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BONDI MASS IN CLASSICAL FIELD THEORY

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We discuss three classical field theories based on the wave equation: scalar field, electrodynamics and linearized gravity. Certain generating formula on a hyperboloid and on a null surface are derived for them. The linearized Einstein equations are analyzed around the null infinity. It is shown how the dynamics can be reduced to gauge invariant quanitities in a quasi-local way. The quasi-local gauge-invariant "density" of the Hamiltonian is derived on the hyperboloid and on the future null infinity $\mathcal{J}^+$. The result gives a new interpretation of the Bondi mass loss formula. We show also how to define the angular momentum. Starting from an affine approach for Einstein equations we obtain variational formulae for Bondi-Sachs type metrics related to energy and angular momentum generators. The original van der Burg asymptotic hierarchy is revisited and the relations between linearized and asymptotic nonlinear situations are established. We discuss also supertranslations, Newman-Penrose charges and Janis solutions.

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1. Introduction

In the papers [2–4], from the series "Gravitational waves in general relativity" Bondi, van der Burg, Metzner and Sachs have analyzed asymptotic behaviour of the gravitational field at null infinity. The energy in this regime, so called Bondi mass, was defined and the main property — loss of the energy — was proved. See also discussion on p. 127 in the last paper [10] in this series. The energy at null infinity was also proposed by Trautman [1] and it will be called the Trautman-Bondi energy (or TB energy).

We interprete their result from symplectic point of view and we show that the concept of Trautman-Bondi energy arises not only in gravity but can be also defined for other fields. In this case the TB energy can be treated formally as a "Hamiltonian" and the loss of energy formula has a natural interpretation given by (2.21). We apply similar technique to define the angular momentum.

We introduce here the language of *generating functions* which simplifies enormously our calculations. This point of view on dynamics is due to Tulczyjew (see [29]).

We start from an example of a scalar field for which we define TB energy as a "Hamiltonian" on a hyperboloid. The motivation for considering hyperboloids in gravitation one can find in [12, 15] and [16].

In Section 3 we give an example from electrodynamics.

Next we prove analogous formulae for the linearized gravity. The result is formulated in a nice gauge-independent way. We show how the formula (2.21) can be related to the original Bondi-Sachs result – mass loss equation (35) of [2] (cf. also equations (4.16) in [3], (13) in [4] and (3.8) in [10]). Our result is an important gauge-independent generalization of this original mass loss equation. It shows the straightforward relation between the Weyl tensor on the scri and the flux of the radiation energy through it. We show how to define the angular momentum from this point of view.

In Section 8 we give "spherically covariant" formulation of the asymptotic equations from [4]. We discuss several features of the theory like supertranslations, charges etc. and also the relations between linear and nonlinear theory.

1.1. New results and propositions

We give a list of problems and results which seem to be important and are discussed in this paper.

- Hamiltonian formula on a hyperboloid and at the future null infinity for the scalar field
- application of the proposed method in electrodynamics
- natural outcome in linearized gravity, non-conservation laws, invariants
- analysis of the symplectic structure proposed by Kijowski in Bondi-Sachs coordinates, non-conservation law for the energy at null infinity gives Bondi formula, symplectic structure on scri gives the result proposed by Ashtekar *et al.*
- application of the method for angular momentum, Hamiltonian formula and non-conservation law for it
- covariant formulation on a sphere of the Bondi–van der Burg–Metzner–Sachs asymptotic hierarchy
- transformation laws with respect to the supertranslations in general case without axial symmetry, hypothesis for the angular momentum and static moment

- TB Four-momentum for any (cross-)section of the future null infinity in terms of the BvBMS asymptotics
- simple relation between asymptotics on scri in full nonlinear theory and linearized gravity
- simple interpretation of the Newman–Penrose constant and their analogy in the linearized theory.

2. Scalar field

Consider a scalar field theory derived from the density of a Lagrangian $L = L(\varphi, \varphi_\mu)$, where $\varphi_\mu := \partial_\mu \varphi$. The entire information about field dynamics may be encoded in the equation

$$\delta L(\varphi, \varphi_\mu) = \partial_\mu (p^\mu \delta\varphi) = (\partial_\mu p^\mu)\delta\varphi + p^\mu \delta\varphi_\mu \; . \tag{2.1}$$

The above generating formula is equivalent to the system of equations

$$\partial_\mu p^\mu = \frac{\partial L}{\partial \varphi} \, , \tag{2.2}$$

$$p^\mu = \frac{\partial L}{\partial \varphi_\mu} \, . \tag{2.3}$$

It is obvious that the system of equations (2.2)–(2.3) is equivalent to the Euler-Lagrange equations in the usual form

$$\partial_\mu \frac{\partial L}{\partial \varphi_\mu} = \frac{\partial L}{\partial \varphi} \, .$$

Hamiltonian description of the theory is based on a chronological analysis, *i.e.* on a (3+1)-foliation of space-time. Treating separately the time derivative and the space derivatives, we rewrite (2.1) as

$$\delta L = (p\delta\varphi)^{\cdot} + \partial_k (p^k \delta\varphi) \; , \tag{2.4}$$

where we denoted $p := p^0$. Integrating over a 3-dimensional space–volume V we obtain

$$\delta \int_V L \; = \; \int_V (\dot{p}\delta\varphi + p\delta\dot{\varphi}) + \int_{\partial V} p^\perp \delta\varphi = \int_V (\dot{p}\delta\varphi - \dot{\varphi}\delta p + \delta(p\dot{\varphi})) + \int_{\partial V} p^\perp \delta\varphi \, , \tag{2.5}$$

where by $p^\perp$ we denote the normal part of the momentum p^k. Hence, the Legendre transformation between p and $\dot{\varphi}$ gives us

$$-\delta \int_V H(\varphi, p) = \int_V (\dot{p}\delta\varphi - \dot{\varphi}\delta p) + \int_{\partial V} p^\perp \delta\varphi \ , \tag{2.6}$$

where the density of the Hamiltonian is

$$H := p\dot{\varphi} - L \tag{2.7}$$

and the Hamiltonian we denote by $\mathcal{H} := \int_V H$. Equation (2.6) is equivalent to the Hamilton equations

$$\dot{p} = -\frac{\delta \mathcal{H}}{\delta \varphi} ; \qquad \dot{\varphi} = \frac{\delta \mathcal{H}}{\delta p} , \tag{2.8}$$

provided no boundary terms remain when the integration by parts is performed. To get rid of these boundary terms we restrict ourselves to an infinitely dimensional functional space of initial data (φ, p), which are defined on V and fulfill the Dirichlet boundary conditions $\varphi|_{\partial V} \equiv f$ on its boundary. Imposing these conditions, we kill the boundary integral in (2.6), because $\delta\varphi \equiv 0$ within the space of fields fulfilling boundary conditions. In this way the formula (2.6) becomes an infinitely dimensional Hamiltonian formula. Without any boundary conditions, the field dynamics in V *can not* be formulated in terms of any Hamiltonian system, because the evolution of initial data in V may be influenced by the field outside of V.

Physically, a choice of boundary conditions corresponds to an insulation of a physical system composed of a portion of the field contained in V. The choice of Dirichlet conditions is not unique. Performing *e.g.* the Legendre transformation between φ and $p^\perp$ in the boundary term of (2.6), we obtain

$$\int_{\partial V} p^\perp \delta\varphi = \delta \int_{\partial V} p^\perp \varphi - \int_{\partial V} \varphi \delta p^\perp \ . \tag{2.9}$$

Hence, we have

$$-\delta\overline{\mathcal{H}} = \int_V (\dot{p}\delta\varphi - \dot{\varphi}\delta p) - \int_{\partial V} \varphi\delta p^\perp \ . \tag{2.10}$$

The new Hamiltonian

$$\overline{\mathcal{H}} = \mathcal{H} + \int_{\partial V} p^\perp \varphi \tag{2.11}$$

generates formally the same partial differential equations governing the dynamics, but the evolution takes place in a *different phase space*. Indeed, to derive the Hamiltonian equations (2.8) from (2.10) we have now to kill $\delta p^\perp$ at the boundary. For this purpose we have to impose the Neumann boundary condition $p^\perp|_{\partial V} = \tilde{f}$. The space of fields fulfilling this condition becomes now our infinite dimensional phase space, different from the previous one.

The difference between the above two dynamical systems is similar to the difference between the evolution of a thermodynamical system in two different regimes: in an adiabatic insulation and in a thermal bath (see [22]). As another example we may consider the dynamics of an elastic body: the Dirichlet conditions mean controlling exactly the position of its surface, whereas the Neumann conditions mean controlling only the forces applied to the surface. We see that the same field dynamics may lead to different Hamiltonian systems according to the way we control the boundary behaviour of the field. Without imposing boundary conditions the field dynamics *can not* be formulated in terms of a Hamiltonian system.

2.1. Coordinates in Minkowski space

We shall consider the flat Minkowski metric of the following form in spherical coordinates

$$\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)\,.$$

The Minkowski space M has a natural structure of spherical foliation around null infinity, more precisely, the neighbourhood of $\mathscr{J}^+$ looks like $S^2 \times M_2$. We shall use several coordinates on M_2: $s, t, r, \rho, \omega, v, u$. They are defined as follows

$$\begin{aligned} r &= \sinh\omega = \rho^{-1}\,, \\ t &= s + \cosh\omega = s + \rho^{-1}\sqrt{1+\rho^2}\,, \\ u &= t - r = s + \frac{\rho}{1+\sqrt{1+\rho^2}}\,, \\ v &= t + r = s + \rho^{-1}(\sqrt{1+\rho^2}+1)\,. \end{aligned}$$

The hypersurfaces s =const., u =const. and v =const. correspond to the lines in the space M_2. The two pictures below show them schematically.

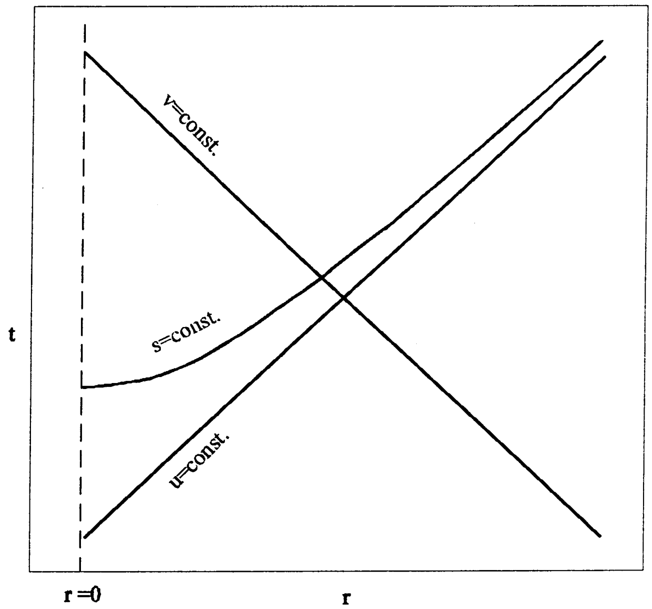

Fig. 1. A piece of M_2 close to the center in usual coordinates: radial r and temporal t.

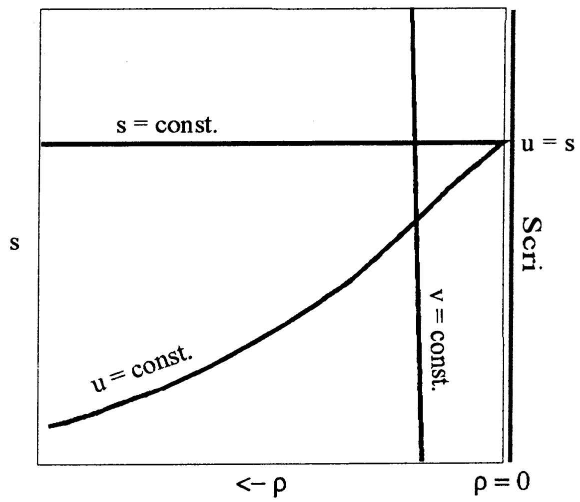

Fig. 2. A piece of M_2 close to null infinity in coordinates ρ, s.

2.2. Scalar field on a hyperboloid

We shall consider a scalar field φ in a flat Minkowski space M with the metric

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \rho^{-2}\left(-\rho^2\mathrm{d}s^2 + \frac{2\mathrm{d}s\mathrm{d}\rho}{\sqrt{1+\rho^2}} + \frac{\mathrm{d}\rho^2}{1+\rho^2} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right). \tag{2.12}$$

Let us fix a coordinate chart (x^μ) on M such that $x^1 = \theta$, $x^2 = \phi$ (spherical angles), $x^3 = \rho$ and $x^0 = s$, and let us denote by $\overset{\circ}{\gamma}_{AB}$ a metric on a unit sphere ($\overset{\circ}{\gamma}_{AB}\mathrm{d}x^A\mathrm{d}x^B := \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$).

We shall consider an initial value problem on a hyperboloid Σ

$$\Sigma_s := \left\{x \in M \mid x^0 = s = \text{const.}\right\}$$

for our scalar field φ with a density of the Lagrangian (corresponding to the wave equation)

$$\begin{aligned} L :&= -\frac{1}{2}\sqrt{-\det \eta_{\mu\nu}}\eta^{\mu\nu}\varphi_\mu\varphi_\nu = -\frac{1}{2}\rho^{-2}\sin\theta \\ &\times \left[\rho^2(\varphi_3)^2 - \frac{(\varphi_0)^2}{1+\rho^2} + \frac{2\varphi_3\varphi_0}{\sqrt{1+\rho^2}} + \overset{\circ}{\gamma}{}^{AB}\varphi_A\varphi_B\right]. \end{aligned}$$

We use the following convention for indices: Greek indices $\mu, \nu, \ldots$ run from 0 to 3; $k, l, \ldots$ are coordinates on a hyperboloid Σ_s and run from 1 to 3; $A, B, \ldots$ are coordinates on $S(s,\rho)$ and run from 1 to 2, where $S(s,\rho) := \left\{x \in \Sigma_s \mid x^3 = \rho = \text{const.}\right\}$.

The generating formula (2.1) can be written for any $V \subset \Sigma$

$$\delta \int\limits_V L = \int\limits_V (p^0\delta\varphi)_{,0} + \int\limits_{\partial V} p^3\delta\varphi$$

and in particular the definition (2.2) of the canonical momenta p^μ gives the time and radial components of it

$$p^0 = \frac{\partial L}{\partial \varphi_0} = \rho^{-2}\sin\theta\left(\frac{\varphi_0}{1+\rho^2} - \frac{\varphi_3}{\sqrt{1+\rho^2}}\right),$$

$$p^3 = -\frac{\partial L}{\partial \varphi_3} = \rho^{-2}\sin\theta\left(\frac{1}{\sqrt{1+\rho^2}}\varphi_0 + \rho^2\varphi_3\right).$$

Let us observe that in general the integral $\int\limits_V L$ is not convergent on Σ if we assume that $\varphi = O(\rho)$ and $\varphi_3 = O\left(1\right)$. The same problem with "infinities" we meet in p^0 and p^3. We can "renormalize" L adding a full divergence

$$\begin{aligned} \overline{L}: &= -\frac{1}{2}\sin\theta\left[\rho^2(\psi_3)^2 - \frac{1}{1+\rho^2}(\psi_0)^2 + \frac{2}{\sqrt{1+\rho^2}}\psi_3\psi_0 + \overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B\right] \\ &= L + \frac{1}{2}\partial_0\left(\sin\theta\frac{\rho^{-3}}{\sqrt{1+\rho^2}}\varphi^2\right) - \frac{1}{2}\partial_3\left(\sin\theta\rho^{-1}\varphi^2\right), \end{aligned} \tag{2.13}$$

where we have introduced a new field variable $\psi := \rho^{-1}\varphi$ which is natural close to the null infinity. The generating formula takes the following form with respect to the new variable ψ

$$\delta\int\limits_V \overline{L} = \int\limits_V \left(\pi^0\delta\psi\right)_{,0} + \int\limits_{\partial V}\pi^3\delta\psi$$

and the Euler-Lagrange equations (2.2)–(2.3) we write explicitly

$$\begin{aligned} \pi^0 &= \frac{\partial\overline{L}}{\partial\psi_0} = \sin\theta\left(\frac{\psi_0}{1+\rho^2} - \frac{\psi_3}{\sqrt{1+\rho^2}}\right), \\ \pi^3 &= \frac{\partial\overline{L}}{\partial\psi_3} = -\sin\theta\left(\frac{1}{\sqrt{1+\rho^2}}\psi_0 + \rho^2\psi_3\right), \\ \pi^A &= \frac{\partial\overline{L}}{\partial\psi_A} = -\sin\theta\,\overset{\circ}{\gamma}{}^{AB}\psi_B\,, \\ \partial_\mu\pi^\mu &= \frac{\partial\overline{L}}{\partial\psi} = 0\,. \end{aligned}$$

It is easy to check that all terms are finite at null infinity, provided $\psi = O(1)$ and $\psi_3 = O(1)$.

From the above equations one can easily obtain the wave equation

$$\bar{\Box}\,\psi = 0\,, \tag{2.14}$$

where the wave operator $\bar{\Box}$ is defined with respect to the metric

$$\overline{\eta}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu := -\rho^2\mathrm{d}s^2 + \frac{2\mathrm{d}s\mathrm{d}\rho}{\sqrt{1+\rho^2}} + \frac{\mathrm{d}\rho^2}{1+\rho^2} + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2 = \rho^2\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{2.15}$$

which is conformally related to the original flat metric $\eta_{\mu\nu}$.

Remark. Let us observe that

$$\overline{L} = -\frac{1}{2}\sqrt{-\det \overline{\eta}_{\mu\nu}}\overline{\eta}^{\mu\nu}\psi_\mu\psi_\nu = L + \frac{1}{2}\left(\sqrt{-\det \overline{\eta}_{\mu\nu}}(\ln\rho)_{,\nu}\overline{\eta}^{\mu\nu}\psi^2\right)_{,\mu}$$

so we are not surprised that (2.14) holds. It can be easily checked that the equation (2.14) is equivalent to the original wave equation

$$\Box\varphi = 0 \tag{2.16}$$

by the usual conformal transformation for the conformally invariant operator $\bar{\Box} + \frac{1}{6}\overline{R}$ because the scalar curvature $\overline{R}$ of the metric $\overline{\eta}_{\mu\nu}$ vanishes.

From the Legendre transformation between π^0 and ψ_0 we can define the "Hamiltonian" density

$$H := \frac{1}{2}\sin\theta\left[(\rho\psi_{,3})^2 + \frac{1}{1+\rho^2}(\psi_0)^2 + \overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B\right] = \pi^0\psi_0 - \overline{L}$$

and the following variational relation holds

$$-\delta\int_V H = \int_V\left(\dot{\pi}\delta\psi - \dot{\psi}\delta\pi\right) + \int_{\partial V}\pi^3\delta\psi\ , \tag{2.17}$$

where here $\pi := \pi^0$.

Remark. The relation between (2.17) coming from $\overline{L}$ and (2.6) with respect to L gives the same result for the numerical value of the Hamiltonian $\mathcal{H} := \int_\Sigma H$ and this can be easily seen from the following observations

$$\begin{aligned} \dot{\pi}\delta\psi - \dot{\psi}\delta\pi &= \dot{p}^0\delta\varphi - \dot{\varphi}\delta p^0\ , \\ \pi^3\delta\psi - p^3\delta\varphi &= \frac{1}{2}\delta\sin\theta\rho\psi^2\ . \end{aligned}$$

So the formulae give the same Hamiltonian because $\rho\psi^2$ vanishes on $\mathcal{J}^+$.

Unfortunately, if we integrate the relation (2.17) over hyperboloid Σ, we quickly realize that the boundary term

$$\int_{\partial\Sigma_s}\pi^3\delta\psi = \int_{S(s,0)}\sin\theta\dot{\psi}\delta\psi$$

does not vanish for the usual asymptotics of the field ψ. If we want to have a closed Hamiltonian system, we have to assume that $\dot{\psi}|_{\partial\Sigma} = 0$ and then the energy will be conserved in time. But we would like to describe the situation with any data $\psi|_{\mathcal{J}^+}$. In this case we can define the Trautman-Bondi energy, but it would be no longer conserved, formally we can treat it as a "Hamiltonian" of the opened Hamiltonian system and the formula (2.17) is useful as a definition of the Trautman-Bondi energy together with its changes in time. In our case the boundary condition f depends on time (see disscussion after formula (2.8)) and an interesting case for us is to compare the data with different boundary conditions. Although the energy defined on a hyperboloid is not a Hamiltonian in a usual sense, it plays an important role for the description of the radiation at null infinity. The method is useful for the construction of the other generators of the Poincaré group and will be applied for the angular momentum.

We should express our Hamiltonian as a functional of (π, ψ)

$$H := \frac{1}{2} \sin\theta \left[(\rho\psi_3)^2 + \left(\frac{\pi\sqrt{1+\rho^2}}{\sin\theta} + \psi_3 \right)^2 + \overset{\circ}{\gamma}{}^{AB} \psi_A \psi_B \right] \tag{2.18}$$

and the Hamilton equations (2.8) are the following

$$\dot{\psi} = \frac{\pi}{\sin\theta}(1+\rho^2) + \psi_3\sqrt{1+\rho^2}\,, \tag{2.19}$$

$$\dot{\pi} = (\pi\sqrt{1+\rho^2})_{,3} + \left[(1+\rho^2)\sin\theta\psi_3\right]_{,3} + (\sin\theta\, \overset{\circ}{\gamma}{}^{AB}\psi_B)_{,A}\,, \tag{2.20}$$

and they correspond to the wave equation (2.14).

The variational formula (2.17) describes an open Hamiltonian system because in our case there is no possibility to kill the boundary term. Our "Hamiltonian" is not conserved in time

$$-\partial_0 \left(\int_{\Sigma} H \right) = \int_{\partial\Sigma} \pi^3 \dot{\psi} = \int_{S(s,0)} \sin\theta (\dot{\psi})^2 \tag{2.21}$$

(we remind that $\partial\Sigma = S(s,0)$ is odd oriented). Formally, the result (2.21) can be obtained from (2.17) if we replace variation δ with ∂_0 but it can be also checked by a direct computation using equations (2.19) and (2.20) together with the definition (2.18) of the density H.

Nevertheless this formal calculation is very useful. For example, we can easily define the angular momentum along the z-axis as a generator for the vector field $\frac{\partial}{\partial\phi}$

$$\int_{\Sigma} (\pi_{,\phi}\delta\psi - \psi_{,\phi}\delta\pi) = -\delta \int_{\Sigma} \pi\psi_{,\phi} = -\delta J_z\,.$$

Using equations of motion, we can check that the angular momentum is not conserved in time

$$-\partial_0 J_z = -\partial_0 \left(\int\limits_{\Sigma} \pi \psi_{,\phi} \right) = \int\limits_{\partial\Sigma} \pi^3 \psi_{,\phi} = \int\limits_{S(s,0)} \sin\theta \dot{\psi} \psi_{,\phi} \,. \tag{2.22}$$

We will show in the sequel that the formulae (2.21) and (2.22) can be written for the linearized gravity and have the interpretation of the TB mass loss formula and angular momentum loss equation.

Let us formulate the following theorem:

Theorem. If the TB mass is conserved then the angular momentum is conserved too.

This means that it is impossible to radiate away the angular momentum without a loss of mass. The proof is a simple consequence of (2.21) and (2.22). If the TB mass is conserved then (from (2.21)) $\dot{\psi}$ has to vanish on $\mathcal{J}$ and from (2.22) we get that the angular momentum is conserved.

We shall see in the sequel that this theorem also holds for Bondi-Sachs type metrics describing asymptotically flat solutions at null infinity for the full (nonlinear) Einstein equations.

2.3. Scalar field on a null cone

We shall consider an initial value problem on a null surface N defined as follows

$$N := \left\{ x \in M \mid v = s + \rho^{-1}(1 + \sqrt{1+\rho^2}) = \text{const.} \right\} , \tag{2.23}$$

where we have introduced a null coordinate $v := s + \rho^{-1}(1 + \sqrt{1+\rho^2})$ which plays the role of time in our analysis. Formally, $\mathcal{J}^+$ corresponds to the surface $\rho = 0$. Let us rewrite the Minkowski metric (2.12) using new coordinates $v, \overline{u}$ instead of s, ρ

$$\eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \rho^{-2} \left(-\rho^2 \mathrm{d}v^2 - \rho^2 \mathrm{d}v \mathrm{d}\overline{u} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2 \right) .$$

The relation between coordinates $(v, \overline{u})$ and (x^0, x^3) used in the previous subsection is the following

$$\begin{aligned} v &= x^0 + \rho^{-1}(1 + \sqrt{1+\rho^2}) \,, \quad \overline{u} = -2\rho^{-1} \,, \quad \rho = x^3 \,, \\ \partial_0 &= \partial_v \,, \quad \partial_3 = 2\rho^{-2}\partial_{\overline{u}} - \rho^{-2} \left(1 + \frac{1}{\sqrt{1+\rho^2}} \right) \partial_v \,, \\ \mathrm{d}x^0 &= \mathrm{d}v + \frac{1}{2} \left(1 + \frac{1}{\sqrt{1+\rho^2}} \right) \mathrm{d}\overline{u} \,, \quad \mathrm{d}x^3 = \frac{1}{2}\rho^2 \mathrm{d}\overline{u} \,. \end{aligned}$$

The density of the Lagrangian takes the form

$$\overline{L} := -\frac{1}{2}\sqrt{-\det\overline{\eta}_{\mu\nu}}\,\overline{\eta}^{\mu\nu}\psi_\mu\psi_\nu = \sin\theta\left[\psi_{\overline{u}}\psi_v - \psi_{\overline{u}}^2 - \frac{1}{4}\rho^2\,\overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B\right] .$$

The formula (2.1) on the null surface N can be written as follows

$$\delta\int_N \overline{L} = \int_N (\pi^v\delta\psi)_{,v} + \int_{\partial N}\pi^{\overline{u}}\delta\psi$$

and the corresponding components of the canonical momenta are

$$\pi^v = \frac{\partial L}{\partial\psi_v} = \sin\theta\psi_{\overline{u}}\,,$$

$$\pi^{\overline{u}} = \frac{\partial L}{\partial\psi_{\overline{u}}} = \sin\theta\left(\psi_v - 2\psi_{\overline{u}}\right) = -\sin\theta\left(\rho^2\psi_3 + \frac{\psi_0}{\sqrt{1+\rho^2}}\right) .$$

If we perform Legendre transformation, we obtain the density of the Hamiltonian on the cone N

$$H = \pi^v\psi_v - \overline{L} = \sin\theta\left[(\psi_{\overline{u}})^2 + \frac{\rho^2}{4}\,\overset{\circ}{\gamma}{}^{AB}\psi_A\psi_B\right] .$$

Let us observe that on the limiting surface $\mathcal{J}^+$ (parallel to N) we get

$$\lim_{\rho\to 0^+} H = \sin\theta(\psi_{\overline{u}})^2 = \sin\theta\dot{\psi}^2 \tag{2.24}$$

and the Hamiltonian $\int\limits_{\mathcal{J}^+} H$ describes the total flux of energy through $\mathcal{J}^+$. Moreover, the symplectic structure on N has also a natural limit on scri[1]

$$\int_N \pi\delta\psi = \int_N \sin\theta\psi_{\overline{u}}\delta\psi \overset{\rho\to 0^+}{\longrightarrow} \int_{\mathcal{J}^+}\sin\theta\dot{\psi}\delta\psi\,\mathrm{d}\overline{u}\mathrm{d}\theta\mathrm{d}\phi\,. \tag{2.25}$$

We will show in the sequel that the above formulae exist in electrodynamics and linearized gravity. Similarly, the equation

$$\int_N \pi\psi_{,\phi} = \int_N \sin\theta\psi_{\overline{u}}\psi_{,\phi} \overset{\rho\to 0^+}{\longrightarrow} \int_{\mathcal{J}^+}\sin\theta\dot{\psi}\psi_{,\phi}\mathrm{d}\overline{u}\mathrm{d}\theta\mathrm{d}\phi \tag{2.26}$$

describes the flux of angular momentum through $\mathcal{J}^+$.

