1}$$

$$\sim 1.88 \times 10^{-2}\frac{\alpha^{(\nu)}}{\mu^{(\nu)}} \tag{71}$$

as well as

$$M_{11}^{(\nu)} = \frac{320}{261}\mu^{(\nu)} = 1.23\mu^{(\nu)} \quad, \quad M_{22}^{(\nu)} = \frac{14976}{725}\mu^{(\nu)} = 20.7\mu^{(\nu)},$$

$$|M_{12}^{(\nu)}| = \frac{8\sqrt{3}}{29}\beta^{(\nu)} \sim 9.70\mu^{(\nu)} \tag{72}$$

and

$$m_1 \sim -2.78\mu^{(\nu)}, \; m_2 \sim 24.7\mu^{(\nu)}, \; X \sim \sqrt{2} - 1 = 0.414. \tag{73}$$

The perturbative parameter $\alpha^{(\nu)}/\mu^{(\nu)}$ is free. In Eqs (71)–(73), the sign "$\sim$" denotes the estimate valid in the case of our input $4X^2(1 + X^2)^{-2} \sim 1/2$ leading to the relation (59) for $\beta^{(\nu)}/\mu^{(\nu)}$ *i.e.*, $\beta^{(\nu)}/\mu^{(\nu)} \sim 20.3$. Another input is Eq. (60) giving for $\mu^{(\nu)}$ the value (61), $\mu^{(\nu)} \sim (0.71 \text{ to } 4.1) \times 10^{-3}$ eV.

We can see from Eqs (68)–(70) and (71)–(73) that the corrections to the neutrino–oscillation probabilities (58) (in the vacuum) are very small (for $\alpha^{(\nu)}/\mu^{(\nu)} < 1$). The largest of them is $\delta P\left(\nu_\mu \to \nu_\tau, t\right)$.

8. Conclusions and a proposal

In this paper, starting with the generic form (24) of lepton mass matrix, following from our texture dynamics expressed by Eqs (18)–(21), we concentrated mainly on neutrinos. For the parameters involved in this form we considered two options: either (i) among the neutrinos ν_e, ν_μ, ν_τ practically only the neighbours mix and do it weakly, or (ii) practically only ν_μ and ν_τ mix and do it strongly. In both cases, we evaluated the neutrino masses, the lepton Cabibbo–Kobayashi–Maskawa matrix and the neutrino-oscillation probabilities (in the vacuum), expressing all these quantities in terms of few parameters determined essentially from the experimental data. In the second case, we calculated also the lowest-order perturbative corrections to these quantities, caused by possible weak mixing of ν_e with ν_μ and ν_τ.

The second option turned out to be consistent with the experiments for atmospheric neutrinos [3–5] which seem to indicate a large $\nu_\mu \to \nu_\tau$ oscillation amplitude of the order $O(1)$. Then, very small $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ oscillation amplitudes were implied and so, apparently, must be much strengthened in the Sun matter [6,4,5] in order to avoid the problem for solar neutrinos. In these calculations, it was tentatively assumed that the nondiagonal part of the charged-lepton mass-matrix, which is experimentally consistent with zero, is really zero.

Let us add a remark bearing on the last question and concerning the values of parameters involved in our generic form of mass matrix, when it is applied also to the quarks u, c, t and d, s, b. Such an application, as it was made in the second Ref. [1], led to the values

$$\alpha^{(u)} \simeq 1740 \text{ MeV}, \quad \alpha^{(d)} + \beta^{(d)} \simeq 405 \text{ MeV}, \tag{74}$$

when they were fitted to the experimental data for $|V_{cb}|$ and $|V_{ub}/V_{cb}|$. If $\alpha^{(u)} : \alpha^{(d)} = |Q^{(u)}| : |Q^{(d)}| = 2$, as was conjectured there, then

$$\alpha^{(d)} \simeq 870 \text{ MeV}, \quad \beta^{(d)} \simeq -465 \text{ MeV}, \tag{75}$$

what leaves $\beta^{(u)}$ unknown, unless also $\beta^{(u)} : \beta^{(d)} = 2$ giving $\beta^{(u)} \simeq -930$ MeV (at present, $\beta^{(u)}$ cannot be determined from the data directly). In the spirit of the relation $\alpha^{(u)} : \alpha^{(d)} = |Q^{(u)}| : |Q^{(e)}|$ for quarks, the analogical conjecture $\alpha^{(\nu)} : \alpha^{(e)} = |Q^{(\nu)}| : |Q^{(e)}| = 0$ would be natural for leptons, leaving now $\alpha^{(e)}$ as well as $\beta^{(\nu)}$ and $\beta^{(e)}$ free (to be determined from the neutrino and charged-lepton experiments).