[1] These observations has been applied by Ashtekar *et al.* [31–33] for the description of the space of radiative modes in exact relativity, see also equation (7.17) in this article.

2.4. ADM mass

We have tried to treat separately the hyperboloid and scri and we have learned that there is no possibility to get a nice Hamiltonian system. Let us denote by N (only in this subsection) a "piece" of $\mathscr{J}^+$ between Σ and spatial i^0. If we take the surface $\Sigma \cup N$ together

$$-\delta\left(\int\limits_{\Sigma} H+\int\limits_{N} H\right)=\int\limits_{\Sigma}\left(\dot{\pi}\delta\psi-\dot{\psi}\delta\pi\right)+\int\limits_{N}\left(\dot{\pi}\delta\psi-\dot{\psi}\delta\pi\right)+\int\limits_{\partial\Sigma}\pi^3\delta\psi+\int\limits_{\partial N}\pi^{\overline{u}}\delta\psi \tag{2.27}$$

we will obtain a Hamiltonian system provided we can kill the boundary term. This can be achieved, assuming for example that

$$\lim_{u\to-\infty}\dot{\psi}=0$$

which simply means that $\dot{\psi}$ is vanishing at spatial infinity. This usually happens for initial data on Cauchy surface t =const with compact support or vanishing sufficiently fast at spatial infinity. The following relations confirm our theorem

$$\begin{aligned}\pi^3\big|_{\partial\Sigma} &= -\sin\theta\dot{\psi}=\pi^{\overline{u}}\,,\\ \partial\Sigma &= S(s,0),\quad \partial N=S(s,0)\cup S(-\infty,0)\,,\\ -\delta m_{\mathrm{ADM}} &= \int\limits_{\Sigma\cup N}\left(\dot{\pi}\delta\psi-\dot{\psi}\delta\pi\right)+\int\limits_{\partial(\Sigma\cup N)}\sin\theta\dot{\psi}\delta\psi\,,\end{aligned} \tag{2.28}$$

where $m_{\mathrm{ADM}} := \int\limits_{\Sigma\cup N} H$. Let us note that here $N=\bigcup\limits_{u\le s} S(u,0)$ but we can also consider $N=\bigcup\limits_{u\in[s,s_0]} S(u,0)$ and then (2.28) leads to the Bondi mass on $\sum_{s_0}$ as a Hamiltonian [34].

2.4.1. One-parameter family of Hamiltonian systems and their limit

$$\begin{aligned}\Sigma_{\tau,\varepsilon} &:= \{s=\tau,\ \rho\ge\varepsilon\}\ ,\\ N_{\tau,\varepsilon} &:= \left\{v=\tau+\frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon},\ \frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}}\le\rho\le\varepsilon\right\}\ ,\\ I_{\tau,\varepsilon} &:= \left\{t=\tau,\ 0<\rho\le\frac{\varepsilon}{1+\sqrt{1+\varepsilon^2}}\right\}\ ,\\ \lim_{\varepsilon\to0^+}\Sigma_{\tau,\varepsilon} &= \Sigma_\tau\ ,\end{aligned}$$

$$
\begin{aligned}
\lim_{\varepsilon\to 0^+} N_{\tau,\varepsilon} &= N_\tau \subset \mathcal{J}^+ \,, \\
\lim_{\varepsilon\to 0^+} I_{\tau,\varepsilon} &= i^0 \,, \\
N_{\tau,\varepsilon} &= \left\{ v = \tau + \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon},\ \frac{-2\sqrt{1+\varepsilon^2}}{\varepsilon} + \tau \le u \le \tau \right\} , \\
I_{\tau,\varepsilon} &= \left\{ t = \tau,\ r \ge \frac{1+\sqrt{1+\varepsilon^2}}{\varepsilon} \right\} .
\end{aligned}
$$

$\Sigma_{\tau,\varepsilon} \cup N_{\tau,\varepsilon} \cup I_{\tau,\varepsilon}$ is an explicit example of a one-parameter family of surfaces (with respect to τ) and the Hamiltonian related to this family is an ADM mass. On the other hand, the Hamiltonian system (2.27) is a limit of these systems with respect to the second parameter ε ($\varepsilon \to 0^+$). In this way we have certain "finite" procedure for the Hamiltonian system (2.27) at infinity.

2.5. Energy–momentum tensor

Let us consider the energy–momentum tensor for the scalar field φ

$$T^\mu{}_\nu = \frac{1}{\sqrt{-\eta}} \left(p^\mu \varphi_\nu - \delta^\mu{}_\nu L \right) ,$$

where $\eta := \det \eta_{\mu\nu}$ and by $\delta^\mu{}_\nu$ we have denoted Kronecker's delta. For the Lagrangian L desribing scalar field φ the canonical energy momentum is symmetric. From the Noether theorem we have

$$\partial_\mu \left(\sqrt{-\eta} T^\mu{}_\nu X^\nu \right) = 0$$

for a Killing vector field X^μ and integrating the above formula we obtain

$$\partial_0 \int_\Sigma \sqrt{-\eta} T^0{}_\nu X^\nu = - \int_{\partial\Sigma} \sqrt{-\eta} T^3{}_\nu X^\nu \,. \tag{2.29}$$

Usually, when Σ is a spacelike surface with the end at spatial infinity, the boundary term on the right-hand side vanishes and the equation (2.29) expresses conservation law for the appriopriate generator related to the vector field X. On the contrary, for the hyperboloid the right-hand side does not vanish and (2.29) expresses *non-conservation* law. It can be easily verified that for the energy and angular momentum we have respectively

$$\int_\Sigma \sqrt{-\eta} T^0{}_0 = \int_\Sigma H \,,$$

$$\int_{\Sigma} \sqrt{-\eta} T^0{}_{\phi} = \int_{\Sigma} \pi \psi_{,\phi} \,.$$

The boundary terms arising in (2.29) for the energy can be expressed in terms of energy–momentum tensor

$$-\int_{\partial\Sigma} T^3{}_0 \rho^{-4} \sin\theta \mathrm{d}\theta \mathrm{d}\phi = -\frac{1}{2}\int_{\partial\Sigma} T^{\overline{u}}{}_v \rho^{-2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \quad \left(= \int_{\partial\Sigma\subset\mathcal{J}^+} \dot{\psi}^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \right) ,$$

$$\int_N T^v{}_v \rho^{-4} \mathrm{d}\rho \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \int_N \mathrm{d}\overline{u} \left(\frac{1}{2} T^v{}_v \rho^{-2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \right)$$

and

$$\frac{1}{2}\rho^{-2} T^v{}_v \Big|_{\mathcal{J}^+} = \dot{\psi}^2 \,.$$

Similarly for angular momentum

$$-\int_{\partial\Sigma} T^3{}_{\phi} \rho^{-4} \sin\theta \mathrm{d}\theta \mathrm{d}\phi = -\frac{1}{2}\int_{\partial\Sigma} T^{\overline{u}}{}_{\phi} \rho^{-2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \quad \left(= \int_{\partial\Sigma\subset\mathcal{J}^+} \dot{\psi}\psi_{,\phi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \right) ,$$

$$\int_N T^v{}_{\phi} \rho^{-4} \mathrm{d}\rho \sin\theta \mathrm{d}\theta \mathrm{d}\phi = \int_N \mathrm{d}\overline{u} \left(\frac{1}{2} T^v{}_{\phi} \rho^{-2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \right)$$

and

$$\frac{1}{2}\rho^{-2} T^v{}_{\phi} \Big|_{\mathcal{J}^+} = \dot{\psi}\psi_{,\phi} \,.$$

It is easy to verify that the result is compatible with (2.21)–(2.22) and (2.24)–(2.26).

This calculation shows that *quasi-local* density $\int_{S^2} \dot{\psi}^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ of the energy on $\mathcal{J}^+$ has two different interpretations. It is a boundary term which describes non-conservation of the "Hamiltonian" on a hyperboloid Σ or a density of a "Hamiltonian" on $\mathcal{J}^+$. More precisely, it is a density with respect to the parameter $\overline{u}$ but integrated over a sphere. This is an example of an object which is local on M_2 but non-local on S^2. We call such objects *quasi-local*. It will be shown in the sequel that this concept of *quasi-locality* is useful in electrodynamics and gravitation.

The equations (2.21) and (2.22) are examples of the general formula which has the following form for any Killing vector field $X = X^{\mu}\partial_{\mu}$

$$\partial_0 \left(\int_{\Sigma} \sqrt{-\eta} T^0{}_{\nu} X^{\nu} \right) = -\int_{\partial\Sigma} \pi^3 X^A \psi_A + \pi^3 X^0 \dot{\psi} \,. \qquad (2.30)$$

Equation (2.21) corresponds to the vector field $X_H := \partial_0$ and (2.22) to the $X_J := \partial_\phi$.

Remark. One can check by a direct computation that $X^3|_{\mathcal{J}^+}=0$, which simply means that the (Poincaré group related) Killing field X is tangent to the $\mathcal{J}^+$.

The vector field corresponding to the linear momentum in z direction

$$X_P := -\frac{\cos\theta}{\sqrt{1+\rho^2}}\partial_0 - \rho^2\cos\theta\partial_3 - \rho\sin\theta\partial_\theta\ , \quad X_P|_{\mathcal{J}^+} = -\cos\theta\partial_0$$

gives the loss formula

$$-\partial_0 P_z = \int_{\partial\Sigma} \pi^3 X_P^0 \dot{\psi} = -\int_{S(s,0)} \sin\theta\cos\theta(\dot{\psi})^2\,, \tag{2.31}$$

where $P_z := \int_\Sigma \sqrt{-\eta}T^0_\mu X^\mu_P$.

Similarly, we can take a boost generator along z-axis

$$X_K := -\rho\sqrt{1+\rho^2}\cos\theta\partial_3 - \sqrt{1+\rho^2}\sin\theta\partial_\theta + sX_P\ , \quad X_K|_{\mathcal{J}^+} = sX_P|_{\mathcal{J}^+} + \hat{\partial}_\phi,$$

where $\hat{\partial}_A := \varepsilon_A{}^B\partial_B$, and the formula (2.30) takes the form

$$-\partial_0 K_z = \int_{\partial\Sigma} \pi^3 X_K^0\dot{\psi} + \pi^3 X_K^A\psi_A = -s\partial_0 P_z - \int_{S(s,0)} \sin^2\theta\dot{\psi}\psi_\theta \tag{2.32}$$

for $K_z := \int_\Sigma T^0_\mu X^\mu_K$ or

$$-\partial_0 K_z + s\partial_0 P_z = \int_{S(s,0)} \sin\theta\dot{\psi}\hat{\partial}_\phi\psi\,.$$

Equations (2.21), (2.22), (2.32) and (2.31) express the *non-conservation* law of the Poincaré group generators defined at null infinity.

3. Electrodynamics

This section should convince the reader that the TB mass and angular momentum at null infinity can be described in classical electrodynamics in a similar way as for the scalar field in previous section.

The field equations for linear electrodynamics may be written as follows

$$\delta L = \partial_\mu(\mathcal{F}^{\nu\mu}\delta A_\nu) = \partial_\mu(\mathcal{F}^{\nu\mu})\delta A_\nu + \mathcal{F}^{\nu\mu}\delta A_{\nu\mu}\,, \tag{3.1}$$

where $A_{\nu\mu} := \partial_\mu A_\nu$ and L is the Lagrangian density of the theory. The above formula (see [25]) is a convenient way to write the Euler-Lagrange equations

$$\partial_\mu \mathcal{F}^{\nu\mu} = \frac{\delta L}{\delta A_\nu} \tag{3.2}$$

together with the relation between the electromagnetic field $f_{\mu\nu} = A_{\nu\mu} - A_{\mu\nu}$ and the electromagnetic induction density $\mathcal{F}^{\nu\mu}$ describing the momenta canonically conjugate to the potential

$$\mathcal{F}^{\nu\mu} = \frac{\delta L}{\delta A_{\nu\mu}}\,. \tag{3.3}$$

For the linear Maxwell theory the Lagrangian density is given by the standard formula

$$L = -\frac{1}{4}\sqrt{-\eta}f^{\mu\nu}f_{\mu\nu} \tag{3.4}$$

and relation (3.3) reduces in this case to $\mathcal{F}^{\mu\nu} := \sqrt{-\eta}\eta^{\mu\alpha}\eta^{\nu\beta}f_{\alpha\beta}$.

Integrating (3.1) over V we obtain

$$\begin{aligned}\delta\int_V L &= \int_V \partial_0(\mathcal{F}^{k0}\delta A_k) + \int_{\partial V}\mathcal{F}^{\nu 3}\delta A_\nu = \\ &= \int_V \partial_0(\mathcal{F}^{B0}\delta A_B + \mathcal{F}^{30}\delta A_3) + \int_{\partial V}(\mathcal{F}^{B3}\delta A_B + \mathcal{F}^{03}\delta A_0)\,.\end{aligned} \tag{3.5}$$

We assume that the charge e defined by the surface integral

$$e := \int_{S(s,\rho)} \mathcal{F}^{03} \tag{3.6}$$

vanishes. The situation with $e \neq 0$ can be described Similarly as in [24] but we are interested in "wave" degrees of freedom and we are going to show, how the volume part of (3.5) can be reduced to the gauge-invariant quantities.

Let $\overset{\circ}{\Delta} := \rho^{-2}\triangle_{S(s,\rho)}$, where $\triangle_{S(s,\rho)}$ denotes the 2-dimensional Laplace-Beltrami operator on a sphere $S(s,\rho)$. One can easily check that the operator $\overset{\circ}{\Delta}$ does not depend on ρ and is equal to the Laplace-Beltrami operator on the unit sphere $S(1)$. Operator $\overset{\circ}{\Delta}$ is invertible on the space of monopole–free functions (functions with a vanishing mean value on each $S(s,\rho)$).

Let us denote by ε^{AB} the Levi–Civita antisymmetric tensor on a sphere $S(s,\rho)$. We can rewrite (3.5), provided that the electric charge e vanishes, in the following way

$$\delta \int_V L =$$
$$\int_V \partial_0 \left[\mathcal{F}^{0B}{}_{,B} \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}A^B{}_{||B} + \mathcal{F}^{30}\delta A_3 + \mathcal{F}^{0B}{}_{,C}\,\varepsilon_B^{\ C} \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}(\varepsilon^{AB}A_{A||B})\right]$$
$$+\int_{\partial V} (\mathcal{F}^{3B}{}_{,B} \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}A^B{}_{||B} + \mathcal{F}^{03}\delta A_0 + \mathcal{F}^{3B}{}_{,C}\,\varepsilon_B^{\ C} \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}(\varepsilon^{AB}A_{A||B}))\,. \tag{3.7}$$

Here, by "$||$" we denote the 2-dimensional covariant derivative on each sphere $S(s,\rho)$. Using identities $\partial_B\mathcal{F}^{B0}+\partial_3\mathcal{F}^{30}=0$ and $\partial_B\mathcal{F}^{B3}+\partial_0\mathcal{F}^{03}=0$ implied by the Maxwell equations and integrating again by parts we finally obtain

$$\delta \int_V L =$$
$$\int_V \partial_0 \left[\mathcal{F}^{30} \stackrel{\circ}{\Delta}{}^{-1}\delta(\stackrel{\circ}{\Delta} A_3 - (\rho^{-2}A^B{}_{||B})_{,3}) + (\mathcal{F}^{0B||C}\varepsilon_{BC}) \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}(\varepsilon^{AB}A_{A||B})\right]$$
$$+\int_{\partial V} \left[\mathcal{F}^{03} \stackrel{\circ}{\Delta}{}^{-1}\delta(\stackrel{\circ}{\Delta} A_0 - \rho^{-2}A^B{}_{||B,0}) + (\mathcal{F}^{3B||C}\varepsilon_{BC}) \stackrel{\circ}{\Delta}{}^{-1}\delta\rho^{-2}(\varepsilon^{AB}A_{A||B})\right]\,. \tag{3.8}$$

The quantities $\stackrel{\circ}{\Delta} A_0 - \rho^{-2}A^B{}_{||B,0}$ and $(\stackrel{\circ}{\Delta} A_3 - (\rho^{-2}A^B{}_{||B})_{,3})$ are gauge invariant and it may be easily checked that

$$\sin\theta\left(\stackrel{\circ}{\Delta} A_0 - \rho^{-2}A^B{}_{||B,0}\right) = \rho^2\mathcal{F}^A{}_{0||A}\ (=\pi^3)$$

and

$$\sin\theta\left[\stackrel{\circ}{\Delta} A_3 - (\rho^{-2}A^B{}_{||B})_{,3}\right] = \rho^2\mathcal{F}^A{}_{3||A}\ (=\pi)\,.$$

Let us introduce the following gauge invariants

$$\begin{aligned}
\psi &:= \mathcal{F}^{30}/\sin\theta\\
\psi &:= \rho^{-2}\varepsilon^{AB}A_{B||A} = -\mathcal{F}^{30}/\sin\theta\\
\pi &:= -\rho^2\mathcal{F}_3{}^A{}_{||A}\\
*\pi &:= \mathcal{F}^{0B||C}\varepsilon_{BC} = \rho^2 *\mathcal{F}_3{}^A{}_{||A}\,,
\end{aligned}$$

where

$$*\mathcal{F}^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\lambda\sigma}\mathcal{F}_{\lambda\sigma} \, .$$

Now we will show how the vacuum Maxwell equations

$$\partial_\mu \mathcal{F}^{\mu\nu} = 0 \, , \quad \partial_\mu * \mathcal{F}^{\mu\nu} = 0$$

allow to introduce equations for gauge-invariants. The result is analogous to (2.19) and (2.20) describing scalar field

$$\dot{\psi} = \frac{\pi}{\sin\theta}(1+\rho^2) + \psi_3\sqrt{1+\rho^2} \, , \tag{3.9}$$

$$\dot{\pi} = (\pi\sqrt{1+\rho^2})_{,3} + \left[(1+\rho^2)\sin\theta\psi_3\right]_{,3} + \sin\theta \, \mathring{\Delta}\, \psi \, , \tag{3.10}$$

$$*\dot{\psi} = \frac{*\pi}{\sin\theta}(1+\rho^2) + *\psi_3\sqrt{1+\rho^2} \, , \tag{3.11}$$

$$*\dot{\pi} = (*\pi\sqrt{1+\rho^2})_{,3} + \left[(1+\rho^2)\sin\theta *\psi_3\right]_{,3} + \sin\theta \, \mathring{\Delta} *\psi \, . \tag{3.12}$$

The proof of (3.9) is based on the observations that

$$\sin\theta\psi_{,0} = \mathcal{F}^{30}{}_{,0} = -\mathcal{F}^{3A}{}_{,A} = \frac{\pi}{\sin\theta}(1+\rho^2) + \psi_3\sqrt{1+\rho^2} \, ,$$

$$-\pi = \rho^2 \mathcal{F}_3{}^A{}_{||A} = \frac{\mathcal{F}^{3A}{}_{||A}}{1+\rho^2} + \frac{\mathcal{F}^{0A}{}_{||A}}{\sqrt{1+\rho^2}}$$

and

$$\sin\theta\psi_3 = \mathcal{F}^{30}{}_{,3} = \mathcal{F}^{0A}{}_{||A} \, .$$

Similarly for (3.10) we have the following relations

$$-\left(*\mathcal{F}^{0A}{}_{||B}\varepsilon_A{}^B\right)_{,0} - \left(*\mathcal{F}^{3A}{}_{||B}\varepsilon_A{}^B\right)_{,3} + (*\mathcal{F}^{AB}{}_{||BC}\varepsilon_A{}^C)_{,0} = 0 \, ,$$

$$\begin{aligned} *\mathcal{F}^{0A} &= -\rho^2\varepsilon^{AB}\mathcal{F}_{3B} \, , \\ *\mathcal{F}^{3A} &= \rho^2\varepsilon^{AB}\mathcal{F}_{0B} \, , \\ *\mathcal{F}^{AB} &= \rho^2\varepsilon^{AB}\mathcal{F}_{03} \, , \\ \mathcal{F}_{03} &= -\rho^{-4}\mathcal{F}^{03} \, , \end{aligned}$$

which allow to get the equation

$$\left(\frac{\mathcal{F}^{3A}{}_{||A}}{1+\rho^2} + \frac{\mathcal{F}^{0A}{}_{||A}}{\sqrt{1+\rho^2}}\right)_{,0} + \left(\rho^2\mathcal{F}^{0A}{}_{||A} - \frac{\mathcal{F}^{3A}{}_{||A}}{\sqrt{1+\rho^2}}\right)_{,3} + \mathring{\Delta}\mathcal{F}^{30} = 0 \, ,$$

which is equivalent to (3.10). For $(*\psi, *\pi)$ the proof is the same, provided we apply the Hodge dual $*$ for the variables and equations:

$$(\pi, \psi) \xrightarrow{-*} (*\pi, *\psi) \xrightarrow{*} (\pi, \psi) \, .$$

Now we will show, how our variables appear in formula (3.8). Let us perform the Legendre transformation in the volume V

$$-\mathcal{F}^{03}\delta\left[A_3 - \overset{\circ}{\Delta}{}^{-1}(\rho^{-2}A^B{}_{||B})_{,3}\right] = -\delta\left[\mathcal{F}^{03}\left(A_3 - \overset{\circ}{\Delta}{}^{-1}(\rho^{-2}A^B{}_{||B})_{,3}\right)\right] + \left[A_3 - \overset{\circ}{\Delta}{}^{-1}(\rho^{-2}A^B{}_{||B})_{,3}\right]\delta\mathcal{F}^{03}$$

and on the boundary ∂V

$$\mathcal{F}^{03}\delta\left(A_0 - \overset{\circ}{\Delta}{}^{-1}\rho^{-2}A^B{}_{||B,0}\right) = \delta\left[\mathcal{F}^{03}\left(A_0 - \overset{\circ}{\Delta}{}^{-1}\rho^{-2}A^B{}_{||B,0}\right)\right] - \left(A_0 - \overset{\circ}{\Delta}{}^{-1}\rho^{-2}A^B{}_{||B,0}\right)\delta\mathcal{F}^{03}\,.$$

This way the formula (3.8) may be written as

$$\begin{aligned}\delta\int_V &\left[L - \partial_0(\psi\overset{\circ}{\Delta}{}^{-1}\pi) - \partial_3(\psi\overset{\circ}{\Delta}{}^{-1}\rho^2\mathcal{F}^A{}_{0||A})\right] \\ &= -\int_V \partial_0\left[\pi\overset{\circ}{\Delta}{}^{-1}\delta\psi + *\pi\overset{\circ}{\Delta}{}^{-1}\delta * \psi\right] \\ &+ \int_{\partial V}\left[\rho^2\mathcal{F}^A{}_{0||A}\overset{\circ}{\Delta}{}^{-1}\delta\psi - \mathcal{F}^{3A||B}\varepsilon_{AB}\delta * \psi\right]\,. \end{aligned} \tag{3.13}$$

Finally we obtain the following variational principle

$$\delta\int_V \overline{L} = -\int_V \partial_0(\pi\overset{\circ}{\Delta}{}^{-1}\delta\psi + *\pi\overset{\circ}{\Delta}{}^{-1}\delta * \psi) + \int_{\partial V}\pi^3\overset{\circ}{\Delta}{}^{-1}\delta\psi + *\pi^3\overset{\circ}{\Delta}{}^{-1}\delta*\psi\,, \tag{3.14}$$

where the Lagrangian $\overline{L}$ is defined by

$$\overline{L} = L - \partial_0\left(\psi\overset{\circ}{\Delta}{}^{-1}\pi\right) - \partial_3\left(\psi\overset{\circ}{\Delta}{}^{-1}\rho^2\mathcal{F}^A{}_{0||A}\right) \tag{3.15}$$

and boundary momenta are

$$\pi^3 := -\rho^2\mathcal{F}_0{}^A{}_{||A} = \rho^2\mathcal{F}^{0A}{}_{||A} - \frac{\mathcal{F}^{3A}{}_{||A}}{\sqrt{1+\rho^2}} = \sin\theta\left(\frac{\dot{\psi}}{\sqrt{1+\rho^2}} + \rho^2\psi_3\right)\,,$$

$$*\pi^3 := -\mathcal{F}^{3A||B}\varepsilon_{AB} = \rho^2 * \mathcal{F}_0{}^A{}_{||A} = \sin\theta\left(\frac{*\dot{\psi}}{\sqrt{1+\rho^2}} + \rho^2 * \psi_3\right)\,.$$

From Lagrangian relation (3.14) we immediately obtain the Hamiltonian one, performing the Legendre transformation

$$
\begin{aligned}
-\delta\int_V H \;=\; & -\int_V \dot{\pi}\,\mathring{\Delta}^{-1}\delta\psi - \dot{\psi}\,\mathring{\Delta}^{-1}\delta\pi + {*\dot{\pi}}\,\mathring{\Delta}^{-1}\delta *\psi - {*\dot{\psi}}\,\mathring{\Delta}^{-1}\delta *\pi \\
& + \int_{\partial V} \pi^3\,\mathring{\Delta}^{-1}\delta\psi + {*\pi^3}\,\mathring{\Delta}^{-1}\delta *\psi\,, \qquad (3.16)
\end{aligned}
$$

where

$$
H := -\pi\,\mathring{\Delta}^{-1}\dot{\psi} - {*\pi}\,\mathring{\Delta}^{-1} *\dot{\psi} - \overline{L} \qquad (3.17)
$$

is the density of the Hamiltonian of the electromagnetic field on the hyperboloid.

The value of $\int_V H$ is equal to the amount of electromagnetic energy contained in a volume V and defined by the energy–momentum tensor

$$
T^{\mu}{}_{\nu} = f^{\mu\lambda}f_{\lambda\nu} + \frac{1}{4}\delta^{\mu}_{\nu}f^{\kappa\lambda}f_{\kappa\lambda}\,.
$$

We are not surprised that the quantity H is related to $T^0{}_0$ by

$$
\int_{S(s,\rho)} H\mathrm{d}\theta\mathrm{d}\phi = \int_{S(s,\rho)} \sqrt{-\eta}\,T^0{}_0\mathrm{d}\theta\mathrm{d}\phi
$$

and to prove it we can use the following identity

$$
\begin{aligned}
&\rho^{-4}\sin\theta\left[{*\pi}\,\mathring{\Delta}^{-1} *\dot{\psi} - \dot{\pi}\,\mathring{\Delta}^{-1}\psi - \partial_3\left(\psi\,\mathring{\Delta}^{-1}\rho^2\mathcal{F}^A{}_{0||A}\right)\right] \\
&= \mathcal{F}^{03}\mathcal{F}_{03} - \mathcal{F}^{0A}{}_{||A}\,\mathring{\Delta}^{-1}\rho^{-2}\mathcal{F}_0{}^A{}_{||A} - \mathcal{F}^{0A||B}\varepsilon_{AB}\,\mathring{\Delta}^{-1}\mathcal{F}_{0A||B}\varepsilon^{AB}\,.
\end{aligned}
$$

The non-conservation law for the energy we can write as follows

$$
\begin{aligned}
-\partial_0\int_\Sigma H \;&=\; \int_{\partial\Sigma}\sin\theta\left(\dot{\psi}\,\mathring{\Delta}^{-1}\dot{\psi} + {*\dot{\psi}}\,\mathring{\Delta}^{-1}{*\dot{\psi}}\right) \\
&=\; -\int_{S(s,0)}\left(\dot{\psi}\,\mathring{\Delta}^{-1}\dot{\psi} + {*\dot{\psi}}\,\mathring{\Delta}^{-1}{*\dot{\psi}}\right)\sin\theta\mathrm{d}\theta\mathrm{d}\phi\,. \qquad (3.18)
\end{aligned}
$$

For angular momentum defined by

$$
J_z := -\int_\Sigma \pi\,\mathring{\Delta}^{-1}\psi_{,\phi} + {*\pi}\,\mathring{\Delta}^{-1}{*\psi}_{,\phi}
$$

we have a similar formula

$$\begin{aligned} -\partial_0 J_z &= \partial_0 \int\limits_{\Sigma} \pi \mathring{\Delta}^{-1} \psi_{,\phi} + *\!\pi \mathring{\Delta}^{-1} *\!\psi_{,\phi} \\ &= -\int\limits_{S(s,0)} \sin\theta \left(\dot{\psi} \mathring{\Delta}^{-1} \psi_{,\phi} + *\!\dot{\psi} \mathring{\Delta}^{-1} *\!\psi_{,\phi} \right) \mathrm{d}\theta \mathrm{d}\phi \,, \end{aligned} \tag{3.19}$$

but the relation with symmetric energy-momentum tensor is not so obvious.

$$\tilde{J}_z := \int\limits_{\Sigma} \sqrt{-\eta} T^0{}_\phi = \int\limits_{\Sigma} \mathcal{F}^{03} f_{3\phi} + *\mathcal{F}^{03} * f_{3\phi} \,.$$

Using the relations

$$\begin{aligned} \pi &= -\rho^{-2} \sin\theta f_3{}^A{}_{||A} \,, \qquad \psi_{,3} = -\rho^{-2} \varepsilon^{AB} f_{3A||B} \,, \\ *\!\pi &= \rho^{-2} \sin\theta * f_3{}^A{}_{||A} \,, \qquad *\!\psi_{,3} = -\rho^{-2} \varepsilon^{AB} * f_{3A||B} \,, \end{aligned}$$

we can express $\tilde{J}_z$ in terms of $(\pi, \psi, *\!\pi, *\!\psi)$ as

$$\begin{aligned} \tilde{J}_z &= \int\limits_{\Sigma} \psi \left(\mathring{\Delta}^{-1} \pi_{,\phi} + \sin\theta \hat{\partial}_\phi \mathring{\Delta}^{-1} *\!\psi_{,3} \right) + \int\limits_{\Sigma} *\!\psi \left(\mathring{\Delta}^{-1} *\!\pi_{,\phi} - \sin\theta \hat{\partial}_\phi \mathring{\Delta}^{-1} *\!\psi_{,3} \right) \\ &= -\int\limits_{\Sigma} \pi \mathring{\Delta}^{-1} \psi_{,\phi} + *\!\pi \mathring{\Delta}^{-1} *\!\psi_{,\phi} + \int\limits_{\partial\Sigma} \sin\theta \psi \hat{\partial}_\phi \mathring{\Delta}^{-1} *\!\psi \,, \end{aligned}$$

where $\hat{\partial}_A := \varepsilon_A{}^B \partial_B$ and we have used the identity

$$\int\limits_{S^2} \sin\theta \psi_{,3} \hat{\partial}_\phi \mathring{\Delta}^{-1} * \psi = -\int\limits_{S^2} \sin\theta *\!\psi \hat{\partial}_\phi \mathring{\Delta}^{-1} \psi_{,3} \,.$$

The boundary term $\int\limits_{\partial\Sigma} \sin\theta \psi \hat{\partial}_\phi \mathring{\Delta}^{-1} *\!\psi$ usually has to vanish, if we want to interprete the integral $\int\limits_{\Sigma} \sqrt{-\eta} T^0{}_\phi$ as an angular momentum generator, but in our case ψ, $*\!\psi$ do not vanish on $\mathcal{J}^+$ and we obtain in general two different definitions of the angular momenta J_z and $\tilde{J}_z$.

Let us observe that

$$A_\phi = \left(\rho^{-2} \mathring{\Delta}^{-1} A^B{}_{||B} \right)_{,\phi} - \hat{\partial}_\phi (\mathring{\Delta}^{-1} * \psi)$$

and

$$\int\limits_{V} \left(\mathcal{F}^{\lambda 0} A_\phi \right)_{,\lambda} = \int\limits_{\partial V} \sin\theta \psi \mathring{\Delta}^{-1} \rho^{-2} \left(A^B{}_{||B} \right)_{,\phi} - \int\limits_{\partial V} \sin\theta \psi \hat{\partial}_\phi \mathring{\Delta}^{-1} * \psi$$

so the angular momentum J_z is related rather to the canonical energy-momentum tensor with gauge $A^B{}_{||B} = 0$ than to the symmetric one. More precisely, the canonical energy-momentum density $\mathcal{T}^\mu{}_\nu$ is related to the symmetric one as follows

$$\mathcal{T}^\mu{}_\nu = \mathcal{F}^{\lambda\mu}A_{\lambda,\nu} - \frac{1}{4}\delta^\mu{}_\nu L = \sqrt{-\eta}T^\mu{}_\nu + \left(\mathcal{F}^{\lambda\mu}A_\nu\right)_{,\lambda} .$$

For the angular momentum we obtain

$$\begin{aligned} \int\limits_\Sigma \mathcal{T}^0{}_\phi &= -\int\limits_\Sigma \pi \overset{\circ}{\Delta}{}^{-1}\psi_{,\phi} + *\pi \overset{\circ}{\Delta}{}^{-1}*\psi_{,\phi} - \int\limits_{\partial\Sigma} \sin\theta\psi_{,\phi} \overset{\circ}{\Delta}{}^{-1}\rho^{-2}A^B{}_{||B} \\ &= J_z + \int\limits_{S(s,0)} \psi_{,\phi} \overset{\circ}{\Delta}{}^{-1}\rho^{-2}A^B{}_{||B}\sin\theta \mathrm{d}\theta\mathrm{d}\phi . \end{aligned}$$

Let us observe that if $\psi_{,\phi}$ and $*\psi_{,\phi}$ are vanishing on $\mathcal{J}^+$ then J_z is well defined in terms of the canonical energy-momentum tensor density $\mathcal{T}^0{}_\phi$ and is conserved.

3.1. Electrodynamics on a null surface

Now we will show, how the formula (2.25) can be obtained in the classical electrodynamics. Let us consider a volume $V \subset N$, where N has been already defined by (2.23). Let us integrate infinitesimal symplectic relation

$$\begin{aligned} \int\limits_V \mathcal{F}^{v\nu}\delta A_\nu \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi &= \int\limits_{\partial V} \mathcal{F}^{v\rho} \overset{\circ}{\Delta}{}^{-1}\delta\rho^{-2}A^B{}_{||B}\mathrm{d}\theta\mathrm{d}\phi \\ &+ \int\limits_V \mathcal{F}^{v\rho}\delta\left[A_\rho - \overset{\circ}{\Delta}{}^{-1}(\rho^{-2}A^B{}_{||B})_{,\rho}\right]\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \\ &- \int\limits_V \mathcal{F}^{vA||B}\varepsilon_{AB} \overset{\circ}{\Delta}{}^{-1}\delta\rho^{-2}A_{A||B}\varepsilon^{AB}\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \end{aligned}$$

and Similarly to the considerations on Σ we have

$$\overset{\circ}{\Delta} A_\rho - (\rho^{-2}A^B{}_{||B})_{,\rho} = \frac{1}{\sin\theta}\mathcal{F}^{vA}{}_{||A} = \psi_{,\rho} = 2\rho^{-2}\psi_{,\overline{u}} ,$$

where $\overline{u} = -\frac{2}{\rho}$ and $\partial_\rho = 2\rho^{-2}\partial_{\overline{u}}$. For dual degree of freedom holds the analogical relation

$$\mathcal{F}^{vA||B}\varepsilon_{AB} = *\mathcal{F}^{vA}{}_{||A} = 2\sin\theta\rho^{-2} *\psi_{,\overline{u}}$$

and finally we obtain gauge–independent part + boundary term + full variation

$$
\begin{aligned}
\int\limits_V \mathcal{F}^{v\nu}\delta A_\nu \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \;=\; & \int\limits_{\partial V} \mathcal{F}^{v\rho}\, \mathring{\Delta}^{-1}\delta\rho^{-2}A^B{}_{||B}\mathrm{d}\theta\mathrm{d}\phi \\
& -\delta\int\limits_V \sin\theta\psi\, \mathring{\Delta}^{-1}\psi_{,u}\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi \\
& +\int\limits_V \sin\theta\left(\psi_{,u}\, \mathring{\Delta}^{-1}\delta\psi + *\psi_{,u}\, \mathring{\Delta}^{-1}\delta *\psi\right)\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi\,.
\end{aligned}
\tag{3.20}
$$

This equality means that, modulo the boundary term and full variation, we can reduce our symplectic form on $\mathcal{J}^+$ to the invariants. The final form is similar to (2.25) and posseses a *quasi-local* character. Let us notice that on the surface v =const. we can use the coordinate u as well as $\overline{u}$ or in other words $\partial_u|_N = \partial_{\overline{u}}|_N$, $\mathrm{d}u|_N = \mathrm{d}\overline{u}|_N$ and this observation refers obviously to the objects on N but not on M.

Now we will show, how the flux of energy through $\mathcal{J}^+$ is related to the energy-momentum tensor, similarly as in Subsection 2.5.

$$
T^v{}_v = \frac{1}{2}\rho^{-4}(f^{v\rho})^2 + \frac{1}{2}\rho^{-4}(*f^{v\rho})^2 + \frac{1}{2}\eta_{AB}f^{vA}f^{vB}\,,
$$

$$
\begin{aligned}
\int\limits_V T^v{}_v\rho^{-4}\sin\theta\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi = \; & -\int\limits_V \sin\theta\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi\left(\psi_{,u}\, \mathring{\Delta}^{-1}\psi_{,u} + *\psi_{,u}\, \mathring{\Delta}^{-1} *\psi_{,u}\right) \\
& +\frac{1}{4}\int\limits_V \rho^2\sin\theta\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi\left(\psi^2 + *\psi^2\right)\,.
\end{aligned}
$$

The last term vanishes on the scri

$$
\rho^2\left(\psi^2 + *\psi^2\right) \overset{\rho\to 0^+}{\longrightarrow} 0
$$

so that

$$
\int\limits_{\mathcal{J}^+} \sqrt{-\eta}T^v{}_v = -\int\limits_{\mathcal{J}^+} \mathrm{d}u\left[\sin\theta\mathrm{d}\theta\mathrm{d}\phi\left(\psi_{,u}\, \mathring{\Delta}^{-1}\psi_{,u} + *\psi_{,u}\, \mathring{\Delta}^{-1} *\psi_{,u}\right)\right]\,.
\tag{3.21}
$$

The integral on a sphere in quadratic brackets represents the *quasi-local* density of the flux of the energy through $\mathcal{J}^+$. The main difference with a scalar field is that here there is no possibility to work with the local density because of the operator $\overset{\circ}{\Delta}{}^{-1}$ and only a quasi-local object assigned to a sphere can be well defined. However, if we introduce "quasi-local vector"

$$\Psi_A := \partial_A(\overset{\circ}{\Delta}{}^{-1}\psi) + \varepsilon_A{}^B\partial_B(\overset{\circ}{\Delta}{}^{-1} *\psi)\,,$$

then the flux of energy through $\mathcal{J}^+$ can be described by a "local density"

$$-\int\limits_{S^2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \left(\psi_{,u} \overset{\circ}{\Delta}{}^{-1}\psi_{,u} + *\psi_{,u} \overset{\circ}{\Delta}{}^{-1} *\psi_{,u}\right) = \int\limits_{S^2} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \, \overset{\circ}{\gamma}{}^{AB}\dot{\Psi}_A\dot{\Psi}_B\,,$$

where $\overset{\circ}{\gamma}{}^{AB}$ is the inverse metric to the standard metric on a unit sphere ($\overset{\circ}{\gamma}_{AB}\mathrm{d}x^A\mathrm{d}x^B = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$). Similarly, the symplectic structure (3.20) takes the form

$$\int\limits_V \mathcal{F}^{v\nu}\delta A_\nu \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}\phi \sim \int\limits_V \sin\theta \mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi\dot{\Psi}^A\delta\Psi_A\,,$$

where $\Psi^A := \overset{\circ}{\gamma}{}^{AB}\Psi_B$ and symbol "$\sim$" denotes equality modulo full variation and boundary term.

4. Linearized gravity on a hyperboloid

We start from the ADM formulation of the initial value problem for Einstein equations [5]. In Subsection 4.1 we introduce the hyperboloidal slicing and in Subsection 4.2 we consider an initial value problem for the linearized Einstein equations on it. In Subsection 4.3 we discuss "charges" on the hyperboloid and in the next two subsections we introduce invariants, which describe reduced dynamics. In Subsection 4.6 we derive the "Hamiltonian" in terms of gauge invariant quantities.

4.1. Hyperboloidal conventions

The flat Minkowski metric of the following form in spherical coordinates

$$\eta_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{4.1}$$

with $r = \sinh\omega$, $t = s + \cosh\omega$ already defined in Section 2, can be expressed in the coordinates s, ω well adopted to a "hyperboloidal" slicing of Minkowski spacetime M

$$\eta_{\mu\nu}\mathrm{d}y^\mu\mathrm{d}y^\nu = -\mathrm{d}s^2 - 2\sinh\omega\,\mathrm{d}s\,\mathrm{d}\omega + \mathrm{d}\omega^2 + \sinh^2\omega(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2). \tag{4.2}$$

In this section we use the different coordinate ω instead of ρ used previously, but at the end we will return to ρ to compare the results for the scalar field and linearized gravity. Let us fix a coordinate chart (y^μ) on M, such that $y^1=\theta$, $y^2=\phi$ (spherical angles), $y^3=\omega$ and $y^0=s$. So we have

$$\Sigma_s := \{y \in M:\ y^0=s\} = \bigcup_{\omega\in[0,\infty[} S_s(\omega) \quad \text{where } S_s(\omega) := \{y\in\Sigma_s : y^3=\omega\} \tag{4.3}$$

and Σ_s is a three–dimensional hyperboloid, $S_s(\omega)=S(s,\frac{1}{\sinh\omega})$ and $\partial\Sigma_s = S_s(\infty)=S(s,0)$.

We use the similar convention for indices (as for coordinates (x^μ)), namely: greek indices $\mu,\nu,\ldots$ run from 0 to 3; $k,l,\ldots$ are coordinates on Σ and run from 1 to 3; $A,B,\ldots$ are coordinates on $S(r)$ and run from 1 to 2.

The hyperboloid Σ has a very simple geometry. The induced Riemannian metric η_{kl} on Σ in our coordinates takes the form

$$\eta_{kl}\mathrm{d}y^k\mathrm{d}y^l = \mathrm{d}\omega^2 + \sinh^2\omega(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2)\,. \tag{4.4}$$

The hypersurface Σ is a constant curvature space and the three–dimensional curvature tensor of Σ can be expressed by the metric

$${}^3R_{ijkl} = \eta_{jk}\eta_{il} - \eta_{ik}\eta_{jl}\,. \tag{4.5}$$

4.2. ADM formulation for linearized gravity on a hyperboloid

Let (g_{kl}, P^{kl}) be the Cauchy data for Einstein equations on a three–dimensional hyperboloid Σ. This means that g_{kl} is a Riemannian metric on Σ and P^{kl} is a symmetric tensor density, which we identify with the ADM momentum [5], *i.e.*

$$P^{kl} = \sqrt{\det g_{mn}}(g^{kl}\mathrm{Tr}K - K^{kl})\,,$$

where K_{kl} is the second fundamental form (external curvature) of the imbedding of Σ into a spacetime M, which is now curved.
The 12 functions (g_{kl}, P^{kl}) must fulfill 4 Gauss–Codazzi constraints

$$P_i{}^l{}_{|l} = 8\pi\sqrt{\det g_{mn}}\,T_{i\mu}n^\mu\,, \tag{4.6}$$

$$(\det g_{mn})\mathcal{R} - P^{kl}P_{kl} + \frac{1}{2}(P^{kl}g_{kl})^2 = 16\pi(\det g_{mn})T_{\mu\nu}n^\mu n^\nu\,, \tag{4.7}$$

where $T_{\mu\nu}$ is an energy momentum tensor of the matter, by $\mathcal{R}$ we denote the (three–dimensional) scalar curvature of g_{kl}, n^μ is a future timelike four–vector normal to the hypersurface Σ and the calculations have been made with respect to the three–metric g_{kl} ("|" denotes the covariant derivative, indices are raised and lowered etc.).

The Einstein equations and the definition of the metric connection imply the first order (in time) differential equations for g_{kl} and P^{kl} (see [5] or [6] p. 525) and contain the lapse function N and the shift vector N^k as parameters

$$\dot{g}_{kl} = \frac{2N}{\sqrt{g}}\left(P_{kl} - \frac{1}{2}g_{kl}P\right) + N_{k|l} + N_{l|k}\,, \tag{4.8}$$

where $g := \det g_{mn}$ and $P := P^{kl}g_{kl}$

$$\begin{aligned}\dot{P}^{kl} = \quad & -N\sqrt{g}\left(\mathcal{R}^{kl} - \frac{1}{2}g^{kl}\mathcal{R}\right) - \frac{2N}{\sqrt{g}}\left(P^{km}P_m{}^l - \frac{1}{2}PP^{kl}\right) + \left(P^{kl}N^m\right)_{|m} \\ & + \frac{N}{2\sqrt{g}}g^{kl}\left(P^{kl}P_{kl} - \frac{1}{2}P^2\right) - N^k{}_{|m}P^{ml} - N^l{}_{|m}P^{mk} \\ & + \sqrt{g}\left(N^{|kl} - g^{kl}N^{|m}{}_{|m}\right) + 8\pi N\sqrt{g}T_{mn}g^{km}g^{ln}\,. \end{aligned} \tag{4.9}$$

We want to consider an initial value problem for the linearized Einstein equations on the hyperboloidal slicing, introduced in the previous section. For this purpose let us first check that on this slicing the ADM momentum P^{kl} for the background flat Minkowski spacetime on each hyperboloid Σ_s is no longer trivial

$$P^{kl} = -2\sqrt{g}g^{kl} \tag{4.10}$$

and

$$g_{kl}\mathrm{d}y^k\mathrm{d}y^l = \eta_{kl}\mathrm{d}y^k\mathrm{d}y^l = \mathrm{d}\omega^2 + \sinh^2\omega(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\,, \tag{4.11}$$

where g^{kl} is the three–dimensional inverse of g_{kl}.

Let us define the linearized variations $(h_{kl}, \mathcal{P}^{kl})$ of the full nonlinear Cauchy data (g_{kl}, P^{kl}) around background data (4.10), (4.11)

$$h_{kl} := g_{kl} - \eta_{kl}\,, \quad \mathcal{P}^{kl} := P^{kl} + 2\Lambda\eta^{kl}\,, \tag{4.12}$$

where $\Lambda := \sqrt{\det\eta_{kl}}$ $(= \sinh^2\omega\sin\theta)$.

We should now rewrite equations (4.6)–(4.9) in a linearized form in terms of $(h_{kl}, \mathcal{P}^{kl})$. Let us denote $\mathcal{P} := \eta_{kl}\mathcal{P}^{kl}$ and $h := \eta^{kl}h_{kl}$. The vector constraint (4.6) can be linearized as follows

$$P_i{}^l{}_{|l} \approx \mathcal{P}_i{}^l{}_{|l} - 2\Lambda h_i{}^k{}_{|k} + \Lambda h_{|i}\,. \tag{4.13}$$

Let us stress that the symbol "|" has different meanings on the left-hand side and on the right-hand side of the above formula. It denotes the covariant derivative with respect to the full nonlinear metric g_{kl} when applied to the

P^{kl}, but on the right-hand side it means the covariant derivative with respect to the background metric η_{kl}. The scalar constraint (4.7) after linearization takes the form

$$\sqrt{g}\mathcal{R} - \frac{1}{\sqrt{g}}\left(P^{kl}P_{kl} - \frac{1}{2}(P^{kl}g_{kl})^2\right) \approx \Lambda\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} - 2\mathcal{P}\,. \tag{4.14}$$

The linearized constraints for vacuum ($T_{\mu\nu} = 0$) have the following form

$$\mathcal{P}_l{}^k{}_{|k} - 2\Lambda h_l{}^k{}_{|k} + \Lambda h_{|l} \;=\; 0 (= 8\pi\Lambda T_{l\mu}n^\mu)\,, \tag{4.15}$$

$$\Lambda\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} - 2\mathcal{P} \;=\; 0 (= 16\pi\Lambda T_{\mu\nu}n^\mu n^\nu)\,. \tag{4.16}$$

The linearization of (4.8) leads to the equation

$$\begin{aligned}\dot{h}_{kl} \;=\; & \frac{2N}{\Lambda}\left(\mathcal{P}_{kl} - \frac{1}{2}\eta_{kl}\mathcal{P}\right) + h_{0k|l} + h_{0l|k} + 2N\eta_{kl}\left(n + \frac{1}{2}h\right) - 2Nh_{kl} \\ & - N^m(h_{mk|l} + h_{ml|k} - h_{kl|m})\,,\end{aligned} \tag{4.17}$$

where $N := \frac{1}{\sqrt{-\eta^{00}}} = \cosh\omega$, $N_3 = \eta_{03} = -\sinh\omega$, $N_A = \eta_{0A} = 0$ are the lapse and shift for the background and $n := \frac{h^{00}}{2\eta^{00}}$ is the linearized lapse. Finally the linearization of (4.9) takes the form

$$\begin{aligned}\dot{\mathcal{P}}^{kl} &= -N\Lambda h^{kl} + N\mathcal{P}^{kl} + N^m\mathcal{P}^{kl}{}_{|m} + 2\Lambda\left(h_0{}^{k|l} + h_0{}^{l|k} - \eta^{kl}h_0{}^m{}_{|m}\right) \\ &+ \Lambda\left[(Nn)^{|kl} - \eta^{kl}(Nn)^{|m}{}_{|m}\right] - \frac{\Lambda}{2}N\left(h^{mk|l}{}_m + h^{ml|k}{}_m - h^{kl|m}{}_m - h^{|kl}\right) \\ &- \frac{\Lambda}{2}N_m\left[h^{kl|m} + 3h^{ml|k} + 3h^{mk|l} - \eta^{kl}(h^{|m} + 2h^{mn}{}_{|n})\right]\,.\end{aligned} \tag{4.18}$$

It is well known (see for example [8]) that linearized Einstein equations are invariant with respect to the "gauge" transformation:

$$h_{\mu\nu} \to h_{\mu\nu} + \xi_{\mu;\nu} + \xi_{\nu;\mu}\,, \tag{4.19}$$

where ξ_μ is a covector field, pseudoriemannian metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ and ";" denotes four–dimensional covariant derivative with respect to the flat Minkowski metric $\eta_{\mu\nu}$. There is no (3+1)-splitting of the gauge for hyperboloidal slicing, similar to the situation described in [8]. The (3+1)-decomposition of the gauge acts on Cauchy data in the following way

$$\begin{aligned}\Lambda^{-1}\mathcal{P}^{kl} \;\to\; & \Lambda^{-1}\mathcal{P}^{kl} + N\xi^{0|kl} - N\eta^{kl}\xi^{0|m}{}_m - 2N\xi^0\eta^{kl} - N^k\xi^{0|l} - N^l\xi^{0|k} \\ & + 2\eta^{kl}N_m\xi^{0|m} + 2\xi^{k|l} + 2\xi^{l|k} - 2\eta^{kl}\xi^m{}_{|m}\,,\end{aligned} \tag{4.20}$$

$$h_{kl} \;\to\; h_{kl} + \xi_{l|k} + \xi_{k|l} + 2N\eta_{kl}\xi^0\,. \tag{4.21}$$

It can be easily checked that the scalar constraint (4.16) and the vector constraint (4.15) are invariant with respect to the gauge transformations (4.20) and (4.21). The Cauchy data (h_{kl}, $\mathcal{P}^{kl}$) and ($\overline{h}_{kl}$, $\overline{\mathcal{P}}^{kl}$) on Σ are equivalent to each other if they can be related by the gauge transformation ξ_μ. The evolution of canonical variables $\mathcal{P}^{kl}$ and h_{kl} given by equations (4.17), (4.18) is not unique unless the lapse function n and the shift vector ${h^0}_k$ are specified.