In Sections 6 and 7 we allowed for $\alpha^{(\nu)}$ to be different from zero, but small ($\alpha^{(\nu)}/\mu^{(\nu)} < 1$). Now, using partly the suggestion that $\alpha^{(\nu)} : \alpha^{(e)} = 0$, we might expect rather the inequality $0 \le \alpha^{(\nu)} : \alpha^{(e)} \ll 1$. If so, a

new perturbation $\delta\widehat{V}^{(e)}$ of the unperturbed $\widehat{V} = \widehat{U}^{(\nu)\dagger}\widehat{U}^{(e)}$ with the trivial $\widehat{U}^{(e)} = \widehat{1}$ would arise, when $\widehat{U}^{(e)} \to \widehat{1} + \delta\widehat{U}^{(e)}$ with a $\delta\widehat{U}^{(e)}$ proportional to $\alpha^{(e)}$ or $\alpha^{(e)} + \beta^{(e)}$. Such a new $\delta\widehat{V}^{(e)}$ should be more significant than our previous perturbation $\delta\widehat{V}^{(\nu)} = \left(\delta\widehat{U}^{(\nu)}\right)^{\dagger}$ proportional to $\alpha^{(\nu)}$ (and discussed in detail in Section 7).

As our last item in this paper, let us evaluate the perturbation

$$\delta\widehat{V}^{(e)} = \widehat{U}^{(\nu)\dagger}\delta\widehat{U}^{(e)}\,, \tag{76}$$

and also the related corrections to the neutrino-oscillation probabilities (in the vacuum). Then,

$$\widehat{V} \to \widehat{V} + \delta\widehat{V}^{(e)} + \delta\widehat{V}^{(\nu)} = \widehat{V} + \delta\widehat{V}^{(e)} \tag{77}$$

under the conjecture of comparatively negligible or even vanishing $\delta\widehat{V}^{(\nu)}$.

Since $\alpha^{(e)}/\mu^{(e)}$ and $\beta^{(e)}/\mu^{(e)}$ can be treated as perturbative parameters, we obtain from Eqs (25), (24) and (28) in the lowest (linear) order in $\alpha^{(e)}/\mu^{(e)}$ and $(\alpha^{(e)} + \beta^{(e)})/\mu^{(e)}$ the unitary matrix $\widehat{1} + \delta\widehat{U}^{(e)}$ diagonalizing the mass matrix $\widehat{M}^{(e)} + \delta\widehat{M}^{(e)}$, where

$$\delta\widehat{U}^{(e)} = \frac{1}{29}\begin{pmatrix} 0 & 2\frac{\alpha^{(e)}}{m_\mu}\mathrm{e}^{i\varphi^{(e)}} & 0 \\ -2\frac{\alpha^{(e)}}{m_\mu}\mathrm{e}^{-i\varphi^{(e)}} & 0 & 8\sqrt{3}\frac{(\alpha^{(e)}+\beta^{(e)})}{m_\tau}\mathrm{e}^{i\varphi^{(e)}} \\ 0 & -8\sqrt{3}\frac{(\alpha^{(e)}+\beta^{(e)})}{m_\tau}\mathrm{e}^{-i\varphi^{(e)}} & 0 \end{pmatrix} \tag{78}$$

(what is consistent with Eqs (44) if there $\alpha^{(\nu)} = 0 = \beta^{(\nu)}$ formally). Here, $\widehat{M}^{(e)} = \mathrm{diag}(m_e\,,\, m_\mu\,,\, m_\tau)$ contains the unperturbed charged-lepton masses

$$m_e = \frac{\mu^{(e)}}{29}\varepsilon^{(e)\,2}\,,\; m_\mu = \frac{\mu^{(e)}}{29}\frac{4}{9}\left(80 + \varepsilon^{(e)\,2}\right)\,,\; m_\tau = \frac{\mu^{(e)}}{29}\frac{24}{25}\left(624 + \varepsilon^{(e)\,2}\right) \tag{79}$$