Let us define the "new momentum" p^{kl} as

$$p^{kl} := \mathcal{P}^{kl} - \Lambda\left(2h^{kl} - \eta^{kl}h\right) \qquad (p := \mathcal{P} + \Lambda h)$$

and notice that this object can be also introduced in full nonlinear theory as $P^{kl} + 2\sqrt{g}g^{kl}$ and after linearization gives p^{kl}, *i.e.*

$$P^{kl} + 2\sqrt{g}g^{kl} \approx p^{kl}\,.$$

Let us also observe that the new momentum is trivial for flat Minkowski data. Moreover, the symplectic structure is preserved

$$\mathrm{d}P^{kl} \wedge \mathrm{d}g_{kl} - \mathrm{d}\left(P^{kl} + 2\sqrt{g}g^{kl}\right) \wedge \mathrm{d}g_{kl} = -4\mathrm{d}^2\sqrt{g} = 0$$

and the gauge transformation for p^{kl} is simpler than for $\mathcal{P}^{kl}$

$$\Lambda^{-1}p^{kl} \;\to\; \Lambda^{-1}p^{kl} + N\xi^{0|kl} - N\eta^{kl}{\xi^{0|m}}_m - N^k\xi^{0|l} - N^l\xi^{0|k} + 2\eta^{kl}N_m\xi^{0|m}\,. \tag{4.22}$$

The vector constraint has a familiar form

$$ {p_k}^l{}_{|l} = 0 (= 8\pi\Lambda T_{l\mu}n^\mu)\,. \tag{4.23}$$

We can also rewrite the dynamical equation (4.18) in terms of the new momentum

$$\begin{aligned}\dot{p}^{kl} \;=\; & N^m p^{kl}_{|m} + N\left(\eta^{kl}p - 3p^{kl}\right) + \Lambda\left[(Nn)^{|kl} - \eta^{kl}{(Nn)^{|m}}_{|m} + 2Nn\eta^{kl}\right] \\ & + N\Lambda\left(\eta^{kl}h - 3h^{kl}\right) - \frac{\Lambda}{2}N\left({h^{mk|l}}_m + {h^{ml|k}}_m - {h^{kl|m}}_m - h^{|kl}\right) \\ & + \frac{\Lambda}{2}N_m\left[h^{ml|k} + h^{mk|l} - h^{kl|m} + \eta^{kl}(h^{|m} - 2{h^{mn}}_{|n})\right]\,. \end{aligned} \tag{4.24}$$

We will show in the sequel that it is possible to define a reduced dynamics in terms of invariants, which is no longer sensitive on gauge conditions. The construction is analogous to the analysis given in [7].

4.3. "Charges" on a hyperboloid

The vector constraint (4.23) allows to introduce "charges" related to the symmetries of the hyperboloid. There are six generators of the Lorentz group, which are simultaneously Killing vectors on the hyperboloid Σ. Let us denote this Killing field by X^k. It is defined by the equation

$$X_{k|l} + X_{l|k} = 0 \,. \tag{4.25}$$

Let $V \subset \Sigma$ be a compact region in Σ. For example $V := \bigcup_{r\in[r_0,r_1]} S_s(r)$ and $\partial V = S_s(r_0) \cup S_s(r_1)$. From (4.23) and (4.25) we get

$$(8\pi \int_V \Lambda T_{l\mu} n^\mu) = 0 = \int_V p^{kl}{}_{|l} X_k = \int_V (p^{kl} X_k)_{|l} = \int_{\partial V} p^{3k} X_k \,. \tag{4.26}$$

The equation (4.26) expresses the "Gauss" law for the charge "measured" by the flux integral.

In particular for angular momentum, when $X = \partial/\partial\phi$, we can show the relation of this charge to the dipole part of invariant $\mathbf{y}$, which we will introduce in the sequel (Subsection 4.4).

$$\begin{aligned} 16\pi s^z \, : \; &= \; 16\pi j^{xy} = -2 \int_{\partial V} p^3{}_\phi = -2 \int_{\partial V} p^3{}_A (r^2 \varepsilon^{AB} \cos\theta)_{||B} \\ &= \; 2 \int_{\partial V} r^2 p^3{}_{A||B} \varepsilon^{AB} \cos\theta = \int_{\partial V} \Lambda \mathbf{y} \cos\theta \,. \end{aligned} \tag{4.27}$$

The time translation defines a mass charge as follows

$$\begin{aligned} (16\pi \int_V \Lambda T_{0\mu} n^\mu) \; &= \; 0 = \int_V N \left[\Lambda \left(h^{kl}{}_{|l} - h^{|k} \right)_{|k} - 2\pi \right] + 2N_k p^{kl}{}_{|l} \\ &= \int_V \left[2N_k p^{kl} + N\Lambda \left(h^{lk}{}_{|k} - h^{|l} \right) + \Lambda \left(N_k h^{kl} - N^l h \right) \right]_{|l} \\ &= \int_{\partial V} 2N_k p^{k3} + \Lambda \left(N h^{3k}{}_{|k} - N h^{|3} + N_k h^{k3} - N^3 h \right) \end{aligned} \tag{4.28}$$

and it can be related to the monopole part of an invariant $\mathbf{x}$ (Subsection 4.4).

$$
\begin{aligned}
16\pi p^0 &= \int_{\partial V} 2N_k p^{k3} + \Lambda\left(N h^{3k}{}_{|k} - N h^{|3} + N_k h^{k3} - N^3 h\right) \\
&= \int_{\partial V} \frac{\Lambda}{\sinh\omega}\left(2\cosh^2\omega h^{33} - \cosh\omega\sinh\omega H,_3 - H - \frac{2\sinh^2\omega}{\Lambda} p^{33}\right) \\
&= \int_{\partial V} \frac{\Lambda}{r}\mathbf{x}\,. \qquad (4.29)
\end{aligned}
$$

Remark. The traceless part of h_{kl} and p^{kl} have nice properties with respect to the gauge transformation (4.19), which splits into 0-component (transversal to Σ) which acts on p^{kl} and space components (tangent to Σ) which act on $h_{kl} - \frac{1}{3}\eta_{kl}h$. The traces h and $\mathcal{P}$ remain nontrivial unless we impose gauge conditions. The most popular gauge condition, which allows to obtain the scalar constraint (4.16) as a full divergence (see (4.30) below), is to assume that $\mathcal{P} = 0$. Assuming such gauge we can define another "mass" charge as a surface integral coming from the scalar constraint (4.16) (but we obtain totally nonlocal object). More precisely, one can analyze the scalar constraint (4.16) (in the same way as (4.26) for the vector one)

$$
2\int_V \mathcal{P} = \int_V \Lambda\left(h^{kl}{}_{|l} - h^{|k}\right)_{|k} = \int_{\partial V} \Lambda\left(h^{3l}{}_{|l} - h^{|3}\right) \qquad (4.30)
$$

but there is no "Gauss" law for the "mass" defined by the surface integral on the right-hand side of (4.30), unless we impose gauge condition $\mathcal{P} = 0$. This means that such definition of the mass charge, measured by the flux integral at null infinity, is *not* gauge invariant like the ADM mass at spatial infinity. This consideration should convince the reader that the definition (4.29) is better than (4.30) together with vanishing $\mathcal{P}$.

4.4. The (2+1)-decomposition and reduction

Now we introduce reduced gauge invariant data on Σ for the gravitational field, similar to the invariants introduced in [7]. For this purpose we use a spherical foliation of Σ (see equations (4.3) and (4.4)).

In this section we present mainly results without detailed proofs as in the section about electrodynamics. See also Appendix A where we give explicit formulae used in this section.

Let $\kappa := \coth\omega$. The gauge (4.21) splits in the following way

$$
h_{33} \;\to\; h_{33} + 2\xi_{3,3} + 2N\xi^0\,, \qquad (4.31)
$$

$$h_{3A} \;\to\; h_{3A} + \xi_{3,A} + \xi_{A,3} - 2\kappa\xi_A\,, \tag{4.32}$$

$$h_{AB} \;\to\; h_{AB} + \xi_{A||B} + \xi_{B||A} + 2\kappa\eta_{AB}\xi_3 + 2N\eta_{AB}\xi^0\,, \tag{4.33}$$

where by "$||$" we denote the covariant derivative with respect to the two–metric η_{AB} on $S(r)$. Similarly, the gauge (4.22) can be splitted as follows

$$\Lambda^{-1}p^{33} \;\to\; \Lambda^{-1}p^{33} + N\xi^{0|33} - N\xi^{0|m}{}_m = \Lambda^{-1}p^{33} - N\xi^{0||A}{}_A - 2N\kappa\xi^{0,3}\,, \tag{4.34}$$

$$\Lambda^{-1}p^{3A} \;\to\; \Lambda^{-1}p^{3A} + N\xi^{0|3A} - N^3\xi^{0|A} = \Lambda^{-1}p^{3A} + N\xi^{0,3A} - \frac{1}{\sinh\omega}\xi^{0,A}\,, \tag{4.35}$$

$$\begin{aligned}\Lambda^{-1}p^{AB} \;\to\;& \Lambda^{-1}p^{AB} + N\xi^{0|AB} - N\eta^{AB}\xi^{0|m}{}_m + 2\eta^{AB}N_m\xi^{0|m} = \Lambda^{-1}p^{AB}\\ &+N\xi^{0||AB} - N\eta_{AB}\left(\xi^{0,3}{}_3 + \xi^{0||C}{}_C + (\kappa - 2N^3)\xi^0{}_{,3}\right)\,.\end{aligned} \tag{4.36}$$

It is also quite easy to rewrite the (2+1)-decomposition of (4.17)

$$\begin{aligned}\dot h_{33} &= \frac{2N}{\Lambda}\left(p_{33} - \frac{1}{2}p\right) + 2h_{03|3} + 2N(n + h_{33}) - N^3h_{33|3}\\ &= \frac{2N}{\Lambda}\left(p_{33} - \frac{1}{2}p\right) + 2h_{03,3} + 2Nn + 2Nh_{33} - N^3h_{33,3}\,,\end{aligned} \tag{4.37}$$

$$\begin{aligned}\dot h_{3A} &= \frac{2N}{\Lambda}p_{3A} + h_{03|A} + h_{0A|3} + 2Nh_{3A} - N^3h_{33|A}\\ &= \frac{2N}{\Lambda}p_{3A} + h_{03,A} + h_{0A,3} - 2\kappa h_{0A} - N^3h_{33,A}\,,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\dot h_{AB} &= \frac{2N}{\Lambda}\left(p_{AB} - \frac{1}{2}\eta_{AB}p\right) + h_{0A|B} + h_{0B|A} + 2N\eta_{AB}n + 2Nh_{AB}\\ &\quad -N^3(h_{3A|B} + h_{3B|A} - h_{AB|3}) = \frac{2N}{\Lambda}\left(p_{AB} - \frac{1}{2}\eta_{AB}p\right) + h_{0A||B}\\ &\quad +h_{0B||A} + 2\kappa\eta_{AB}h_{03} + 2N\eta_{AB}(n + h_{33}) + 2Nh_{AB}\\ &\quad -N^3(h_{3A||B} + h_{3B||A} - h_{AB,3})\,.\end{aligned} \tag{4.39}$$

The vector constraint (4.23) can be splitted in the similar way

$$p_3{}^k{}_{|k} \;=\; p_3{}^3{}_{,3} + p_3{}^A{}_{||A} - \kappa p^{AB}\eta_{AB} = 0\,, \tag{4.40}$$

$$p_A{}^k{}_{|k} \;=\; p_A{}^3{}_{,3} + p_A{}^B{}_{||B} = p_{3A,3} + S_A{}^B{}_{||B} + \frac{1}{2}S_{||A} = 0\,, \tag{4.41}$$

where $S := p^{AB}\eta_{AB}$ and $S^{AB} := p^{AB} - \frac{1}{2}\eta^{AB}S$. Similarly, let us denote $H := h_{AB}\eta^{AB}$ and $\chi_{AB} := h_{AB} - \frac{1}{2}\eta_{AB}H$. The invariants are defined as follows

$$\begin{aligned}
\mathbf{x} : &= 2\cosh^2\omega h^{33} + 2\cosh\omega\sinh\omega h^{3C}{}_{||C} + \sinh^2\omega\chi^{AB}{}_{||AB} \\
&\quad - \cosh\omega\sinh\omega H,_3 - \frac{1}{2}(\overset{\circ}{\Delta} + 2)H - \frac{2\sinh^2\omega}{\Lambda}p^{33}\,, \qquad (4.42)\\
\mathbf{X} : &= 2\sinh^2\omega S^{AB}{}_{||AB} + 2\cosh\omega\sinh\omega p^{3A}{}_{||A} + \overset{\circ}{\Delta} p^{33}\,, \qquad (4.43)\\
\mathbf{y} : &= 2\Lambda^{-1}\sinh^2\omega p^{3A||B}\varepsilon_{AB}\,, \qquad (4.44)\\
\mathbf{Y} : &= \Lambda(\overset{\circ}{\Delta} + 2)h^{3A||B}\varepsilon_{AB} - \sinh^2\omega(\Lambda\chi^C{}_{A||CB}\varepsilon^{AB}),_3\,. \qquad (4.45)
\end{aligned}$$

The (2+1)-decomposition of the scalar constraint (4.16) can be written in the form

$$\begin{aligned}
&\Lambda\left(h^{|l}{}_l - h^{kl}{}_{|lk}\right) + 2\mathcal{P}\\
&= \left[\Lambda(H,_3 - 2h^{3A}{}_{||A} - 2\kappa h^{33} + \kappa H)\right],_3 - 2\Lambda(h^{33} + H)\\
&+2(p^{33} + S) - \Lambda(\chi^{AB}{}_{||BA} + 2\kappa h^{3A}{}_{||A}) + \Lambda\left(h^{33||A}{}_A + \frac{1}{2}H^{||C}{}_C\right) = 0\,.
\end{aligned} \qquad (4.46)$$

The dynamical equations (4.24) take the following (2+1)-form:

$$\begin{aligned}
&\Lambda^{-1}\dot{p}^{33} = \Lambda^{-1}N_3(p^{33},_3 - \kappa S) - (Nn)^{||A}{}_A - 2\kappa Nn,_3\\
&+\frac{N}{2}\left[h^{33||A}{}_A + H,_{33} + 2\kappa H,_3 - 2\kappa(2h^{3A}{}_{||A} + h^{33},_3) - (2h^{3A}{}_{||A}),_3\right]\\
&+\frac{1}{2}N_3\left[H,_3 - 2h^{3A}{}_{||A}\right]\,, \qquad (4.47)\\
&\Lambda^{-1}\dot{p}^{3A} = \Lambda^{-1}N^3(p^{3A},_3 + 2\kappa p^{3A}) + \left[(Nn),_3 - \kappa Nn\right]^{||A}\\
&+\frac{N}{2}\left[H,_3{}^{||A} - \kappa h^{33||A} + \frac{2}{\sinh^2\omega}h^{3A} - h^{AB}{}_{||B},_3\right.\\
&\left.-2\kappa h^{AB}{}_{||B} - h^{3B||A}{}_{||B} + h^{3A}{}_{||B}{}^{||B}\right] + \frac{1}{2}N_3 h^{33||A}\,, \qquad (4.48)\\
&\Lambda^{-1}\dot{p}_{AB} = \Lambda^{-1}N^3 p_{AB},_3 + \Lambda^{-1}N\left[\eta_{AB}(p^{33} + S) + p_{AB}\right]\\
&+N(n_{||AB} - \eta_{AB}n^{||C}{}_C) - N\eta_{AB}(n,_{33} + \kappa n,_3) + 2\eta_{AB}N^3 n,_3 + \frac{N^3}{2}\left[h^3{}_{A||B}\right.\\
&\left.+h^3{}_{B||A} - \eta_{AB}h^{3C}{}_{||C} + \eta_{AB}\left(\frac{1}{2}H,_3 - h^{3A}{}_{||A} - h^{33},_3\right) - \chi_A{}^C,_3\eta_{CB}\right]\\
&+\frac{N}{2}\left[(\chi^C{}_B,_3\eta_{CA}),_3 + \chi_{AB}{}^{||C}{}_{||C} - \chi^C{}_{A||BC} - \chi^C{}_{B||AC} + h^{33}{}_{||AB} + \frac{1}{2}\eta_{AB}H_{||C}{}^C\right.\\
&+\eta_{AB}\frac{1}{\sinh^2\omega}(h^{33} + H) + \eta_{AB}(\kappa H,_3 - 2\kappa h^{3A}{}_{||A} - 2\kappa^2 h^{33}) + \frac{2}{\sinh^2\omega}\chi_{AB}\\
&\left.-\left(\eta_{AB}(\kappa h^{33} - \frac{1}{2}H,_3) + h^3{}_{A||B} + h^3{}_{B||A}\right),_3\right]\,. \qquad (4.49)
\end{aligned}$$

We can check the reduced field equations for our invariants

$$
\begin{aligned}
\dot{\mathbf{x}} &= \frac{N}{\Lambda}\mathbf{X} + (N^3\mathbf{x}),_3 \,, && (4.50)\\
\dot{\mathbf{X}} &= N^3\mathbf{X},_3 + \Lambda N \triangle_\Sigma \mathbf{x} - \Lambda N^3(\mathbf{x},_3 + 2\kappa\mathbf{x}) \,, && (4.51)\\
\dot{\mathbf{y}} &= \frac{N}{\Lambda}\mathbf{Y} + \frac{N^3}{\Lambda}(\Lambda\mathbf{y}),_3 \,, && (4.52)\\
\dot{\mathbf{Y}} &= \Lambda(N^3\Lambda^{-1}\mathbf{Y}),_3 + \Lambda N \triangle_\Sigma \mathbf{y} - \Lambda N^3(\mathbf{y},_3 + 2\kappa\mathbf{y}) \,, && (4.53)
\end{aligned}
$$

where $\triangle_\Sigma$ is a Laplacian on a hyperboloid Σ.

It can be easily verified that the invariants $\mathbf{x}$ and $\mathbf{y}$ fulfill the usual d'Alembert equation (as a consequence of the above dynamical equations)

$$
\begin{aligned}
\bar{\Box}\ \mathbf{x} &= 0 \,,\\
\bar{\Box}\ \mathbf{y} &= 0 \,.
\end{aligned}
$$

Let us notice that $\mathbf{x}$ and $\mathbf{y}$ are scalars on each sphere $S_s(r)$ with respect to the coordinates y^A.

For the scalar f on a sphere we can define a "monopole" part $\mathrm{mon}(f)$ and a "dipole" part $\mathrm{dip}(f)$ as a corresponding component with respect to spherical harmonics on S^2. Similarly, the "dipole" part of a vector v^A corresponds to the dipole harmonics for the scalars $v^A{}_{||A}$ and $\varepsilon^{AB}v_{A||B}$. Let us denote by $\underline{f}$ "mono–dipole–free" part of f. According to this decomposition we have

$$
\begin{aligned}
\mathbf{x} &= \mathrm{mon}(\mathbf{x}) + \mathrm{dip}(\mathbf{x}) + \underline{\mathbf{x}} \,,\\
\mathbf{y} &= \mathrm{mon}(\mathbf{y}) + \mathrm{dip}(\mathbf{y}) + \underline{\mathbf{y}} \,.
\end{aligned}
$$

Then the mono-dipole part of each scalar can be solved explicitly from the equations (4.50)–(4.53) and the solution has the form

$$
\begin{aligned}
\mathbf{x} - \underline{\mathbf{x}} &= \frac{4\mathbf{m}}{\sinh\omega} + \frac{12\mathbf{k}}{\sinh^2\omega} \,,\\
\mathbf{y} - \underline{\mathbf{y}} &= \frac{12\mathbf{s}}{\sinh^2\omega} \,.
\end{aligned}
$$

Let $\mathbf{p} := \dot{\mathbf{k}}$. We obtain

$$\dot{\mathbf{m}} = \dot{\mathbf{p}} = \dot{\mathbf{s}} = 0$$

and

$$\mathbf{k} = \mathbf{p}(s + \cosh\omega) + \mathbf{k}_0 \,.$$

Moreover, $\overset{\circ}{\Delta} m = 0$, $(\overset{\circ}{\Delta} + 2)\mathbf{p} = (\overset{\circ}{\Delta} + 2)\mathbf{k}_0 = (\overset{\circ}{\Delta} + 2)\mathbf{s} = 0$, which simply means that $\mathbf{m}$ is a monopole and $\mathbf{k}_0$, $\mathbf{p}$, $\mathbf{s}$ are dipoles and they are constant on M_2. They correspond to the charges introduced in [8]. Let us rewrite the solution in coordinates u, r, which will be more useful in the sequel

$$\mathbf{x} = \underline{\mathbf{x}} + \frac{4\mathbf{m} + 12\mathbf{p}}{r} + \frac{12(\mathbf{k}_0 + \mathbf{p}u)}{r^2}\,, \tag{4.54}$$

$$\mathbf{y} = \underline{\mathbf{y}} + \frac{12\mathbf{s}}{r^2}\,. \tag{4.55}$$

Let us also remind the relation between spatial constant three-vectors in cartesian coordinates and dipole harmonics

$$\mathbf{k}_0 = \frac{j^{l0} z_l}{r}\,, \quad \mathbf{p} = \frac{p^l z_l}{r}\,, \quad \mathbf{s} = \frac{s^l z_l}{r}\,,$$

where z_l are cartesian coordinates and j^{l0}, p^l, s^l are corresponding three-vectors representing our charges (see [8]).

4.5. Reduction of the symplectic form on a hyperboloid

We want to show the relation between the symplectic structure and the invariants introduced in the previous subsection. Let (p^{kl}, h_{kl}) be the Cauchy data on a hyperboloid.

The quadratic form $\int_V p^{kl}\delta h_{kl}$ can be decomposed into monopole part, dipole part and the remainder in a natural way.

From the considerations given in the Appendix B we can easily see that

$$\int_V \underline{p}^{kl}\delta\underline{h}_{kl} \sim \int_V \underline{\mathbf{X}}\, \mathring{\Delta}^{-1}(\mathring{\Delta} + 2)^{-1}\delta\underline{\mathbf{x}} + \underline{\mathbf{Y}}\, \mathring{\Delta}^{-1}(\mathring{\Delta} + 2)^{-1}\delta\underline{\mathbf{y}}\,,$$

where symbol "$\sim$" denotes equality modulo full variation and boundary term. Moreover, the "mono-dipole" part has the form

$$\begin{aligned}
\text{mon}\ \left(\int_V p^{kl}\delta h_{kl}\right) &\sim \int_V \frac{1}{2\cosh^2\omega} p^{33}\delta\text{mon}(\mathbf{x})\,,\\
\text{dip}\ \left(\int_V p^{kl} q_{kl}\right) &\sim \int_V \frac{1}{2\cosh^2\omega} p^{33}\delta\text{dip}(\mathbf{x}) + \int_V \Lambda(h_{3A||B}\varepsilon^{AB})\, \mathring{\Delta}^{-1}\delta\text{dip}(\mathbf{y})\\
&+ \int_V \left(\tanh\omega\, \mathring{\Delta}^{-1} h^{3A}{}_{||A} - \frac{H}{4\cosh^2\omega}\right)\delta\text{dip}(\mathbf{X})\,.
\end{aligned} \tag{4.56}$$

The mono-dipole part of invariants: $\text{mon}(\mathbf{x}), \text{dip}(\mathbf{x}), \text{dip}(\mathbf{X}), \text{dip}(\mathbf{y})$ represents 10 charges which are supposed to be fixed, they are analogous to the electric charge in electrodynamics. If we assume that there is no matter inside volume V then all of them are vanishing (this is included in (4.54) and (4.55) as the regularity conditions at $r=0$). In particular on hyperboloid Σ we obtain that mono-dipole part vanishes for linearized vaccum Einstein equations and the symplectic structure can be reduced to the invariants

$$\int_{\Sigma} p^{kl}\delta h_{kl} \sim \int_{\Sigma} \underline{\mathbf{X}}\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{x}} + \underline{\mathbf{Y}}\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\underline{\mathbf{y}}\,. \tag{4.57}$$

4.6. Non-conservation laws on a hyperboloid with the end at $\mathcal{J}^+$

Let us return to the coordinate $\rho := \frac{1}{\sinh\omega}$. The metric on M takes the starting form (2.12). It is convenient to introduce new canonical field variables similar to the variables for the scalar field and electrodynamics

$$\begin{aligned} \Psi_x : &= \rho^{-1}\underline{\mathbf{x}}\,, \quad \Psi_y := \rho^{-1}\underline{\mathbf{y}}\,, \\ \Pi_x : &= \frac{\underline{\mathbf{X}}}{\sqrt{1+\rho^2}}\,, \quad \Pi_y := \frac{\underline{\mathbf{Y}} - \Lambda\underline{\mathbf{y}}}{\sqrt{1+\rho^2}}\,. \end{aligned}$$

Equations of motion are the same for both degrees of freedom

$$\begin{aligned} \frac{1}{\sqrt{1+\rho^2}}\dot{\Psi}_\iota - \Psi_{\iota},_\rho &= \frac{\Pi_\iota\sqrt{1+\rho^2}}{\sin\theta}\,, \qquad \iota = x, y\,, \\ \dot{\Pi}_\iota - (\Pi_\iota\sqrt{1+\rho^2}),_\rho &= \sin\theta\left[\,\overset{\circ}{\Delta}\Psi_\iota + ((1+\rho^2)\Psi_{\iota},_\rho\,),_\rho\right]\,, \end{aligned}$$

and they are similar to (2.19), (2.20) for the scalar field and (3.9), (3.10) for electrodynamics.

The reduction of the symplectic form (from the previous section) allows to formulate the Hamiltonian relation in terms of the new canonical variables

$$\begin{aligned} &\sum_{\iota=x,y}\int_V \dot{\Psi}_\iota\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\Pi_\iota - \dot{\Pi}_\iota\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\Psi_\iota = 16\pi\delta\mathcal{H} \\ &-\sum_{\iota=x,y}\int_{\partial V}\left[\Pi_\iota\sqrt{1+\rho^2} + \sin\theta(1+\rho^2)\Psi_{\iota},_\rho\right]\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta\Psi_\iota\,, \end{aligned} \tag{4.58}$$

where

$$16\pi\mathcal{H} := \frac{1}{2}\sum_{\iota=x,y}\int_V \left(\frac{\Pi_\iota\sqrt{1+\rho^2}}{\sin\theta} + \Psi_{\iota},_\rho\right)\, \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left(\frac{\Pi_\iota\sqrt{1+\rho^2}}{\sin\theta} + \Psi_{\iota},_\rho\right)$$

$$+\frac{1}{2}\sum_{\iota=x,y}\int_V \rho^2\sin\theta\Psi_{\iota,\rho}\ \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\Psi_{\iota,\rho}-\sin\theta\Psi_\iota(\overset{\circ}{\Delta}+2)^{-1}\Psi_\iota\,. \tag{4.59}$$

Similarly for angular momentum we propose the following expression

$$16\pi J_z=\sum_{\iota=x,y}\int_\Sigma \Pi_\iota\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\Psi_{\iota,\phi}\,. \tag{4.60}$$

The non-conservation laws for the energy and angular momentum

$$\begin{aligned}
-16\pi\partial_0\mathcal{H} &= \sum_{\iota=x,y}\int_{S(0)} \sin\theta\dot{\Psi}_\iota\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\dot{\Psi}_\iota\,,\\
-16\pi\partial_0 J_z &= \sum_{\iota=x,y}\int_{S(0)} \sin\theta\dot{\Psi}_\iota\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\Psi_{\iota,\phi}\ ,
\end{aligned}$$

are similar to (2.21), (2.22) and (3.18), (3.19). It should be also possible to formulate linear momentum P_z in a similar way as (2.31)

$$-16\pi\partial_0 P_z=\sum_{\iota=x,y}\int_{S(0)} \sin\theta\cos\theta\left[\dot{\Psi}_\iota\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\dot{\Psi}_\iota\right]\ ,$$

but this will be analyzed in a separate paper[2]. It is obvious that all these formulae are *quasi-local*.

5. Linearized gravity in null coordinates

We are going to follow the idea from Subsection 2.2 and apply it to linearized gravity.

5.1. Minkowski metric in null coordinates

Let us define the null coordinates: $u := t-r$, $v := r+t$ together with the index a corresponding to the coordinates (u,v). The spherical foliation is the same as previously and the coordinates on a sphere (x^A), $(A=1,2)$, $(x^1=\theta, x^2=\phi)$ are the same.

[2] The meaning of the expression in quadratic brackets is not obvious and it should be rather $\dot{\psi}^{AB}\dot{\psi}_{AB}$ where ψ_{AB} are introduced in Section 7. Also in electrodynamics the definition of P_z is not obvious, however in electromagnetic case we have energy-momentum tensor.

For convenience we need also some more denotations: $\rho := r^{-1} = \frac{2}{v-u}$, $\rho_{,a} = \rho^2 \varepsilon_a$, where $\varepsilon_u := \frac{1}{2}$, $\varepsilon_v := -\frac{1}{2}$, $\eta^{ab}\varepsilon_a\varepsilon_b = 1$. We will also need $\varepsilon^a := \eta^{ab}\varepsilon_b$ and we can check that $\varepsilon^u = 1$, $\varepsilon^v = -1$, $\eta_{ab}\varepsilon^a\varepsilon^b = 1$.

The explicit formulae for the components of Minkowski metric can be denoted as follows

$$\eta_{AB} = \rho^{-2} \overset{\circ}{\gamma}_{AB} \,, \quad \eta_{ab} = -\frac{1}{2}|E_{ab}| \,, \quad \eta_{aA} = 0 \,,$$

where $E_{uu} = 0 = E_{vv}$ and $E_{uv} = 1 = -E_{vu}$. Similarly, the inverse metric has the following components

$$\eta^{AB} = \rho^2 \overset{\circ}{\gamma}{}^{AB} \,, \quad \eta^{ab} = -2|E^{ab}| \,, \quad \eta^{aA} = 0 \,,$$

where $E^{uu} = 0 = E^{vv}$ and $E^{uv} = 1 = -E^{vu}$. We shall also need the derivatives

$$\eta^{AB}{}_{,a} = 2\rho\varepsilon_a\eta^{AB} \,, \quad \eta_{AB,a} = -2\rho\varepsilon_a\eta_{AB}$$

and finally the nonvanishing Christoffel symbols except $\Gamma^A{}_{BC}$ are the following

$$\Gamma^a{}_{AB} = \rho\varepsilon^a\eta_{AB} \,, \quad \Gamma^A{}_{aB} = -\rho\varepsilon_a\delta^A{}_B \,.$$

5.2. Riemann tensor in null coordinates

We need to derive the linearized Riemann tensor in null coordinates

$$\begin{aligned}
2R_{abcd} &= h_{ad,bc} - h_{bd,ac} + h_{bc,ad} - h_{ac,bd} \,, \\
2R_{abcD} &= h_{aD,bc} - h_{bD,ac} + h_{bc,aD} - h_{ac,bD} \\
&\quad +\rho\varepsilon_b\left(h_{aD,c} + h_{cD,a} - h_{ac,D}\right) - \rho\varepsilon_a\left(h_{bD,c} + h_{cD,b} - h_{bc,D}\right) \,, \\
2R_{AbCd} &= h_{dA||C,b} + h_{bC||A,d} - h_{bd||AC} - h_{AC,bd} \\
&\quad +\rho\varepsilon_b\left(h_{dA||C} - h_{dC||A} - h_{AC,d}\right) - \rho\varepsilon_d\left(h_{bC||A} - h_{bA||C} - h_{AC,b}\right) \\
&\quad +\rho\eta_{AC}\varepsilon^a\left(h_{bd,a} - h_{ad,b} - h_{ab,d}\right) - 2\rho^2\varepsilon_b\varepsilon_d h_{AC} \,, \\
2R_{ABCd} &= h_{dA||BC} + h_{BC||A,d} - h_{Bd||AC} - h_{AC||B,d} \\
&\quad +2\rho\varepsilon_d\left(h_{bC||A} - h_{AC||B}\right) \\
&\quad +\rho\eta_{BC}\varepsilon^a\left(h_{aA,d} - h_{dA,a} + h_{ad,A} + 2\rho\varepsilon_d h_{aA}\right) \\
&\quad -\rho\eta_{AC}\varepsilon^a\left(h_{aB,d} - h_{dB,a} + h_{ad,B} + 2\rho\varepsilon_d h_{aB}\right) \,, \\
2R_{abCD} &= h_{aD||C,b} - h_{bD||C,a} + h_{bC||D,a} - h_{aC||D,b} \\
&\quad +2\rho\varepsilon_b\left(h_{aD||C} - h_{aC||D}\right) + 2\rho\varepsilon_a\left(h_{bC||D} - h_{bD||C}\right) \,, \\
2R_{ABCD} &= h_{AD||BC} + h_{BC||AD} - h_{BD||AC} - h_{AC||BD}
\end{aligned}$$

$$
\begin{aligned}
&+\rho\eta_{AC}\varepsilon^a\left(h_{BD,a}-h_{aB||D}-h_{aD||B}\right)\\
&+\rho\eta_{BD}\varepsilon^a\left(h_{AC,a}-h_{aC||A}-h_{aA||C}\right)\\
&-\rho\eta_{BC}\varepsilon^a\left(h_{AD,a}-h_{aA||D}-h_{aD||A}\right)\\
&-\rho\eta_{AD}\varepsilon^a\left(h_{BC,a}-h_{aB||C}-h_{aC||B}\right)\\
&+\rho^2\left(h_{BD}\eta_{AC}+h_{AC}\eta_{BD}-h_{AD}\eta_{BC}-h_{BC}\eta_{AD}\right)\\
&+2\rho^2\varepsilon^a\varepsilon^b h_{ab}\left(\eta_{AC}\eta_{BD}-\eta_{BC}\eta_{AD}\right)\,.
\end{aligned}
$$

5.3. Ricci tensor in null coordinates

The Ricci tensor takes the following form

$$
\begin{aligned}
2R_{ab} &= h^c{}_{b,ac}+h_a{}^c{}_{,cb}-h_{ab}{}^{,c}{}_c-h^c{}_{c,ab}\\
&\quad +h_{aA,b}{}^{||A}+h_{bA,a}{}^{||A}-h_{ab}{}^{||A}{}_A-H_{,ab}\\
&\quad +\rho\varepsilon_a H_{,b}+\rho\varepsilon_b H_{,a}+2\rho\varepsilon^c\left(h_{ab,c}-h_{ac,b}-h_{bc,a}\right)\,,\\
2R_{aB} &= h^b{}_{B,ab}-h_{aB}{}^{,c}{}_c+h_a{}^c{}_{,cB}-h^c{}_{c,aB}\\
&\quad +h_a{}^A{}_{||BA}-h_{aB}{}^{||A}{}_A+\chi_B{}^A{}_{||A,a}-\tfrac{1}{2}H_{||B,a}\\
&\quad +\rho\varepsilon_a\left(2h^b{}_{B,b}-h^b{}_{b,B}\right)-2\rho\varepsilon^b h_{bB,a}-2\rho^2\varepsilon_a\varepsilon^b h_{bB}\,,\\
2R_{AB} &= \left(h^a{}_{A||B}+h^a{}_{B||A}\right)_{,a}-h^a{}_{a||AB}-\chi_{AB}{}^{,a}{}_a\\
&\quad -2\rho\varepsilon^a\chi_{AB,a}+\chi_A{}^C{}_{||CB}+\chi_B{}^C{}_{||CA}-\chi_{AB}{}^{||C}{}_C\\
&\quad +\eta_{AB}\Big[-\tfrac{1}{2}(H^{||C}{}_C+H^{,a}{}_a)+2\rho\varepsilon^a\left(H_{,a}-h_a{}^A{}_{||A}\right)\\
&\quad +\rho^2(2\varepsilon^a\varepsilon^b h_{ab}-H)\Big]\,.
\end{aligned}
$$

5.4. Gauge in null coordinates

The gauge transformation ξ_μ

$$h_{\mu\nu}\to h_{\mu\nu}+\xi_{(\mu;\nu)}$$

splits in the following way

$$
\begin{aligned}
h_{ab} &\longrightarrow h_{ab}+\xi_{a,b}+\xi_{b,a}\,,\\
h_{aA} &\longrightarrow h_{aA}+\xi_{a,A}+\xi_{A,a}+2\rho\varepsilon_a\xi_A\,,\\
h_{AB} &\longrightarrow h_{AB}+\xi_{A||B}+\xi_{B||A}-2\rho\eta_{AB}\varepsilon^a\xi_a\,.
\end{aligned}
$$

The following formulae will also be useful.

$$\begin{aligned}\chi_{AB} &\longrightarrow \chi_{AB}+\xi_{A||B}+\xi_{B||A}-\eta_{AB}\xi^{C}{}_{||C}\,,\\ \frac{1}{2}H &\longrightarrow \frac{1}{2}H+\xi^{A}{}_{||A}-2\rho\varepsilon^{a}\xi_{a}\,,\\ h_{a}{}^{A} &\longrightarrow h_{a}{}^{A}+\xi_{a}{}^{||A}+\xi^{A}{}_{,a}\,.\end{aligned}$$

They are straightforward consequences of the previous ones.

5.5. Invariants

Let us introduce the following gauge invariant quantities

$$\mathbf{y}_a := (\overset{\circ}{\Delta}+2)h_{aA||B}\varepsilon^{AB}-(\rho^{-2}\chi_{A}{}^{C}{}_{||CB}\varepsilon^{AB})_{,a}\,, \tag{5.1}$$

$$\mathbf{y} := 2\rho^{-2}(h_{bB||A}\varepsilon^{AB})_{,a}E^{ab}\,, \tag{5.2}$$

$$\mathbf{x} := \rho^{-2}\chi^{AB}{}_{||BA}-\frac{1}{2}\overset{\circ}{\Delta}H+\rho^{-1}\varepsilon^{a}H_{,a}-H+2\varepsilon^{a}\varepsilon^{b}h_{ab}-2\rho^{-1}\varepsilon^{a}h_{a}{}^{A}{}_{||A}\,, \tag{5.3}$$

$$\begin{aligned}\mathbf{x}_{ab} := {}& \overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2)h_{ab}-(\overset{\circ}{\Delta}+2)\left[(\rho^{-2}h_{a}{}^{A}{}_{||A})_{,b}+(\rho^{-2}h_{b}{}^{A}{}_{||A})_{,a}\right]\\ &+\left[\rho^{-2}(\rho^{-2}\chi^{AB}{}_{||AB})_{,a}\right]_{,b}+\left[\rho^{-2}(\rho^{-2}\chi^{AB}{}_{||AB})_{,b}\right]_{,a}\,.\end{aligned} \tag{5.4}$$

They fullfill the following equations

$$\begin{aligned}&(\rho^{-2}\mathbf{y}^{a})_{,a}=0\,, \qquad 2\rho^{-2}\mathbf{y}_{b,a}E^{ab}=(a+2)\mathbf{y}\,,\\ &2E^{ab}(\rho^{-2}\mathbf{y})_{,b}+\rho^{-2}\mathbf{y}^{a}=0\,,\\ &[\rho^{-4}(\mathbf{y}_{a,b}-\mathbf{y}_{b,a})]^{,b}+\rho^{-2}(a+2)\mathbf{y}_{a}=0\,,\\ &(\rho^{-1}\mathbf{y})^{,a}{}_{a}+\rho^{-1}a\mathbf{y}=0\,,\\ &(\rho^{-1}\mathbf{x})^{,a}{}_{a}=-\rho^{-1}a\mathbf{x}\,,\\ &\rho^{-2}\mathbf{x}^{ab}{}_{,ab}=\overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2)\mathbf{x}\,,\\ &\eta^{ab}\mathbf{x}_{ab}=0\,,\\ &\mathbf{x}_{ab}=2(\rho^{-2}\mathbf{x})_{,ab}-\eta_{ab}(\rho^{-2}\mathbf{x})^{,c}{}_{c}\,,\end{aligned} \tag{5.5}$$

if we assume vacuum equations $R_{\mu\nu}=0$.

5.6. Reduction of symplectic form on $\mathcal{J}^+$

Now we will show how the linearized symplectic form on the null surface N can be reduced to the invariants in "wave" part Similarly to the "hyperboloidal" case. In the full nonlinear theory it was introduced by Kijowski [27] (however he was interested only in spacelike surfaces), see also short reminder in the Section 6.

We shall calculate this form in a convenient gauge but the final result will be gauge-invariant. This way we shall prove that, modulo boundary terms depending on the gauge, the invariant part of the symplectic form can be obtained in the demanded shape in "wave" part.

5.6.1. Gauge conditions

Let us assume the following gauge conditions

$$\chi_{AB} = 0\,, \quad {h_a}^A{}_{||A} = 0\,.$$

It is easy to verify that they are compatible for the "wave" part

$$\begin{aligned}
\rho^{-2}\chi^{AB}{}_{||AB} &\longrightarrow \rho^{-2}\chi^{AB}{}_{||AB} + (\overset{\circ}{\Delta} + 2)\xi^A{}_{||A}\,,\\
\rho^{-2}{\chi_A}^C{}_{||CB}\varepsilon^{AB} &\longrightarrow \rho^{-2}{\chi_A}^C{}_{||CB}\varepsilon^{AB} + (\overset{\circ}{\Delta} + 2)\xi_{A||B}\varepsilon^{AB}\,,\\
{h_a}^A{}_{||A} &\longrightarrow {h_a}^A{}_{||A} + \rho^{-2}\overset{\circ}{\Delta}\xi_a + (\xi^A{}_{||A})_{,a}\,.
\end{aligned}$$

More precisely, mono-dipole-free parts of ξ_a and ξ_A are uniquely defined under these gauge conditions.

5.6.2. Partial reduction to extract gauge invariant part

The linearized $\pi^{\mu\nu}$ has the form

$$\pi^{\mu\nu} = -\Lambda h^{\mu\nu} + \frac{1}{2}\eta^{\mu\nu}(h^a_a + H)\Lambda$$

and it can be simplified in our gauge. Let us observe that invariants (5.1) and (5.4) have simple form in this gauge in terms of $h_{\mu\nu}$. From (5.4) and (5.5) we obtain $\underline{h}^a_a = -4\underline{h}_{uv} = 0$. Moreover, $\underline{\pi}^{AB} = \frac{1}{2}\Lambda\eta^{AB}\underline{h}^a_a - \Lambda\chi^{AB}$ vanishes. Similarly $\pi^{Ab}{}_{||A} = 0$ because $\pi^{Ab} = -\Lambda h^{Ab}$. Finally from the above considerations we obtain

$$\int\limits_{S(s,\rho)} \underline{\pi}^{\mu\nu}\delta\underline{A}^a_{\mu\nu} = \int\limits_{S(s,\rho)} \underline{\pi}^{cd}\delta\underline{A}^a_{cd} - 2\underline{\pi^{bA||B}\varepsilon_{AB}}\rho^{-2}\overset{\circ}{\Delta}{}^{-1}\underline{\delta A^a{}_{bA||B}\varepsilon^{AB}}\,.$$

One can show the following relation

$$\int\limits_V \underline{\pi}^{cd}\delta\underline{A}^v_{cd} \sim \int\limits_V \Lambda\rho^2(\rho^{-1}\mathbf{x}^{ab})_{,u}\overset{\circ}{\Delta}{}^{-2}(\overset{\circ}{\Delta} + 2)^{-2}\delta(\rho^{-1}\mathbf{x}_{ab})\,, \tag{5.6}$$

where $\sim$ denotes equality modulo boundary terms and full variation.

Similarly one can prove

$$\int_V \underline{\pi^{bA||B}\varepsilon_{AB}}\rho^{-2} \overset{\circ}{\Delta}{}^{-1}\delta\underline{A^v{}_{bA||B}\varepsilon^{AB}}$$
$$\sim \int_V \Lambda\rho^2\mathbf{y}^b \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-2}\delta\left[(\rho^{-4}\mathbf{y}^v)_{,b}-(\rho^{-2}\mathbf{y}_b)^{,v}\right] . \tag{5.7}$$

5.6.3. Full reduction to x,y

We would like to obtain a similar formula to (2.25). The "curl" part (5.7) reduces easily to the demanded form

$$\int_V \Lambda\rho^2\mathbf{y}^b \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-2}\delta\left[(\rho^{-4}\mathbf{y}^v)_{,b}-(\rho^{-2}\mathbf{y}_b)^{,v}\right]$$
$$\sim \int_V 2\Lambda\rho^2(\rho^{-1}\underline{\mathbf{y}})_{,u} \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta(\rho^{-1}\underline{\mathbf{y}}) .$$

On the other hand, the second part of (5.6) can be rewritten in the following way

$$\int_V \Lambda\rho^2(\rho^{-1}\mathbf{x}^{ab})_{,u} \overset{\circ}{\Delta}{}^{-2}(\overset{\circ}{\Delta}+2)^{-2}\delta(\rho^{-1}\mathbf{x}_{ab})$$
$$\sim \int_V 2\Lambda\rho^2(\rho^{-1}\underline{\mathbf{x}})_{,u} \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta(\rho^{-1}\underline{\mathbf{x}})$$
$$+\int_V 16\Lambda\rho^2\left[\rho^{-1}(\rho^{-1}\underline{\mathbf{x}})_{,vv}+\frac{1}{2}\overset{\circ}{\Delta}(\rho^{-1}\underline{\mathbf{x}})_{,v}\right] \overset{\circ}{\Delta}{}^{-2}(\overset{\circ}{\Delta}+2)^{-2}\delta(\rho^{-1}\underline{\mathbf{x}}) .$$

Let us observe that the last term vanishes on $\mathcal{J}^+$, more precisely $(\rho^{-1}\mathbf{x})_{,v} = O(\rho^2)$. The presented calculations should convince the reader that the following formula holds

$$\int_N \underline{\pi}^{\mu\nu}\delta\underline{A}^v_{\mu\nu} \sim \int_N 2\Lambda\rho^2\left[(\Psi_x)_{,u} \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta(\Psi_x)\right.$$
$$\left.+(\Psi_y)_{,u} \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\delta(\Psi_y)\right]$$

and this is a *quasi-local* form which is similar to (2.25) and (3.20).

6. Generating formula for Einstein equations

Let us remind some results from [27] which will be useful for the sequel. The variation of the Hilbert Lagrangian

$$L = \frac{1}{16\pi}\sqrt{|g|}\ R \tag{6.1}$$

may be calculated as follows

$$\delta L = \delta\left(\frac{1}{16\pi}\sqrt{|g|}\ g^{\mu\nu}\ R_{\mu\nu}\right) = -\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} + \frac{1}{16\pi}\sqrt{|g|}\ g^{\mu\nu}\delta R_{\mu\nu}\,, \tag{6.2}$$

where

$$\mathcal{G}^{\mu\nu} := \sqrt{|g|}\ (R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)\,. \tag{6.3}$$

It was proved in [27] that the last term in (6.2) is a boundary term (a complete divergence). For this purpose we denote

$$\pi^{\mu\nu} := \frac{1}{16\pi}\sqrt{|g|}\ g^{\mu\nu} \tag{6.4}$$

and

$$A^{\lambda}_{\mu\nu} := \Gamma^{\lambda}_{\mu\nu} - \delta^{\lambda}_{(\mu}\Gamma^{\kappa}_{\nu)\kappa}\,. \tag{6.5}$$

We have

$$\begin{aligned} \partial_\lambda A^{\lambda}_{\mu\nu} &= \partial_\lambda\Gamma^{\lambda}_{\mu\nu} - \partial_{(\mu}\Gamma^{\lambda}_{\nu)\lambda} = R_{\mu\nu} - \Gamma^{\lambda}_{\sigma\lambda}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} \\ &= R_{\mu\nu} + A^{\lambda}_{\mu\sigma}A^{\sigma}_{\nu\lambda} - \frac{1}{3}A^{\lambda}_{\mu\lambda}A^{\sigma}_{\nu\sigma}\,. \end{aligned} \tag{6.6}$$

Hence, we obtain an identity

$$\begin{aligned} \partial_\lambda\left(\pi^{\mu\nu}\delta A^{\lambda}_{\mu\nu}\right) &= \pi^{\mu\nu}\delta R_{\mu\nu} + \pi^{\mu\nu}\delta\left(A^{\lambda}_{\mu\sigma}A^{\sigma}_{\nu\lambda} - \frac{1}{3}A^{\lambda}_{\mu\lambda}A^{\sigma}_{\nu\sigma}\right) + (\partial_\lambda\pi^{\mu\nu})\,\delta A^{\lambda}_{\mu\nu} \\ &= \pi^{\mu\nu}\delta R_{\mu\nu} + (\nabla_\lambda\pi^{\mu\nu})\,\delta A^{\lambda}_{\mu\nu}\,. \end{aligned} \tag{6.7}$$

Due to the metricity of Γ we have $\nabla_\lambda\pi^{\mu\nu} = 0$. This way we obtain

$$\pi^{\mu\nu}\delta R_{\mu\nu} = \partial_\lambda\left(\pi^{\mu\nu}\delta A^{\lambda}_{\mu\nu}\right) = \partial_\kappa\left(\pi_{\lambda}^{\ \mu\nu\kappa}\delta\Gamma^{\lambda}_{\mu\nu}\right)\,, \tag{6.8}$$

where we denote

$$\pi_{\lambda}^{\ \mu\nu\kappa} := \pi^{\mu\nu}\delta^{\kappa}_{\lambda} - \pi^{\kappa(\nu}\delta^{\mu)}_{\lambda}\,. \tag{6.9}$$

Inserting (6.8) into (6.2) we have

$$\delta L = -\frac{1}{16\pi}\mathcal{G}^{\mu\nu}\delta g_{\mu\nu} + \partial_\lambda\left(\pi^{\mu\nu}\delta A^{\lambda}_{\mu\nu}\right)\,. \tag{6.10}$$

We conclude that Euler-Lagrange equations $\mathcal{G}^{\mu\nu} = 0$ are equivalent to the following generating formula, analogous to (2.1) in field theory

$$\delta L = \partial_\lambda \left(\pi^{\mu\nu} \delta A^\lambda_{\mu\nu} \right) \tag{6.11}$$

or, equivalently,

$$\delta L = \partial_\kappa \left(\pi_\lambda^{\ \mu\nu\kappa} \delta \Gamma^\lambda_{\mu\nu} \right) . \tag{6.12}$$

This formula is a starting point for the derivation of canonical gravity. Let us observe, that it is valid not only in the present, purely metric, context but also in any variational formulation of General Relativity. For this purpose let us rewrite (6.10) without using *a priori* the metricity condition $\nabla_\lambda \pi^{\mu\nu} = 0$. This way we obtain the following, universal formula

$$\delta L = -\frac{1}{16\pi} \mathcal{G}^{\mu\nu} \delta g_{\mu\nu} - \left(\nabla_\kappa \pi_\lambda^{\ \mu\nu\kappa} \right) \delta \Gamma^\lambda_{\mu\nu} + \partial_\kappa \left(\pi_\lambda^{\ \mu\nu\kappa} \delta \Gamma^\lambda_{\mu\nu} \right) . \tag{6.13}$$

It may be proved that, in this form, the formula remains valid also in the metric-affine approach and in the purely-affine one. In metric–affine formulation, the vanishing of $\nabla_\lambda \pi^{\mu\nu}$ is not automatic: it is a part of field equations. We see that, again, the entire field dynamics is equivalent to (6.12). Finally, in the purely affine formulation of General Relativity the Einstein equations are satisfied "from the very beginning" whereas the metricity condition for the connection becomes the dynamical equation. We conclude that also in this case the entire information about the field dynamics is contained in the generating formula (6.12).