[undistinguished, as yet, from their experimental values, as seen from Eqs (31) and (32)], whereas

$$\delta\widehat{M}^{(e)} = \frac{1}{29}\begin{pmatrix} 0 & 2\alpha^{(e)}\mathrm{e}^{i\varphi^{(e)}} & 0 \\ 2\alpha^{(e)}\mathrm{e}^{-i\varphi^{(e)}} & 0 & 8\sqrt{3}(\alpha^{(e)} + \beta^{(e)})\mathrm{e}^{i\varphi^{(e)}} \\ 0 & 8\sqrt{3}(\alpha^{(e)} + \beta^{(e)})\mathrm{e}^{-i\varphi^{(e)}} & 0 \end{pmatrix} \tag{80}$$

is the perturbation. Note that in the lowest (quadratic) perturbative order the perturbed masses $m_e + \delta m_e\,,\; m_\mu + \delta m_\mu\,,\; m_\tau + \delta m_\tau$ are given as in Eqs (28).

In the next step we make use of Eqs (55) and (78) to calculate $\delta\widehat{V}^{(e)} \equiv (\delta V_{ij}^{(e)}) = \widehat{U}^{(\nu)\dagger}\delta\widehat{U}^{(e)}$. The result is

$$
\begin{aligned}
\delta V_{01}^{(e)} &= \frac{2}{29}\frac{\alpha^{(e)}}{m_\mu}\mathrm{e}^{i\varphi^{(e)}}, \ \delta V_{10}^{(e)} = -\frac{2}{29\sqrt{1+X^2}}\frac{\alpha^{(e)}}{m_\mu}\mathrm{e}^{-i\varphi^{(e)}}, \\
\delta V_{12}^{(e)} &= \frac{8\sqrt{3}}{29\sqrt{1+X^2}}\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau}\mathrm{e}^{i\varphi^{(e)}} = -\delta V_{21}^{(e)*}, \\
\delta V_{02}^{(e)} &= 0, \ \delta V_{20}^{(e)} = -\frac{2X}{29\sqrt{1+X^2}}\frac{\alpha^{(e)}}{m_\mu}\mathrm{e}^{-i(\varphi^{(\nu)}+\varphi^{(e)})}, \\
\delta V_{00}^{(e)} &= 0, \ \delta V_{11}^{(e)} = \frac{8\sqrt{3}X}{29\sqrt{1+X^2}}\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau}\mathrm{e}^{i(\varphi^{(\nu)}-\varphi^{(e)})} = \delta V_{22}^{(e)*}. \quad (81)
\end{aligned}
$$

Finally, we apply Eqs (16) to the whole lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V}+\delta\widehat{V}^{(e)}$, where $\widehat{V} = \widehat{U}^{(\nu)\dagger}$ and $\delta\widehat{V}^{(e)}$ are given by Eqs (55) and (81), respectively (and $\delta\widehat{V}^{(\nu)} = \left(\delta\widehat{U}^{(\nu)}\right)^\dagger$ is neglected). Then, after some calculations, we obtain in the lowest (linear or quadratic) perturbative order in $\alpha^{(e)}/\mu^{(e)}$ and $(\alpha^{(e)}+\beta^{(e)})/\mu^{(e)}$ the following corrections to the neutrino-oscillation probabilities (in the vacuum):

$$
\delta P\left(\nu_e \to \nu_\mu, t\right) = \frac{16}{841(1+X^2)}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \quad (82)
$$

for the oscillations $\nu_e \to \nu_\mu$,

$$
\delta P\left(\nu_e \to \nu_\tau, t\right) = \frac{16X^2}{841(1+X^2)}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\cos^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) \quad (83)
$$