This formula, compared with (2.1), suggests that the role of field potentials in General Relativity should be rather played by the connection Γ, whereas the metric g should rather remain on the side of canonical momenta. This observation was the origin of the purely affine formulation of the theory. Also in the multisymplectic formulation (*i.e.* formulation in terms of Poincaré-Cartan form – see [26]) the connection appears on the side of field configurations. We stress, however, that the results do not depend upon the choice of a variational formulation.

7. Metrics of Bondi–Sachs type

In this section we shall consider the initial value problem for the curved space-time M with a metric of the form

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -\frac{V}{r} \mathrm{e}^{2\beta} \mathrm{d}u^2 - 2\mathrm{e}^{2\beta} \mathrm{d}u \mathrm{d}r + r^2 \gamma_{AB} (\mathrm{d}x^A - U^A \mathrm{d}u)(\mathrm{d}x^B - U^B \mathrm{d}u) \tag{7.1}$$

on the null cone $C = \{x \in M \mid x^0 = u = \text{const.}\}$ (see [10], [2], [4]) with the boundary ∂C at the future null infinity. We have the following non-vanishing components of the inverse metric $g^{\mu\nu}$

$$\begin{aligned} g^{33} &= \frac{V}{r}\mathrm{e}^{-2\beta}\,, \\ g^{03} &= -\mathrm{e}^{-2\beta}\,, \\ g^{3A} &= -\mathrm{e}^{-2\beta}U^A\,, \\ g^{AB} &= \frac{1}{r^2}\gamma^{AB}\,, \end{aligned}$$

where γ^{AB} is the inverse metric to γ_{AB}.

Let us define the "covector" U_B as

$$U_B := g_{BA}U^A = r^2\gamma_{BA}U^A\,.$$

We have in our coordinate system the following non-vanishing components of the metric $g_{\mu\nu}$

$$\begin{aligned} g_{00} &= -\frac{V}{r}\mathrm{e}^{2\beta} + U_AU^A\,, \\ g_{03} &= -\mathrm{e}^{2\beta}\,, \\ g_{0A} &= -U_A\,, \\ g_{AB} &= r^2\gamma_{AB}\,. \end{aligned}$$

We also assume that

$$\sqrt{\det\gamma_{AB}} = \sin\theta\,.$$

The metric (7.1) implies the following expressions for $16\pi\pi^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and $A^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \delta^\lambda_{(\mu}\Gamma^\sigma{}_{\nu)\sigma}$ defined by (6.4) and (6.5)

$$\begin{aligned} \sqrt{-g} &= \mathrm{e}^{2\beta}r^2\sin\theta\,, \\ 16\pi\pi^{03} &= -r^2\sin\theta\,, \\ 16\pi\pi^{AB} &= \mathrm{e}^{2\beta}\sin\theta\gamma^{AB}\,, \\ 16\pi\pi^{33} &= rV\sin\theta\,, \\ 16\pi\pi^{3A} &= -r^2\sin\theta U^A\,, \\ 16\pi\pi^{03} &= -r^2\sin\theta\,, \\ 16\pi\pi^{AB} &= \mathrm{e}^{2\beta}\sin\theta\gamma^{AB}\,, \\ A^0{}_{33} &= A^0{}_{3A} = 0\,, \\ A^0{}_{03} &= -\beta_{,3} - \frac{1}{r}\,, \end{aligned}$$

$$
\begin{aligned}
A^0{}_{AB} &= \frac{1}{2}\mathrm{e}^{-2\beta}\left(r^2\gamma_{AB}\right)_{,3}\,, \\
A^3{}_{33} &= -\frac{2}{r}\,, \\
A^3{}_{3A} &= \frac{1}{2}\mathrm{e}^{-2\beta}U^B{}_{,3}g_{BA} - \frac{1}{2}\left(\ln\sin\theta\right)_{,A}\,, \\
A^3{}_{03} &= \frac{V}{r}\beta_{,3} + \left(\frac{V}{2r}\right)_{,3} - U^B\beta_{,B} - \dot{\beta} - \frac{1}{2}\mathrm{e}^{-2\beta}U_AU^A{}_{,3}\,, \\
A^3{}_{AB} &= \frac{1}{2}\mathrm{e}^{-2\beta}\left(\dot{g}_{AB} - \frac{V}{r}g_{AB,3} + U_{A||B} + U_{B||A}\right).
\end{aligned}
$$

The following expression below was proposed by Trautman and Bondi and will be called the TB mass:

$$
m_{\mathrm{TB}} := \frac{1}{8\pi}\int\limits_{\partial C} r - V\,. \tag{7.2}
$$

Choose a (3+1)-foliation of space-time and integrate (6.11) over a 3-dimensional null-volume $V \subset C = \{x^0 = \mathrm{const.}\}$

$$
\delta\int\limits_V L = \int\limits_V \left(\pi^{\mu\nu}\delta A^0_{\mu\nu}\right)^{\cdot} + \int\limits_{\partial V}\pi^{\mu\nu}\delta A^3_{\mu\nu}\,. \tag{7.3}
$$

Similarly as in the case of electrodynamics, we use here adapted coordinates; this means that the coordinate x^3 is constant on the boundary ∂V. Adapted coordinates simplify considerably derivation of the final formula. We stress, however, that all our results have an independent, geometric meaning. To rewrite them in a coordinate-independent form it is sufficient to replace "dots" by Lie derivatives $\mathcal{L}_X$, where X is the vector field generating our one-parameter group of transformations, which we are describing. In adapted coordinates $X := \frac{\partial}{\partial x^0}$. Moreover, the upper index "3" has to be replaced everywhere by the sign "$\perp$", denoting the transversal component with respect to the world tube. This way our results have a coordinate-independent meaning as relations between well defined geometric objects and not just their specific components.

Because the translation between these two notations is so simple, we have decided to use much simpler language, based on adapted coordinates. The volume part of the formula (7.3) can be simplified (or reduced) as follows

$$
\begin{aligned}
&16\pi\pi^{\mu\nu}\delta A^0_{\mu\nu} = 16\pi\pi^{kl}\delta A^0_{kl} + 32\pi\pi^{0k}\delta A^0_{0k} + 16\pi\pi^{00}\delta A^0_{00} \\
&= 32\pi\pi^{03}\delta A^0{}_{03} + 16\pi\pi^{AB}\delta A^0{}_{AB} \\
&= -\frac{1}{2}\sin\theta\left(r\gamma_{AB}\right)_{,3}\delta\left(r\gamma^{AB}\right) + \delta\left[2r^4\sin\theta\left(\frac{\beta}{r^2}\right)_{,3}\right].
\end{aligned} \tag{7.4}
$$

The last term in the above formula is a full variation of the quantity, which logarithmically diverges when we try to integrate it, $\beta = O(r^{-2})$ and $2r^4 \sin\theta \left(\frac{\beta}{r^2}\right)_{,3} = O(r^{-1})$. Removing of this term (we can call such procedure: the renormalization of the symplectic form) corresponds to the renormalization of the Lagrangian for scalar field (2.13).

On the other hand, the boundary part in (7.3) can be rewritten as

$$\begin{aligned}
&16\pi\pi^{\mu\nu}\delta {A^3}_{\mu\nu} = 16\pi\pi^{33}\delta {A^3}_{33} + 32\pi\pi^{03}\delta {A^3}_{03} + 32\pi\pi^{3A}\delta {A^3}_{3A} + 16\pi\pi^{AB}\delta {A^3}_{AB} \\
&= 2\sin\theta\left(2V - r^2 {U^B}_{||B}\right)\delta\beta + \sin\theta\gamma^{AB}\delta U_{A||B} - \frac{1}{2}\sin\theta\left(\dot{g}_{AB} - \frac{V}{r}g_{AB,3}\right)\delta\gamma^{AB} \\
&+ r^2\sin\theta \mathrm{e}^{-2\beta}{U^B}_{,3}g_{BA}\delta U^A - \delta\left[2r^2\sin\theta\left(\frac{V}{r^2} + \left(\frac{V}{2r}\right)_{,3} + \frac{V}{r}\beta_{,3} - \dot{\beta} - U^B\beta_{,B}\right)\right],
\end{aligned} \tag{7.5}$$

where by "$||$" we have denoted a covariant derivative with respect to the two-metric g_{AB} on ∂V.

Inserting these results into (7.3) we obtain

$$\begin{aligned}
&16\pi\delta\int_V L = -\int_V \tfrac{1}{2}\sin\theta\left[(r\gamma_{AB})_{,3}\,\delta\left(r\gamma^{AB}\right)\right]_{,0} + \int_{\partial V} r^2\sin\theta \mathrm{e}^{-2\beta}{U^B}_{,3}g_{BA}\delta U^A \\
&+ \int_{\partial V} 2\sin\theta\left(2V - r^2{U^B}_{||B}\right)\delta\beta + \tfrac{1}{2}\sin\theta\left(rV\gamma_{AB,3} - r^2\dot{\gamma}_{AB} - 2U_{A||B}\right)\delta\gamma^{AB} \\
&+ \delta\int_V 4r^2\sin\theta\dot{\beta}_{,3} - \delta\int_{\partial V} r^2\sin\theta\left[\frac{2V}{r^2} + \left(\frac{V}{r}\right)_{,3} + 2\frac{V}{r}\beta_{,3} - 2U^B\beta_{,B}\right]
\end{aligned} \tag{7.6}$$

because 2-dimensional divergencies "$\partial_A f^A$" vanish when integrated over the boundary ∂V.

From (7.4) we get the relation

$$16\pi\pi^{\mu\nu}\dot{A}^0_{\mu\nu} = -\frac{1}{2}\sin\theta\,(r\gamma_{AB})_{,3}\left(r\dot{\gamma}^{AB}\right) + 2r^4\sin\theta\left(\frac{\dot{\beta}}{r^2}\right)_{,3}. \tag{7.7}$$

On the other hand, from [25] we know that

$$\begin{aligned}
&16\pi\int_V \pi^{\mu\nu}\mathcal{L}_X A^0_{\mu\nu} = \int_{\partial V}\sqrt{-g}\left(\nabla^3 X^0 - \nabla^0 X^3\right) \\
&= \int_{\partial V} r^2\sin\theta\left[\left(\frac{V}{r}\right)_{,3} + 2\frac{V}{r}\beta_{,3} - 2U^B\beta_{,B} - 2\dot{\beta} - \mathrm{e}^{-2\beta}U_A U^A_{,3}\right] \\
&(= 2r^2\sin\theta A^3_{03}),
\end{aligned} \tag{7.8}$$

where the last equality can be checked directly for the metric (7.1) and $X^\mu = \delta^\mu_0$. From (7.7) and (7.8) we obtain the final formula

$$16\pi\delta \int_V L = \int_V \frac{1}{2}\sin\theta \left[\left(r\dot{\gamma}^{AB}\right)\delta\left(r\gamma_{AB}\right)_{,3} - \left(r\dot{\gamma}_{AB}\right)_{,3}\delta\left(r\gamma^{AB}\right)\right] - \delta\int_{\partial V} 2V\sin\theta$$
$$+\frac{1}{2}\int_{\partial V}\sin\theta\left(rV\gamma_{AB,3} - r^2\dot{\gamma}_{AB} - 2U_{A||B} + r^2 \mathrm{e}^{-2\beta}U^C_{,3}\gamma_{CA}U_B\right)\delta\gamma^{AB}$$
$$+\int_{\partial V} 2r^2\sin\theta\left(\frac{2V}{r^2} - U^B{}_{||B} + U_A U^A_{,3}\right)\delta\beta - r^2\sin\theta \mathrm{e}^{-2\beta}U_A\delta U^A_{,3}\,. \tag{7.9}$$

Remark. It seems to me that a more natural "control mode" in the above formula corresponds rather to the control of the term $(r^2U^A)_{,3}$ than $U^A_{,3}$ and it can be achieved by the following manipulation

$$-r^2\sin\theta \mathrm{e}^{-2\beta}U_A\delta U^A_{,3} = -\sin\theta \mathrm{e}^{-2\beta}U_A\delta(r^2U^A)_{,3} + \delta\left(r\sin\theta \mathrm{e}^{-2\beta}U_AU^A\right)$$
$$+2r\sin\theta \mathrm{e}^{-2\beta}U_AU^A\delta\beta + \frac{1}{r}\sin\theta \mathrm{e}^{-2\beta}U_AU^A\delta\gamma^{AB}\,.$$

It is convenient to introduce the following asymptotic variables (Π_{AB}, ψ^{AB}) related to asymptotic degrees of freedom

$$\begin{aligned}\psi^{AB} &:= r\gamma^{AB} - r\,\mathring{\gamma}^{AB}\,, \quad \psi_{AB} := r\gamma_{AB} - r\,\mathring{\gamma}_{AB}\,,\\ \Pi_{AB} &:= -\frac{1}{2}\sin\theta\left(r\gamma_{AB}\right)_{,3} + \frac{1}{2}\sin\theta\left(r\,\mathring{\gamma}_{AB}\right)_{,3}\,.\end{aligned}$$

If we pass to the limit, the formula (7.9) takes the form

$$\begin{aligned}-16\pi\delta m_{\mathrm{TB}} &= -\delta\int_{\partial C} 4M\sin\theta\\ &= \int_C \dot{\Pi}_{AB}\delta\psi^{AB} - \dot{\psi}^{AB}\delta\Pi_{AB} - \frac{1}{2}\int_{\partial C}\sin\theta\dot{\psi}_{AB}\delta\psi^{AB}\,,\end{aligned} \tag{7.10}$$

where $V = r - 2M + O(r^{-1})$ and the asymptotic conditions are given in [4] and will be summarized in the next section. We can denote the non-conservation law for the TB mass

$$-16\pi\partial_0 m_{\mathrm{TB}} = -\frac{1}{2}\int_{\partial C}\sin\theta\dot{\psi}_{AB}\dot{\psi}^{AB}\ \left(=\frac{1}{2}\int_{\partial C}\sin\theta\,\mathring{\chi}_{AB,u}\,\mathring{\chi}^{AB}{}_{,u}\right)\,, \tag{7.11}$$

where the last form in the brackets becomes clear when we learn about asymptotics presented in the next section. In particular $\psi_{AB}|_{\mathcal{J}^+} = \overset{\circ}{\chi}_{AB}$ and $\psi^{AB}|_{\mathcal{J}^+} = -\overset{\circ}{\chi}{}^{AB}$.

Similarly, for angular momentum we get the answer from the superpotential proposed by Komar [20]

$$16\pi \int\limits_V \pi^{\mu\nu} \mathcal{L}_X A^0_{\mu\nu} = \int\limits_{\partial V} \sqrt{-g} \left(\nabla^3 X^0 - \nabla^0 X^3 \right) ,$$

where now $X = \partial/\partial\phi$.

The right-hand side can be expressed in terms of the Bondi-Sachs type metric

$$\int\limits_{\partial V} \sqrt{-g} \left(\nabla^3 X^0 - \nabla^0 X^3 \right) = \int\limits_{\partial V} r^4 \sin\theta \mathrm{e}^{-2\beta} \gamma_{\phi A} U^A_{,3} \longrightarrow 16\pi J_z \,.$$

The limit is taken on $\mathcal{J}^+$ and according to the asymptotics presented in the next section we obtain

$$16\pi J_z = -\int\limits_{\partial C} (6N_\phi + \frac{1}{2} \overset{\circ}{\chi}_{\phi B} \overset{\circ}{\chi}{}^{BC}{}_{||C}) \sin\theta \mathrm{d}\theta \mathrm{d}\phi \,. \tag{7.12}$$

But on the other hand

$$16\pi \int\limits_V \pi^{\mu\nu} \mathcal{L}_X A^0_{\mu\nu} = \int\limits_V \pi^{\mu\nu} A^0_{\mu\nu,\phi} = \int\limits_V \varPi_{AB} \psi^{AB}{}_{,\phi}$$

and

$$16\pi\partial_0 \int\limits_C \pi^{\mu\nu} A^0_{\mu\nu,\phi} = \int\limits_C \dot{\varPi}_{AB} \psi^{AB}{}_{,\phi} - \varPi_{AB,\phi} \dot{\psi}^{AB} = \frac{1}{2} \int\limits_{\partial C} \sin\theta \dot{\psi}_{AB} \psi^{AB}{}_{,\phi} \,.$$

We will show in the next section that the non-conservation law for angular momentum agrees in terms of the asymptotics

$$16\pi \dot{J}_z = -\int\limits_{\partial C} \frac{1}{2} \overset{\circ}{\chi}_{AB,u} \overset{\circ}{\chi}{}^{AB}{}_{,\phi} \sin\theta \mathrm{d}\theta \mathrm{d}\phi \,. \tag{7.13}$$

7.1. Symplectic structure on scri

Let us observe that we can use the previous results (7.4) and (7.5) to reduce the form

$$\pi^{\mu\nu}\delta A^{v}_{\mu\nu} = \pi^{\mu\nu}\delta(A^{0}_{\mu\nu} + 2A^{3}_{\mu\nu})\,. \tag{7.14}$$

Let us also remind the coordinate system which should be used to describe the situation in a similar way as in Section 2.3 for scalar field and 3.1 for electrodynamics. $(u,r) \to (v,\overline{u})$, $\overline{u} = -2r$ $v = u + 2r$, $\partial_u = \partial_v$, $\partial_r = -2\partial_{\overline{u}} + \partial_v$, $\mathrm{d}u \wedge \mathrm{d}r = \frac{1}{2}\mathrm{d}\overline{u} \wedge \mathrm{d}v$ and finally $\pi^{\mu\nu}\delta A^{v}_{\mu\nu}\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi = \frac{1}{2}\pi^{\mu\nu}\delta A^{v}_{\mu\nu}\mathrm{d}\overline{u}\mathrm{d}\theta\mathrm{d}\phi$.

If we put

$$16\pi\pi^{\mu\nu}\delta A^{0}_{\mu\nu} = -\frac{1}{2}\sin\theta\,(r\gamma_{AB})_{,3}\,\delta\left(r\gamma^{AB}\right) + \delta\left[2r^4\sin\theta\left(\frac{\beta}{r^2}\right)_{,3}\right] \tag{7.15}$$

and

$$\begin{aligned} 16\pi\pi^{\mu\nu}\delta A^{3}{}_{\mu\nu} &= 2\sin\theta\left(2V - r^2 U^B{}_{||B}\right)\delta\beta + \sin\theta\gamma^{AB}\delta U_{A||B} \\ &-\frac{1}{2}\sin\theta\left(\dot{g}_{AB} - \frac{V}{r}g_{AB,3}\right)\delta\gamma^{AB} + r^2\sin\theta\mathrm{e}^{-2\beta}U^B{}_{,3}g_{BA}\delta U^A \\ &-\delta\left[2r^2\sin\theta\left(\frac{V}{r^2} + \left(\frac{V}{2r}\right)_{,3} + \frac{V}{r}\beta_{,3} - \dot{\beta} - U^B\beta_{,B}\right)\right] \end{aligned} \tag{7.16}$$

into (7.14), assuming asymptotic behaviour on $\mathcal{J}^+$, we obtain the following formula at the future null infinity

$$16\pi\pi^{\mu\nu}\delta A^{v}{}_{\mu\nu}|_{\mathcal{J}^+} = -\sin\theta\dot{\psi}_{AB}\delta\psi^{AB} + 4\delta(\sin\theta M)\,. \tag{7.17}$$

Remark. The symplectic form

$$\int\limits_{\mathcal{J}^+} \mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi\sin\theta\delta\psi^{AB} \wedge \delta\dot{\psi}_{AB}$$

has been considered by Ashtekar *et al.*, see [31, 32] and [33]. Their reformulation in a conformally geometric way (in the spirit of the "universal structure" of Penrose's null infinity) has given the symplectic structure on the space of radiative modes of the non-linear gravitational field in exact general relativity.

Let $N = [u_i, u_f] \times S^2 \subset \mathcal{J}^+$ be here a "finite piece" of $\mathcal{J}^+$. The relation for the TB mass is based on the following observations. First of all from (7.17) we obtain

$$16\pi\int\limits_N \frac{1}{2}\pi^{\mu\nu}A^{v}{}_{\mu\nu,0}\mathrm{d}\overline{u}\mathrm{d}\theta\mathrm{d}\phi = -\frac{1}{2}\int\limits_N \sin\theta\dot{\psi}_{AB}\dot{\psi}^{AB}\mathrm{d}\overline{u}\mathrm{d}\theta\mathrm{d}\phi + 2\int\limits_{\partial N} M\sin\theta\mathrm{d}\theta\mathrm{d}\phi$$

and secondly

$$16\pi \int\limits_N \pi^{\mu\nu} \mathcal{L}_X A^v{}_{\mu\nu} = \int\limits_{\partial N} \sqrt{-g}(\nabla^{\overline{u}} X^v - \nabla^v X^{\overline{u}}) = - \int\limits_{\partial N} 2M \sin\theta \mathrm{d}\theta \mathrm{d}\phi \,,$$

where $X = \partial_0$, so finally

$$-4 \int\limits_{\partial N} M \sin\theta \mathrm{d}\theta \mathrm{d}\phi = -\frac{1}{2} \int\limits_N \sin\theta \dot{\psi}_{AB} \dot{\psi}^{AB} \mathrm{d}\overline{u} \mathrm{d}\theta \mathrm{d}\phi \,.$$

The left-hand side of the above formula represents the change of Bondi mass from initial state u_i to final state u_f ($\partial N = \{u_f\} \times S^2 \cup \{u_i\} \times S^2$) but the right-hand side is a flux of the energy through N which is a piece of $\mathcal{J}^+$ between initial and final state, compare with (8.13) and (8.14).

Similarly, for angular momentum we have

$$\begin{aligned} 16\pi \int\limits_N \pi^{\mu\nu} \mathcal{L}_X A^v{}_{\mu\nu} &= 16\pi \int\limits_N \frac{1}{2} \pi^{\mu\nu} A^v{}_{\mu\nu,\phi} \mathrm{d}\overline{u} \mathrm{d}\theta \mathrm{d}\phi \\ &= -\frac{1}{2} \int\limits_N \sin\theta \dot{\psi}_{AB} \psi^{AB}_{,\phi} \mathrm{d}\overline{u} \mathrm{d}\theta \mathrm{d}\phi \,, \end{aligned}$$

where now $X = \partial_\phi$.

8. Multipole structure of Bondi–van der Burg–Metzner–Sachs equations

The metric (7.1) depends on the six functions: V, β, γ_{AB}, U^A and the asymptotic behaviour of them is described in [3] and [4]. We shall rewrite formulae from van der Burg paper [4] in a "spherically covariant" way. More precisely, we denote:

0. M i V are scalars

1. Pairs of functions U, W and N, P can be combined in two vectors U^A and N^A, respectively

$$\begin{aligned} U_\theta &= U^\theta = U \,, \\ U_\phi &= \sin^2\theta U^\phi = W \sin\theta \,, \\ N_\theta &= N^\theta = N \,, \\ N_\phi &= \sin^2\theta N^\phi = P \sin\theta \,. \end{aligned}$$

2. Pairs of functions c, d, C, H and D, K correspond to the symmetric traceless tensors $\overset{\circ}{\chi}_{AB}$, C_{AB} and D_{AB}:

$$\begin{aligned} \overset{\circ}{\chi}{}^\theta{}_\theta &= -\overset{\circ}{\chi}{}^\phi{}_\phi = 2c \,, \\ \overset{\circ}{\chi}{}^\theta{}_\phi &= \sin^2\theta \, \overset{\circ}{\chi}{}_\theta{}^\phi = 2d \sin\theta \,. \end{aligned}$$

Similarly $C^\theta{}_\theta = C$, $D^\theta{}_\theta = D$ etc. The reason for this notation arises in a natural way, if we change the parameterization of the 2-dimensional metric γ_{AB}. Let us remind that van der Burg in [4] (p. 112) proposed the following parameterization

$$\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B = e^{2\gamma}\cosh(2\delta)\mathrm{d}\theta^2 + 2\sinh(2\delta)\sin\theta\mathrm{d}\theta\mathrm{d}\phi + e^{-2\gamma}\cosh(2\delta)\sin^2\theta\mathrm{d}\phi^2, \tag{8.1}$$

which differs from original Sachs formulation by a linear transformation of the functions γ and δ (see [3] p. 107). Next the used functions γ and δ are expanded in the form

$$\gamma = c/r + \left(C - \frac{1}{6}c^3 - \frac{3}{2}cd^2\right)r^{-3} + Dr^{-4} + O(r^{-5}),$$

$$\delta = d/r + \left(H - \frac{1}{6}d^3 + \frac{1}{2}c^2 d\right)r^{-3} + Kr^{-4} + O(r^{-5}).$$

Let us notice that there is no r^{-2} term, which was analyzed in [10], and vanishing of this term is called "outgoing radiation condition".

We propose to change this parameterization in such a way that for the original Bondi axi-reflection-symmetric metric both formulations are the same. The main advantage of our change is that the expansion terms take a nice geometric form (mainly the term of order r^{-3} takes a nice form).

Let us fix the frame $\mathrm{d}\theta$, $\sin\theta\mathrm{d}\phi$ which is orthonormal with respect to the background metric $\overset{\circ}{\gamma}_{AB}$. The symmetric matrix (close to unity)

$$\begin{pmatrix} e^{2\gamma}\cosh(2\delta) & \sinh(2\delta) \\ \sinh(2\delta) & e^{-2\gamma}\cosh(2\delta) \end{pmatrix} \tag{8.2}$$

with the determinant equal 1 can be also parameterized in a natural way by the exponential mapping

$$\exp(a\sigma_x + b\sigma_z),$$

where σ_x and σ_z are Pauli matrices

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The solution of the matrix equation

$$\begin{pmatrix} e^{2\gamma}\cosh(2\delta) & \sinh(2\delta) \\ \sinh(2\delta) & e^{-2\gamma}\cosh(2\delta) \end{pmatrix} = \exp(a\sigma_x + b\sigma_z) \tag{8.3}$$

leads to the nonlinear relation between a, b and γ, δ in the form

$$a = \sinh(2\gamma)\cosh(2\delta)\frac{\operatorname{arccosh}(\cosh(2\delta)\cosh(2\gamma))}{\sqrt{\sinh^2(2\delta) + \cosh^2(2\delta)\sinh^2(2\gamma)}}$$

$$b = \sinh(2\delta)\frac{\operatorname{arccosh}(\cosh(2\delta)\cosh(2\gamma))}{\sqrt{\sinh^2(2\delta)+\cosh^2(2\delta)\sinh^2(2\gamma)}}$$

but the asymptotic relation for small γ, δ is simpler, namely

$$a = 2\gamma + \frac{8}{3}\gamma\delta^2 + O(\gamma,\delta)^5\,,$$

$$b = 2\delta + \frac{4}{3}\gamma^2\delta + O(\gamma,\delta)^5\,,$$

and it gives only a correction in r^{-3} for our expansion. More precisely,

$$\frac{1}{2}a = c/r + \left(C - \frac{1}{6}(c^2+d^2)c\right)r^{-3} + Dr^{-4} + O(r^{-5})\,,$$

$$\frac{1}{2}b = d/r + \left(H - \frac{1}{6}(d^2+c^2)d\right)r^{-3} + Kr^{-4} + O(r^{-5})\,,$$

and now we can write the expansion in the matrix form

$$\log\gamma_{AB} = \overset{\circ}{\chi}_{AB}/r + \left(2C_{AB} - \frac{1}{48}\overset{\circ}{\chi}_{CD}\overset{\circ}{\chi}{}^{CD}\overset{\circ}{\chi}_{AB}\right)r^{-3} + 2D_{AB}r^{-4} + O(r^{-5})\,, \tag{8.4}$$

where each term of the expansion is a traceless symmetric tensor on a sphere. The indices are raised with respect to the inverse $\overset{\circ}{\gamma}{}^{AB}$ of the background metric (which is a standard metric on a unit sphere). It is diagonal in our coordinates, $\overset{\circ}{\gamma}_{\theta\theta} = 1$ and $\overset{\circ}{\gamma}_{\phi\phi} = \sin^2\theta$. The metric connection of the $\overset{\circ}{\gamma}_{AB}$ has the following non-vanishing components

$$\Gamma^{\theta}{}_{\phi\phi} = -\sin\theta\cos\theta\,, \quad \Gamma^{\phi}{}_{\phi\theta} = \cot\theta\,.$$

We are ready to show the asymptotic expansions for the rest of the quantities, which appear in Bondi–Sachs type metric (7.1). They were introduced in [4] (p.114) but now we can rewrite them in a covariant way on S^2

$$\begin{aligned}
U^A &= -\frac{1}{2r^2}\overset{\circ}{\chi}{}^{AB}{}_{||B} + \frac{2N^A}{r^3} + \frac{1}{r^3}\left[\frac{1}{2}\overset{\circ}{\chi}{}^{A}{}_{B}\overset{\circ}{\chi}{}^{BC}{}_{||C} + \frac{1}{16}\left(\overset{\circ}{\chi}_{CD}\overset{\circ}{\chi}{}^{CD}\right)^{||A}\right],\\
U_A :&= r^2\gamma_{AB}U^B = -\frac{1}{2}\overset{\circ}{\chi}{}^{B}{}_{A||B} + \frac{2N_A}{r} + \frac{1}{16r}\left(\overset{\circ}{\chi}_{CD}\overset{\circ}{\chi}{}^{CD}\right)_{||A},\\
1-\frac{V}{r} &= \frac{2M}{r} + \frac{N^A{}_{||A}}{r^2} - \frac{1}{r^2}\left[\frac{1}{4}\overset{\circ}{\chi}{}^{AB}{}_{||B}\overset{\circ}{\chi}_{A}{}^{C}{}_{||C} + \frac{1}{16}\overset{\circ}{\chi}{}^{CD}\overset{\circ}{\chi}_{CD}\right],\\
\beta &= -\frac{1}{32}\cdot\frac{1}{r^2}\overset{\circ}{\chi}_{AB}\overset{\circ}{\chi}{}^{AB}\,.
\end{aligned}$$

Basic equations (Eq. (13)–(15) in [4]) can be expressed as follows

$$M_{,u} = -\frac{1}{8}\overset{\circ}{\chi}_{AB,u}\overset{\circ}{\chi}{}^{AB}{}_{,u} + \frac{1}{4}\overset{\circ}{\chi}{}^{AB}{}_{||AB,u}\,, \tag{8.5}$$

$$\begin{aligned} 3N^A{}_{,u} = & -M^{||A} - \frac{1}{4}\overset{\circ}{\varepsilon}{}^{AB}\left(\overset{\circ}{\chi}{}_C{}^D{}_{||DE}\overset{\circ}{\varepsilon}{}^{EC}\right)_{||B} \\ & -\frac{3}{4}\overset{\circ}{\chi}{}^A{}_B\overset{\circ}{\chi}{}^{BC}{}_{||C,u} - \frac{1}{4}\overset{\circ}{\chi}{}^{CD}{}_{,u}\overset{\circ}{\chi}{}^A{}_{C||D}\,. \end{aligned} \tag{8.6}$$

The "dynamics" of the further asymptotic hierarchy (Eq. (8)–(9) and (11)–(12) in [4]) takes the form

$$\begin{aligned} & -4\dot{C}_{AB} - \frac{1}{8}\overset{\circ}{\chi}{}^{CD}\overset{\circ}{\chi}_{CD}\overset{\circ}{\chi}_{AB,u} + \frac{1}{4}\overset{\circ}{\chi}_{AB}\overset{\circ}{\chi}{}^{CD}\overset{\circ}{\chi}_{CD,u} \\ = \; & N_{A||B} + N_{B||A} - \overset{\circ}{\gamma}_{AB}N^C{}_{||C} \\ & -M\overset{\circ}{\chi}_{AB} - \frac{1}{4}\overset{\circ}{\varepsilon}_{AC}\overset{\circ}{\chi}{}^C{}_B\overset{\circ}{\chi}{}_E{}^F{}_{||FG}\overset{\circ}{\varepsilon}{}^{EG}\,, \end{aligned} \tag{8.7}$$

$$\begin{aligned} & -4\dot{D}_{AB} = (\overset{\circ}{\Delta} + 4)C_{AB} - (\overset{\circ}{\chi}{}_A{}^C N_B)_{||C} \\ & -(\overset{\circ}{\chi}{}_B{}^C N_A)_{||C} + \overset{\circ}{\gamma}_{AB}(\overset{\circ}{\chi}{}^{CD}N_D)_{||C}\,. \end{aligned} \tag{8.8}$$

Let us observe that mono-dipole but also *quadrupole* part of the right-hand side of (8.8) vanishes. More precisely, let us rewrite this equation in the following way

$$-4\dot{D}_{AB} = (\overset{\circ}{\Delta} + 4)C_{AB} - S_A{}^C{}_{B||C}\,, \tag{8.9}$$

where

$$S_{ABC} := \overset{\circ}{\chi}_{AC}N_B + \overset{\circ}{\chi}_{BC}N_A - \overset{\circ}{\gamma}_{AB}\overset{\circ}{\chi}_{CD}N^D\,.$$

It is easy to check that S_{ABC} is a traceless symmetric tensor (in each pair of indices) and the same holds for $\hat{S}_{ABC} := \varepsilon_{AD}S^D{}_{BC}$. One can prove (see [30]) that $S^{ABC}{}_{||CAB}$ and $\hat{S}^{ABC}{}_{||CAB}$ are orthogonal to the first three eigenvalue spherical harmonics (with $l = 0, 1, 2$).

On the other hand, from the relation

$$C^{AB||C}{}_{CAB} = C^{AB}{}_{||ABC}{}^C + 2C^{AB}{}_{||AB}$$

we obtain that

$$\left[(\overset{\circ}{\Delta} + 4)C^{AB}\right]_{||AB} = (\overset{\circ}{\Delta} + 6)\left(C^{AB}{}_{||AB}\right)$$

and Similarly for $\hat{C}_{AB} := \varepsilon_{AD}C^D{}_B$.

This way we get 10-dimensional space of quadrupole Newman–Penrose charges in D_{AB} which are conserved ([14], [4]). More precisely, quadrupole (and also mono-dipole) parts of $\partial_u D^{AB}{}_{||AB}$ and $\partial_u \hat{D}^{AB}{}_{||AB}$ have to vanish. However, for the polyhomogeneous asymptotics they may be not conserved (see [10]).

Let $v = u + 2r$, than the metric (7.1) takes the following form

$$\begin{aligned} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \;=\; & \left(-\frac{V}{r} + r^2\gamma_{AB}U^AU^B + \mathrm{e}^{2\beta}\right)\mathrm{d}u^2 - \mathrm{e}^{2\beta}\mathrm{d}u\mathrm{d}v \\ & -2r^2\gamma_{AB}U^B\mathrm{d}u\mathrm{d}x^A + r^2\gamma_{AB}\mathrm{d}x^A\mathrm{d}x^B \end{aligned} \tag{8.10}$$

and its linearization can be expressed as follows

$$\begin{aligned} H &\cong \frac{1}{2}\cdot\frac{1}{r^2}\cdot \overset{\circ}{\chi}_{AB}\,\overset{\circ}{\chi}{}^{AB} \cong 0\,, \\ h_{uu} &\cong 1 - \frac{V}{r} + 2\beta + r^2\,\overset{\circ}{\gamma}_{AB}U^AU^B \cong \frac{2M}{r} + \frac{N^A{}_{||A}}{r^2}\,, \\ h_{uv} &\cong -\beta \cong 0\,, \\ h_{uA} &\cong -r^2\,\overset{\circ}{\gamma}_{AB}U^B - r\,\overset{\circ}{\chi}_{AB}U^B \cong -\frac{1}{2}\,\overset{\circ}{\chi}{}^B{}_{A||B} + \frac{2N_A}{r}\,, \\ \chi_{AB} &\cong r\,\overset{\circ}{\chi}_{AB}, \quad h_{AB} \cong r\,\overset{\circ}{\chi}_{AB} + \frac{1}{2}r^2\,\overset{\circ}{\gamma}_{AB}H \cong r\,\overset{\circ}{\chi}_{AB}\,. \end{aligned}$$

The linearized asymptotics of invariants

$$\begin{aligned} \mathbf{x} &\cong \frac{4M}{r} + \frac{6N^A{}_{||A}}{r^2}\,, \\ \mathbf{y} &\cong -\frac{1}{r}\,\overset{\circ}{\chi}_A{}^C{}_{||CB}\,\overset{\circ}{\varepsilon}{}^{AB} + \frac{6}{r^2}N_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB}\,, \\ \varPsi_x = r\mathbf{x} &\cong 4M\,, \quad \dot{\varPsi}_x = r\mathbf{x}_{,u} \cong \overset{\circ}{\chi}{}^{AB}{}_{||AB,u}\,, \\ \varPsi_y = r\mathbf{y} &\cong -\,\overset{\circ}{\chi}_A{}^C{}_{||CB}\,\overset{\circ}{\varepsilon}{}^{AB}, \quad \dot{\varPsi}_y = r\mathbf{y}_{,u} \cong -\left(\overset{\circ}{\chi}_A{}^C{}_{||CB}\,\overset{\circ}{\varepsilon}{}^{AB}\right)_{,u}\,, \end{aligned}$$

give an indication, how to relate linearized theory to the van der Burg asymptotics. This observation will be used in the sequel.

8.1. Supertranslations

Let us consider $\mathcal{J}^+$ as the cartesian product $S^2 \times \mathbb{R}^1$ or rather trivial affine bundle over S^2 with typical fiber $\mathbb{R}^1$; in some more general situations $\mathbb{R}^1$ may be replaced by an interval I. The supertranslation corresponds to the zero section of this affine bundle. On the other hand the boost transformation leads to the nontrivial scaling factor in a fiber and a conformal transformation on a base manifold S^2 (see [3] p. 111).

A prolongation of the supertranslation from scri "to the interior" in Bondi coordinates (metric (7.1)) leads to the following asymptotic relations (see also [3] p. 119)

$$\begin{aligned}
\overline{x}^A &= x^A + \frac{1}{r}\alpha^{||A} - \frac{1}{2r^2}\left(\overset{\circ}{\chi}{}^{AB}\alpha_{||B} - 2\alpha^{||AB}\alpha_{||B} + \Gamma^A{}_{BC}\alpha^{||B}\alpha^{||C} \right) + \dots\,, \\
\overline{u} &= u - \alpha - \frac{1}{2r}\alpha^{||A}\alpha_{||A} + \frac{1}{4r^2}\left[\overset{\circ}{\chi}{}^{AB}\alpha_{||A}\alpha_{||B} - \alpha^{||A}\left(\alpha_{||B}\alpha^{||B}\right)_{||A} \right] + \dots\,, \\
\overline{r} &= r - \frac{1}{2}\overset{\circ}{\Delta}\alpha + \frac{1}{2r}\left[\overset{\circ}{\chi}{}^{AB}{}_{||B}\alpha_{||A} + \frac{1}{2}\overset{\circ}{\chi}{}^{AB}\alpha_{||AB} + \frac{1}{2}\overset{\circ}{\chi}{}^{AB}{}_{,u}\alpha_{||A}\alpha_{||B} \right. \\
&\quad \left. -\frac{1}{2}\alpha^{||AB}\alpha_{||AB} - \alpha_{||A}\alpha^{||A} + \frac{1}{4}(\overset{\circ}{\Delta}\alpha)^2 - \left(\overset{\circ}{\Delta}\alpha\right)^{||A}\alpha_{||A} \right] + \dots\,.
\end{aligned}$$

Now we can check the transformation law for $\overset{\circ}{\chi}$ and M

$$\begin{aligned}
\overline{M} &= M + \frac{1}{2}\overset{\circ}{\chi}{}^{AB}{}_{||B,u}\alpha_{||A} + \frac{1}{4}\overset{\circ}{\chi}{}^{AB}_{,u}\alpha_{||AB} + \frac{1}{4}\overset{\circ}{\chi}{}^{AB}{}_{,uu}\alpha_{||A}\alpha_{||B}\,, \\
\overline{\overset{\circ}{\chi}}_{AB} &= \overset{\circ}{\chi}_{AB} - 2\alpha_{||AB} + \overset{\circ}{\gamma}_{AB}\overset{\circ}{\Delta}\alpha\,, \\
\overline{\partial_A} &= \partial_A + \alpha_{,A}\partial_0
\end{aligned}$$

and finally we obtain that certain combination

$$\overline{4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}} = 4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB} + \overset{\circ}{\Delta}(\overset{\circ}{\Delta} + 2)\alpha \tag{8.11}$$

has a simple transformation law with respect to the supertranslations. Moreover, mono-dipole part of $4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}$ is invariant with respect to the supertranslations. It corresponds to the mass and linear momentum at null infinity. We would like to stress that in general the definition (7.2) is correct only on a $\{u = \text{const.}\}$ cross-section of $\mathcal{J}^+$. Let us consider any (cross-)section $s : S^2 \longrightarrow \mathcal{J}^+$ of the affine bundle $\mathcal{J}^+$.

Definition

$$16\pi m_{\mathrm{TB}} : = \int\limits_{S^2} \left(4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}\right)(s(\theta,\phi))\sin\theta \mathrm{d}\theta\mathrm{d}\phi\,, \tag{8.12}$$

$$16\pi p^k : = \int\limits_{S^2} \left(4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}\right)(s(\theta,\phi))\frac{z^k}{r}\sin\theta \mathrm{d}\theta\mathrm{d}\phi\,. \tag{8.13}$$

The above definition together with (8.11) gives the following:

<u>Theorem 1.</u> The energy-momentum 4-vector at null infinity is invariant with respect to the supertranslations.

On the other hand, the angular momentum defined by (7.12) is not invariant with respect to the supertranslations but obeys the following transformation law

$$16\pi\overline{J}_z = 16\pi J_z + \int_{S^2} 4M\alpha_{,\phi}\,.$$

The definition (8.13) allows us to define the flux of the energy through the piece of $\mathcal{J}^+$ between any two cross-sections of the null infinity in a supertranslation-invariant way. More precisely, let

$$s_i : S^2 \longrightarrow \mathcal{J}^+$$

for $i = 1, 2$ be two different cross-sections of $\mathcal{J}^+$ such that there exists $N \subset \mathcal{J}^+$ with $\partial N = s_2(S^2) \cup s_1(S^2)$. Then one can easily check from the definition (8.13) and the relation (8.5) that

$$\begin{aligned} m_{\mathrm{TB}}(s_2) - m_{\mathrm{TB}}(s_1) &= \frac{1}{16\pi}\int_{\partial N}\left(4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}\right)\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\ &= -\frac{1}{32\pi}\int_{N}\overset{\circ}{\chi}{}_{AB,u}\,\overset{\circ}{\chi}{}^{AB}{}_{,u}\sin\theta\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi \\ &= (\text{flux of energy through } N)\,. \end{aligned} \tag{8.14}$$

Similar formula holds for linear momentum p_k defined by (8.13)

$$\begin{aligned} p^k(s_2) - p^k(s_1) &= \frac{1}{16\pi}\int_{\partial N}\left(4M - \overset{\circ}{\chi}{}^{AB}{}_{||AB}\right)\frac{z^k}{r}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\ &= -\frac{1}{32\pi}\int_{N}\overset{\circ}{\chi}{}_{AB,u}\,\overset{\circ}{\chi}{}^{AB}{}_{,u}\frac{z^k}{r}\sin\theta\mathrm{d}u\mathrm{d}\theta\mathrm{d}\phi \\ &= (\text{flux of linear momentum through } N)\,. \end{aligned} \tag{8.15}$$

<u>Remark</u>. The supertranslation gauge freedom exists also in the linearized theory. The linearized part of the supertranslation corresponds to the gauge condition which preserves five components of the linearized metric: h_{ur}, H, h_{rr}, h_{rA}. More precisely, it is a solution of the gauge conditions

$$\begin{aligned} &\xi^u{}_{,r} = 0\,, \quad \xi^{u||A}{}_A = (\xi^A{}_{||A})_{,r}\,, \quad (\xi_{A||B}\varepsilon^{AB})_{,r} = 0\,, \\ &\xi^A{}_{||A} + \frac{2}{r}\xi^r = 0\,, \quad \xi^r{}_{,r} + \xi^u{}_{,u} = 0\,, \end{aligned}$$

where we use here the Minkowski background metric in the form

$$\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + r^2 \overset{\circ}{\gamma}_{AB}\mathrm{d}x^A\mathrm{d}x^B\,.$$

The solution of the above gauge equations is quite simple

$$\xi^A = \frac{1}{r}\overset{\circ}{\gamma}{}^{AB}\alpha_{||B}\,,\quad \xi^u = -\alpha\,,\quad \xi^r = -\frac{1}{2}\overset{\circ}{\Delta}\alpha\,,$$

where α is any real mapping $\alpha : S^2 \mapsto R$ and mono-dipole part of α corresponds to the usual translations in Minkowski space. The gauge transformation for traceless symmmetric tensor χ_{AB}

$$\chi_{AB} \longrightarrow \chi_{AB} - 2r\alpha_{||AB} + r\overset{\circ}{\gamma}_{AB}\overset{\circ}{\Delta}\alpha$$

is similar to the nonlinear case.

Proposition. How to remove the supertranslation gauge freedom? Assume at time u_0 that $4\underline{M} - \overset{\circ}{\chi}{}^{AB}{}_{||AB} = 0$, then for example the stationary solution becomes a simple stationary solution (the definition is given in Subsection 8.4). This procedure allows to treat $\overset{\circ}{\chi}{}^{AB}$ as the invariant asymptotic degree of freedom and will be considered in Subsection 8.4.

Remark. The Kerr-Newman metric in certain Bondi-Sachs coordinates can be asymptotically represented in such a way that $\underline{M} = 0 = \overset{\circ}{\chi}_{AB}$.

8.2. Hierarchy of asymptotic solution on scri for scalar wave equation

Let us rewrite the wave equation in null coordinates (u, v)

$$\rho^{-1}(\rho^{-1}\varphi)^{,a}{}_a + \overset{\circ}{\Delta}\varphi = 0 \tag{8.16}$$

and suppose we are looking for a solution of the wave equation (8.16) as a series

$$\varphi = \varphi_1\rho + \varphi_2\rho^2 + \varphi_3\rho^3 + \dots\,, \tag{8.17}$$

where each φ_n is a function on scri, $\partial_v\varphi_n = 0$.

If we put the series (8.17) into the wave equation (8.16), we obtain the following recursion

$$\partial_u\varphi_{n+1} = -\frac{1}{2n}[\overset{\circ}{\Delta} + (n-1)n]\varphi_n\,. \tag{8.18}$$

Compare with equations 2, 3, 4 in [2].

Remark. The kernel of the operator $[\overset{\circ}{\Delta} + l(l+1)]$ corresponds to the

l-th spherical harmonics. The right-hand side of (8.18) vanishes on the $n-1$ spherical harmonics subspace. This means that the corresponding multipole in φ_{n+1} does not depend on u. In particular, for $n = 3$ we have quadrupole charge in the fourth order. The nonlinear counterpart of this object is called *Newman-Penrose constant* or *NP charge*. We discuss some features related to NP charges in Section 8.5.

8.3. Linear theory, asymptotic hierarchy, "charges"

Let us first check that the linearized theory can be obtained, if we remove nonlinear terms in the asymptotic hierarchy (8.6)–(8.8)

$$\begin{aligned}
4\dot{M} &= \overset{\circ}{\chi}{}^{AB}{}_{||AB,u}\,, \\
3\dot{N}^A{}_{||A} &= -\overset{\circ}{\Delta} M \quad 3\dot{N}_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB} = -\frac{1}{4}\overset{\circ}{\Delta}\hat{\chi}^{AB}{}_{||AB}\,, \\
-4\dot{C}^{AB}{}_{||AB} &= (\overset{\circ}{\Delta}+2)N^A{}_{||A} \quad -4\hat{C}^{AB}{}_{||AB,u} = (\overset{\circ}{\Delta}+2)N_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB}\,, \\
-4\dot{D}^{AB}{}_{||AB} &= (\overset{\circ}{\Delta}+6)C^{AB}{}_{||AB} \quad -4\hat{D}^{AB}{}_{||AB,u} = (\overset{\circ}{\Delta}+6)\hat{C}^{AB}{}_{||AB}\,, \\
\mathbf{x} &= 4M\rho + 6N^A{}_{||A}\rho^2 + 6C^{AB}{}_{||AB}\rho^3 + 4D^{AB}{}_{||AB}\rho^4 + O(\rho^5)\,, \\
\mathbf{y} &= \hat{\chi}^{AB}{}_{||AB}\rho + 6N_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB}\rho^2 + 6\hat{C}^{AB}{}_{||AB}\rho^3 + 4\hat{D}^{AB}{}_{||AB}\rho^4 + O(\rho^5)\,.
\end{aligned}$$

It is easy to verify full agreement with (8.18) up to the 4-th order for the invariants $\mathbf{x}$, $\mathbf{y}$. Moreover, let us define $m := \mathrm{mon}(M)$, $p := \mathrm{dip}(M)$. Then we get

$$M = m + 3p + \underline{M}\,, \quad 4\dot{\underline{M}} = \overset{\circ}{\chi}{}^{AB}{}_{||AB,u}\,, \quad \dot{m} = \dot{p} = 0$$

and Similarly for N^A we obtain

$$N^A = -p^{||A}u - k_0^{||A} - \overset{\circ}{\varepsilon}{}^{AB} s_{||B} + \underline{N}^A\,,$$

where p, k_0 and s are dipoles. This way we have reconstructed "charges". Let us notice that the mono-dipole parts of invariants

$$\begin{aligned}
\mathbf{x} &= 4m\rho + 12 j^{l0} x_l \cdot \rho^3 + 12 p^l x_l \cdot \rho^3 \cdot \left(u + \frac{1}{\rho}\right)\,, \\
\mathbf{y} &= 12 s^l x_l \cdot \rho^3
\end{aligned}$$

or

$$\mathbf{x} = 4(m+3p)\cdot\rho + 12(k_0 + p\cdot u)\cdot\rho^2\,, \qquad \mathbf{y} = 12 s\cdot\rho^2$$

are the same as in (4.54) and (4.55).

8.3.1. Nonradiating solutions

Suppose $\int_{S^2} \dot{M} = 0$, then from asymptotic equation (8.5) we know that

$$0 = \int_{S^2} \dot{M} = -\frac{1}{8} \int_{S^2} \overset{\circ}{\chi}_{AB,u} \overset{\circ}{\chi}^{AB}{}_{,u}$$

so we get $\overset{\circ}{\chi}_{AB,u} = 0$, and finally also $\dot{M} = 0$. Moreover, equation (8.6) gives the relation

$$3\dot{N}_{A||B} \overset{\circ}{\varepsilon}{}^{AB} = \frac{1}{4}\hat{\chi}^{CD}{}_{||CD}$$

so the dipole part $\mathrm{dip}(\dot{N}_{A||B} \overset{\circ}{\varepsilon}{}^{AB})$ vanishes and this means that the angular momentum is conserved. This way we have proved the theorem formulated at the end of the Subsection 2.2, namely:

Theorem 2. If the TB mass is conserved than angular momentum is conserved too.