for the oscillations $\nu_e \to \nu_\tau$, and

$$
\begin{aligned}
\delta P\left(\nu_\mu \to \nu_\tau, t\right) &= \frac{64\sqrt{3}X}{29(1+X^2)}\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau} \\
&\times \cos\left(\varphi^{(\nu)} - \varphi^{(e)}\right)\sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_\mu}^2}{4|\vec{p}|}t\right) \\
&+ \frac{16X^2}{841(1+X^2)}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2\left[\sin^2\left(\frac{m_{\nu_\mu}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right) - \sin^2\left(\frac{m_{\nu_\tau}^2 - m_{\nu_e}^2}{4|\vec{p}|}t\right)\right]
\end{aligned}
$$
$$
(84)
$$

for the oscillations $\nu_\mu \to \nu_\tau$. Note that the same time-argument appears in Eqs (82) and (83). Also notice the presence of unknown phase factor $\cos(\varphi^{(\nu)} - \varphi^{(e)})$ in Eq. (84) that becomes 1 if $\varphi^{(\nu)} = \varphi^{(e)}$ as *e.g.* for

$$\varphi^{(\nu)} = \varphi^{(e)} = 0. \tag{85}$$

Of course, these corrections are to be added to the unperturbed values (58). In Eqs (82), (83) and (84) there appear the numerical coefficients

$$\frac{16}{841(1+X^2)}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \sim 2.4 \times 10^{-4} \quad, \quad \frac{16X^2}{841(1+X^2)}\left(\frac{\alpha^{(e)}}{m_\mu}\right)^2 \sim 4.1 \times 10^{-5},$$

$$\frac{64\sqrt{3}X}{29(1+X^2)}\frac{\alpha^{(e)}+\beta^{(e)}}{m_\tau} \sim 9.7 \times 10^{-3}. \tag{86}$$

To evaluate these coefficients we put $\beta^{(e)} = 0$ for the sake of simplicity, and then took the central value $(\alpha^{(e)}/\mu^{(e)})^2 = 0.022$ deduced in Eq. (32) from the experimental value of m_τ. We used also our input $4X^2(1+X^2)^{-2} \sim 1/2$ implying $X \sim \sqrt{2} - 1$. When the experimental errors in Eq. (32) are taken into account, the first coefficient (86) becomes $2.4^{+3.6}_{-2.4} \times 10^{-4}$.

We can conclude from Eq. (82) that the predicted oscillations $\nu_e \to \nu_\mu$ (in the vacuum) are very small, and similar in magnitude to those derived in the case of our first option [*cf.* Eqs (45)].Thus, the effect of neutrino oscillations in the Sun matter still appears to be needed. Evidently, the oscillations $\nu_e \to \nu_\mu$ caused by $\delta\widehat{V}^{(\nu)} = \left(\delta\widehat{U}^{(\nu)}\right)^\dagger$ are comparatively negligible or even vanish if $0 \leq \alpha^{(\nu)} \ll \alpha^{(e)}$ [*cf.* Eq. (68)]. Further, from Eq. (84) it follows that the predicted correction to the overwhelming unperturbed oscillations $\nu_\mu \to \nu_\tau$ [*cf.* Eq. (58)] is larger in magnitude than the oscillations $\nu_e \to \nu_\mu$, and also larger than the oscillations $\nu_\mu \to \nu_\tau$ obtained in the case of our first option [*cf.* Eq. (45)]. Again, the correction caused by $\delta\widehat{V}^{(\nu)} = \left(\delta\widehat{U}^{(\nu)}\right)^\dagger$ is comparatively negligible or even vanishes [*cf.* Eq. (70)].

Thus, the atmospheric neutrino experiments, if interpreted in terms of our "texture dynamics", seem to transmit an important message about strong mixing of ν_μ and ν_τ neutrinos and, on the other hand, their weak mixing with ν_e. However, such a strong mixing cannot be really maximal as then the degeneration $m^2_{\nu_\mu} = m^2_{\nu_\tau}$ appears, excluding the experimentally suggested large oscillations $\nu_\mu \to \nu_\tau$. *A priori*, some small oscillations $\nu_e \to \nu_\mu$ (in the vacuum) may be caused by both factor matrices $\widehat{U}^{(\nu)\dagger}$ and $\widehat{U}^{(e)}$ in the lepton Cabibbo–Kobayashi–Maskawa matrix $\widehat{V}$. In this Section of our paper we conjectured that $\widehat{U}^{(e)}$ is practically responsible for such small oscillations (in the vacuum).

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