The general solution of this type (namely $\dot{M} = 0 = \overset{\circ}{\chi}_{AB,u}$) will be called *nonradiating solution* and it has the form

$$\begin{aligned} M &= m + 3p + \underline{M}\,, \\ N^A &= -p^{||A}u - k_0^{||A} - \overset{\circ}{\varepsilon}{}^{AB} s_{||B} + \tilde{N}^A + \frac{u}{3}\left(\frac{1}{4}\overset{\circ}{\varepsilon}{}^{AB}\hat{\chi}^{CD}{}_{||CDB} - \underline{M}^{||A}\right), \end{aligned}$$

where $\underline{\tilde{N}}^A = \tilde{N}^A$ and $\tilde{N}^A_{,u} = 0$. The integration of the equations

$$\begin{aligned} 4\dot{C}_{AB} &= -N_{A||B} - N_{B||A} + \overset{\circ}{\gamma}_{AB} N^C{}_{||C} \\ &\quad + (m + 3p + \underline{M})\overset{\circ}{\chi}_{AB} + \frac{1}{4}\hat{\chi}_{AB}\hat{\chi}^{CD}{}_{||CD}\,, \\ 4\dot{D}_{AB} &= -(\overset{\circ}{\Delta} + 4)C_{AB} + S_A{}^C{}_{B||C}\,, \end{aligned}$$

with

$$S_{ABC} = \overset{\circ}{\chi}_{AC} N_B + \overset{\circ}{\chi}_{BC} N_A - \overset{\circ}{\gamma}_{AB} \overset{\circ}{\chi}_{CD} N^D$$

gives polynomial (with respect to the variable u) solutions of degree 2 for C_{AB} and degree 3 for D_{AB}.

8.4. How to relate linearized theory with van der Burg equations

The "first order" asymptotics of the Bondi-Sachs type metric on $\mathcal{J}^+$ is described by three functions M, $\overset{\circ}{\chi}_{AB}$. We shall try now to relate these data with the boundary value on $\mathcal{J}^+$ of our invariants Ψ_ι in the linearized theory in such a way that the non-conservation laws for the mass and angular momentum are similar in both cases. Suppose we know M, $\overset{\circ}{\chi}_{AB}$ at the moment $u = u_0$ and $\overset{\circ}{\chi}_{AB,u}$ on $\mathcal{J}^+$ (actually we need only in the neighbourhood of u_0). We propose to perform a supertranslation which is related to the data at u_0 in such a way that

$$(4\underline{M} - \overset{\circ}{\chi}{}^{AB}_{||AB})(u_0) = 0\,. \tag{8.19}$$

Remark. The condition (8.19) for flat Minkowski space corresponds to the appriopriate choice of the surface $\{u = u_0 = \text{const.}\}$ which is a true cone with a point-like vertex. Let us call the condition (8.19) the supertranslation gauge at the moment u_0. This way we have removed quasi-locally the supertranslation ambiguity at the moment u_0. We stress that the relation $4\underline{M} - \overset{\circ}{\chi}{}^{AB}_{||AB} = 0$ holds *only* at u_0 because in general $\underline{\text{QF}}(\overset{\circ}{\chi}_{AB,u}) := 4\underline{M}_{,u} - \overset{\circ}{\chi}{}^{AB}_{||AB,u}$ is not vanishing, however for the nonradiating solutions the condition (8.19) may be fullfilled globally for all u. This is the main difference between the linearized theory, where the condition $4\underline{M} - \overset{\circ}{\chi}{}^{AB}_{||AB} = 0$ can be fullfilled globally, and nonlinear data, where we can only demand this condition to be fullfilled at one moment u_0[3]. Nevertheless, these procedure allows us to relate nonlinear data at u_0 with linearized theory, namely

$$\begin{aligned} 4\underline{M} &= \overset{\circ}{\chi}{}^{AB}_{||AB} \to \Psi_x\,, \quad \overset{\circ}{\chi}{}^{AB}_{||AB,u} \to \dot{\Psi}_x\,, \\ \hat{\chi}^{AB}_{||AB} &\to \Psi_y\,, \quad \hat{\chi}^{AB}_{||AB,u} \to \dot{\Psi}_y\,, \end{aligned}$$

and also $\overset{\circ}{\chi}_{AB} = \lim_{\rho\to 0^+}\left(\psi_{AB} - \rho^{-1}\,\overset{\circ}{\gamma}_{AB}\right)$. Now it is easy to verify the analogy. The calculations from the previous sections devoted to the linearized gravity should convince the reader that in linear theory we can believe in the following equations

[3] The choice of the supertranslation corresponds to the choice of the "better" local Bondi-Sachs coordinate system. More precisely, the cross-section $u = u_0$ of $\mathcal{J}^+_g$ for the Einstein metric $g_{\mu\nu}$ in the Bondi-Sachs coordinate system (u, x^A) should be compared rather with the cross-section $\overline{u} = u_0 - \alpha$ of $\mathcal{J}^+_f$ in the flat Minkowski space with coordinates $(\overline{u}, x^A)$ on the future null infinity, where $\overline{u}$ is the usual flat coordinate corresponding to the usual null cone. The mono-dipole freedom in the choice of the supertranslation α (and in the choice of coordinates $(\overline{u}, x^A)$) corresponds to the usual translation freedom in the flat Minkowski space.

$$-16\pi\partial_0 m_{\rm TB} = \int\limits_{S(s,0)} \sin\theta \mathrm{d}\theta\mathrm{d}\phi \left[\dot{\Psi}_x \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\dot{\Psi}_x + \dot{\Psi}_y \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\dot{\Psi}_y\right], \tag{8.20}$$

$$-16\pi\partial_0 J_z = \int\limits_{S(s,0)} \sin\theta \mathrm{d}\theta\mathrm{d}\phi \left[\dot{\Psi}_x \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\Psi_{x,\phi} + \dot{\Psi}_y \overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\Psi_{y,\phi}\right]. \tag{8.21}$$

On the other hand, the energy defined in terms of the asymptotics on $\mathcal{J}^+$ by (8.13) together with (8.5) gives the non-conservation law for the TB mass

$$\begin{aligned} -16\pi\partial_0 m_{\rm TB} &= -\int\limits_{S^2} 4\dot{M}\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \frac{1}{2}\int\limits_{S^2} \overset{\circ}{\chi}_{AB,u}\overset{\circ}{\chi}{}^{AB}{}_{,u}\sin\theta\mathrm{d}\theta\mathrm{d}\phi \\ &= \int\limits_{S^2} \sin\theta\mathrm{d}\theta\mathrm{d}\phi \Big[(\overset{\circ}{\chi}{}^{AB}_{||AB})_{,u}\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\overset{\circ}{\chi}{}^{AB}_{||AB})_{,u} \\ &\quad + (\hat{\chi}^{AB}_{||AB})_{,u}\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\hat{\chi}^{AB}_{||AB})_{,u}\Big]. \end{aligned} \tag{8.22}$$

Similarity between (8.20) and (8.22) is obvious, provided $\overset{\circ}{\chi}{}^{AB}_{||AB,u} \to \dot{\Psi}_x$, $\hat{\chi}^{AB}_{||AB,u} \to \dot{\Psi}_y$.

We propose the following definition of the angular momentum (around z-axis) in terms of the asymptotics:

$$J_z := \frac{3}{8\pi}\int\limits_{S^2} \breve{N}_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB}\cos\theta\sin\theta\mathrm{d}\theta\mathrm{d}\phi\,,$$

where

$$\breve{N}_A := N_A + \frac{1}{12}\overset{\circ}{\chi}_{AB}\overset{\circ}{\chi}{}^{BC}{}_{||C}\,.$$

It is compatible with the earlier proposition given by (7.12). We have promised at the end of the previous section to show the relation (7.13) for angular momentum. The following sequence of equalities holds

$$\begin{aligned} 16\pi\dot{J}_z &= -\int\limits_{S^2} 6\dot{N}_\phi + \frac{1}{2}\partial_u(\overset{\circ}{\chi}_{\phi B}\overset{\circ}{\chi}{}^{BC}{}_{||C}) \\ &= \int\limits_{S^2} \overset{\circ}{\chi}_{\phi B}\overset{\circ}{\chi}{}^{BC}{}_{||C,u} + \frac{1}{2}\overset{\circ}{\chi}{}^{AB}{}_{,u}\overset{\circ}{\chi}_{\phi A||B} - \frac{1}{2}\overset{\circ}{\chi}_{\phi B,u}\overset{\circ}{\chi}{}^{BC}{}_{||C} \\ &= -\frac{1}{2}\int\limits_{S^2} \overset{\circ}{\chi}{}^{AB}{}_{,u}\overset{\circ}{\chi}_{AB,\phi}\,, \end{aligned}$$

where in the middle we have used equation (8.6). The last equality (in the above sequence) is a nontrivial identity and can be denoted more geometrically

$$\int_{S^2} X^A \chi_{AB} \dot{\chi}^{BC}{}_{||C} + \frac{1}{2}\dot{\chi}^{BC}\chi_{AB||C}X^A - \frac{1}{2}X^A\dot{\chi}_{AB}\chi^{BC}{}_{||C}$$
$$= -\frac{1}{2}\int_{S^2} \dot{\chi}^{AB}(X^C\chi_{AB||C} + X^C{}_{||B}\chi_{CA} + X^C{}_{||A}\chi_{CB})\,, \qquad (8.23)$$

where now χ_{AB} and $\dot{\chi}_{AB}$ are any symmetric traceless tensors on a unit sphere, $X^A\partial_A := \partial_\phi$ and

$$\chi_{AB,\phi} = X^C\chi_{AB||C} + X^C{}_{||B}\chi_{CA} + X^C{}_{||A}\chi_{CB}\,.$$

Another form of (8.23) can be transformed in the following way.

$$\frac{1}{2}\int_{S^2} \dot{\chi}^{BC}\left(X^A\chi_{AB||C} + X_C\chi_B{}^A{}_{||A} - X^A\chi_{BC||A}\right) = 0$$

is equivalent to

$$\int_{S^2} \dot{\chi}^{BC}X^A\left(\chi_{BC||A} - \chi_{BA||C}\right) = \int_{S^2} \dot{\chi}^{BC}X_C\chi_B{}^A{}_{||A}$$

and the last equality holds for integrands

$$\begin{aligned}\dot{\chi}^{BC}X^A\left(\chi_{BC||A} - \chi_{BA||C}\right) &= \dot{\chi}^{BC}X^A\varepsilon_{AC}\hat{\chi}_B{}^D{}_{||D} \\ &= X^A\dot{\chi}^{BC}\varepsilon_{AC}\varepsilon_{BF}\chi^{FD}{}_{||D} = X^A\dot{\chi}_{AF}\chi^{FD}{}_{||D}\,.\end{aligned}$$

This way we have proved (8.23) and finally also (7.13). Let us rewrite the final result in the form

$$\begin{aligned}-16\pi\dot{J}_z &= \frac{1}{2}\int_{S^2} \overset{\circ}{\chi}{}^{AB}_{,u}\,\overset{\circ}{\chi}_{AB,\phi} \\ &= \int_{S^2} \sin\theta \mathrm{d}\theta\mathrm{d}\phi \Big[(\overset{\circ}{\chi}{}^{AB}_{||AB})_{,u}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\overset{\circ}{\chi}{}^{AB}_{||AB})_{,\phi} \\ &\quad + (\hat{\chi}^{AB}_{||AB})_{,u}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\hat{\chi}^{AB}_{||AB})_{,\phi}\Big]\,.\end{aligned}$$

The similarity with (8.21) is obvious, provided that the supertranslation ambiguity is removed.

8.4.1. Stationary solutions

Let us introduce in the full nonlinear asymptotics the following objects

$$4M = Q + B^{AB}_{||AB}\,,$$

$$\overset{\circ}{\chi}_{AB} = B_{AB} - 2q_{||AB} + \overset{\circ}{\gamma}_{AB}\,\overset{\circ}{\Delta} q\,,$$

where Q and B_{AB} are invariant with respect to the supertranslations, q represents supertranslation ambiguity

$$\overline{q} = q + \alpha\,, \quad \dot{q} = 0\,,$$

and equation (8.5) is equivalent to

$$\dot{Q} = -\frac{1}{2}\dot{B}_{AB}\dot{B}^{AB} := \mathrm{QF}(\dot{B}_{AB})\,,$$

where quadratic term corresponding to the flux of energy we denote by QF. The supertranslation gauge $\underline{Q}(u_0) = \overset{\circ}{\Delta}(\overset{\circ}{\Delta} + 2)q$ allows to relate Ψ_ι at $\mathcal{J}^+$ with B_{AB}. The decomposition is chosen in a convenient manner for the situation of the so-called "sandwich-wave".

Suppose B_{AB} has compact support on $\mathcal{J}^+$ ($\mathrm{supp}B_{AB} \subset [u_i, u_f] \times S^2 \subset \mathcal{J}^+$). Let us also suppose that below u_i and upper u_f our gravitating system is stationary. These two assumptions define a "sandwich-wave".

In Subsection 8.3.1 we have defined the nonradiating solution. Now let us answer the question: When a nonradiating solution becomes stationary? From $\dot{N}^A = 0$ we obtain $p = \underline{M} = \hat{\chi}^{AB}{}_{||AB} = 0$; m, k_0, s are not restricted but also $\overset{\circ}{\chi}{}^{AB}{}_{||AB}$ does not vanish. From $\dot{D}_{AB} = 0$ we get $C_{AB} = (\overset{\circ}{\Delta} + 4)^{-1} S_A{}^C{}_{B||C}$. Similarly, $\dot{C}_{AB} = 0$ gives $m\,\overset{\circ}{\chi}_{AB} = \tilde{N}_{A||B} + \tilde{N}_{B||A} - \overset{\circ}{\gamma}_{AB}\tilde{N}^C{}_{||C}$ or $\tilde{N}_{A||B}\,\overset{\circ}{\varepsilon}{}^{AB} = 0$ and $\overset{\circ}{\chi}{}^{AB}{}_{||AB} = (\overset{\circ}{\Delta} + 2)\tilde{N}^A{}_{||A}$.

Let us call a *simple stationary* solution the situation when $\overset{\circ}{\chi}_{AB} = 0$ and $M = m = \text{const.}$, described by van der Burg in static case (sec.5 in [4]).

<u>Remark</u> The equations related to the Newman-Penrose charge in static situation presented in [4] at the end of page 119 can be denoted in our notation as

$$(\overset{\circ}{\Delta} + 10)D_{AB} = 15\left(MC_{AB} - N_A N_B + \frac{1}{2}\,\overset{\circ}{\gamma}_{AB} N^C N_C\right)\,.$$

We have defined three categories of special solutions:

simple stationary solutions $\subset$ stationary solutions $\subset$ nonradiating solutions

and let us observe that the supertranslation gauge leads to the conclusion that every stationary solution in supertranslation gauge (8.19) is simple.

(nonradiating solutions in supertranslation gauge) $\cap$ (stationary solutions)
$=$ simple stationary solutions

On the other hand, in the case of the "sandwich wave" the supertranslation gauge at u_i and at u_f in general is not the same. The difference depends on

$$\int_{u_i}^{u_f} \underline{\dot{Q}} \mathrm{d}u = \int_{u_i}^{u_f} \underline{\mathrm{QF}} \mathrm{d}u$$

so in general the initial and final states *cannot* be simple in the same Bondi coordinates.

8.5. Special solutions of asymptotic hierarchy, Newman–Penrose charges

The equations (8.5)–(8.8) represent a nonlinear analogue (up to the fourth order) of the hierarchy (8.18) for the usual wave equation.

We could define, as a generalized NP charge, any solution which starts in the $n+1$-th order from "multipole constant". More precisely, if $\varphi \in \ker[\overset{\circ}{\Delta} + (n-1)n]$ then from (8.18) $\varphi_{n+1,u} = 0$. Let us observe that if this charge vanishes we can derive "finite" Janis solution [13], which is obtained by "cutting the series" and derive hierarchy "upward". More precisely,

$$\begin{aligned}
\varphi &= \varphi_1\rho + \varphi_2\rho^2 + \ldots + \varphi_n\rho^n \,, \\
\varphi_n &\in \ker[\overset{\circ}{\Delta} + n(n-1)] \Longrightarrow \dot{\varphi}_{n+1} = 0 \,, \quad \varphi_n = C(u)Y_{n-1}(\theta,\phi) \,, \\
\varphi_{k-1} &= \frac{2k-2}{n(n-1)-(k-1)(k-2)}\dot{\varphi}_k \,, \quad k \le n \,,
\end{aligned}$$

and Y_l is a spherical harmonics ($[\overset{\circ}{\Delta} + l(l+1)]Y_l = 0$). In particular, when $\dot{\varphi}_1 = 0$, then $C(u)$ is a polynomial of degree $n-1$.

On the other hand, if NP charge is not vanishing, the solution φ can not be stationary. Moreover, monopole and dipole examples show that these solutions are singular (but on $\mathcal{J}^-$). The monopole example is the following

$$\varphi = 2\varphi_2\rho v^{-1} = \varphi_2 \frac{2\rho^2}{2+\rho u} \,, \quad \overset{\circ}{\Delta}\varphi_2 = 0 \,, \quad \partial_u\varphi_2 = 0 \,.$$

Similarly the dipole one

$$\varphi = \frac{4}{3}\varphi_3\rho\frac{3v-u}{v^2(v-u)} = -\varphi_3\frac{4}{3}\frac{\rho^3}{(2+\rho u)^2}, \quad (\overset{\circ}{\Delta}+2)\varphi_3 = 0, \quad \partial_u\varphi_3 = 0,$$

and generally

$$\varphi_{n+1} \in \ker[\overset{\circ}{\Delta} + n(n-1)] \Longrightarrow \dot{\varphi}_{n+1} = 0\,, \quad \varphi_{n+1} = CY_{n-1}(\theta,\phi)\,,$$

but now $\dot{C} = 0$ and $\dot{\varphi}_{n+2} = -\frac{n}{n+1}\varphi_{n+1} \neq 0$.

For gravity we have:

1. $D_{AB} = 0 \Longrightarrow$ Janis solution
2. D_{AB} – pure quadrupole $\Longrightarrow$ NP charge solution.

In the Janis paper there are only linearized solutions. We shall try now to construct an asymptotic quadrupole solution of nonlinear hierarchy (8.5)–(8.8). Let us assume that $\underline{M}$ and $\hat{\chi}^{CD}{}_{||CD}$ are given quadrupoles $(0 = (\overset{\circ}{\Delta}+6)\underline{M} = (\overset{\circ}{\Delta}+6)\hat{\chi}^{CD}{}_{||CD})$.

$$\begin{aligned}
M_{,u} &= \overset{\circ}{\chi}_{AB,u} = 0\,, \\
4\underline{M} &= \tilde{x}(\theta,\phi)\,, \qquad \overset{\circ}{\chi}_C{}^D{}_{||DE}\,\overset{\circ}{\varepsilon}{}^{CE} = \tilde{y}(\theta,\phi)\,, \\
D_{AB} &= 0\,, \\
S_{ABC} &= \overset{\circ}{\chi}_{AC}N_B + \overset{\circ}{\chi}_{BC}N_A - \overset{\circ}{\gamma}_{AB}\,\overset{\circ}{\chi}_{CD}N^D\,, \\
C_{AB} &= (\overset{\circ}{\Delta}+4)^{-1}(S_A{}^C{}_{B||C}) - \frac{1}{24}u^2\left(n_{A||B} + n_{B||A} - \overset{\circ}{\gamma}_{AB}n^C{}_{||C}\right)\,, \\
N^A &= -p^{||A}u - k_0^{||A} - \overset{\circ}{\varepsilon}{}^{AB}s_{||B} + \tilde{N}^A + \frac{u}{3}n^A\,, \\
M &= m + 3p + \underline{M}\,, \\
n^A :&= \frac{1}{4}\overset{\circ}{\varepsilon}{}^{AB}\hat{\chi}^{CD}{}_{||CDB} - \underline{M}^{||A}\,, \\
\tilde{N}_{A||B} &+ \tilde{N}_{B||A} - \overset{\circ}{\gamma}_{AB}\tilde{N}^C{}_{||C} := (m+3p+\underline{M})\,\overset{\circ}{\chi}_{AB} \\
&+ \frac{1}{4}\hat{\chi}_{AB}\hat{\chi}^{CD}{}_{||CD} - 4(\overset{\circ}{\Delta}+4)^{-1}(S_A{}^C{}_{B||C,u})\,.
\end{aligned}$$

This is a special example of the nonradiating asymptotic solution, which was defined by the condition that the TB mass is conserved.

9. Closing remarks

We have shown how to define energy at null infinity and its flux through the $\mathcal{J}^+$ for linear hyperbolic theories like scalar field, electrodynamics and linearized gravity. For a given surface (hyperboloid) which ends on $\mathcal{J}^+$ we have assigned generators like energy and angular momentum. They fulfill non-conservation law which comes in a natural way from the variational formula. The boundary term describing flux through $\mathcal{J}^+$ has been obtained in three ways:

a) from the variational formula on a hyperboloid
b) from the variational formula on the future null infinity surface $\mathcal{J}^+$
c) from the energy-momentum tensor.

In all cases we get the same answer for the scalar field and electrodynamics. For gravity, where there is no energy-momentum tensor, the symplectic method has been successfully applied and it gives the correct answer for the energy at $\mathcal{J}^+$ and the non-conservation law for it. The method is useful for the definition of the angular momentum.

We have explained the relation between linearized theory and Bondi-Sachs asymptotics and discussed the role of the supertranslations.

Appendix A

Explicit formulae on a hyperboloid

We give explicit formulae for the relations which have been used extensively in the Section 4.

$$\begin{aligned}
&\Lambda = \sinh^2\omega \sin\theta, \quad N_{|m} = -N_m, \quad N_{|kl} = -N_{k|l} = N\eta_{kl}, \quad N^{|k}{}_k = -N^k_{|k} = 3N\,,\\
&N = \cosh\omega, \quad N^3 = N_3 = -\sinh\omega, \quad N^A = 0, \quad \eta_{AB,3} = 2\kappa\eta_{AB}\,,\\
&\eta^{AB}{}_{,3} = -2\kappa\eta^{AB}, \quad \varepsilon_{AB,3} = 2\kappa\varepsilon_{AB}, \quad \varepsilon^{AB}{}_{,3} = -2\kappa\varepsilon^{AB}, \quad \kappa = \coth\omega\,,\\
&\Lambda_{,3} = 2\kappa\Lambda, \quad \kappa N_{,3} = -\kappa N_3 = N, \quad \kappa_{,3} = -\frac{1}{\sinh^2\omega}\,,\\
&\Gamma^3{}_{AB} = -\kappa\eta_{AB}, \quad \Gamma^A{}_{B3} = \kappa\delta^A_B, \quad \Gamma^A{}_{BC,3} = 0\,,\\
&{}^2R_{ABCD} = \frac{1}{\sinh^2\omega}\left(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}\right), \quad {}^2R_{AB} = \frac{1}{\sinh^2\omega}\eta_{AB}\,,\\
&\xi^{|33} = \xi^{,33}, \quad \xi^{|3A} = \xi^{,3A} - \kappa\xi^{,A}, \quad \xi^{|AB} = \xi^{||AB} + \kappa\eta^{AB}\xi^{,3}\,,\\
&\xi_{3|3} = \xi_{3,3}, \quad \xi_{3|A} = \xi_{3||A} - \kappa\xi_A, \quad \xi_{A|3} = \xi_{A,3} - \kappa\xi_A\,,\\
&\xi_{A|B} = \xi_{A||B} + \kappa\eta^{AB}\xi_3\,,\\
&h_{33|3} = h_{33,3}, \quad h_{33|A} = h_{33,A} - 2\kappa h_{3A}\,,\\
&h^{AB}{}_{|3} = h^{AB}{}_{,3} + 2\kappa h^{AB}, \quad h_{AB|3} = h_{AB,3} - 2\kappa h_{AB}, \quad h^A{}_{B|3} = h^A{}_{B,3}\,,\\
&h^{3A}{}_{|3} = h^{3A}{}_{,3} + \kappa h^{3A}, \quad h_{3A|3} = h_{3A,3} - \kappa h_{3A}\,,\\
&h_{3A|B} = h_{3A||B} - \kappa h_{AB} + \kappa\eta_{AB}h_{33}\,,\\
&h_{AB|C} = h_{AB||C} + \kappa\eta_{AC}h_{3B} + \kappa\eta_{BC}h_{3A}\,.
\end{aligned}$$

Appendix B

Reduction of the symplectic form on a hyperboloid

Let (p^{kl}, h_{kl}) and (s^{kl}, q_{kl}) denote two pairs of Cauchy data on a hyperboloid. The (2+1)-splitting of the tensor q_{kl} gives the following components on a sphere: $\overset{2}{q} := \eta^{AB} q_{AB}$, q_{33} – scalars on S^2, q_{3A} – vector and $\overset{\circ}{q}_{AB} := q_{AB} - \frac{1}{2}\eta_{AB} \overset{2}{q}$ – symmetric traceless tensor on S^2. Similarly, we can decompose the tensor density p^{kl}. The quadratic form $\int\limits_V p^{kl} q_{kl}$ can be decomposed into monopole part, dipole part and the remainder in a natural way.

The "mono-dipole" part we write separately

$$\begin{aligned} \mathrm{mon}\left(\int\limits_V p^{kl} q_{kl}\right) &= \int\limits_V \frac{1}{2\cosh^2\omega} p^{33} \mathrm{mon}(\xi) + \int\limits_V \frac{\tanh^2\omega}{\Lambda} p^{33} \mathrm{mon}(s^{33}) \\ &\quad + \frac{1}{2} \int\limits_{\partial V} \tanh\omega p^{33} \mathrm{mon}(\overset{2}{q}) \,, \end{aligned} \tag{B.1}$$

$$\begin{aligned} \mathrm{dip}\left(\int\limits_V p^{kl} q_{kl}\right) &= \int\limits_V \frac{1}{2\cosh^2\omega} p^{33} \mathrm{dip}(\xi) \\ &\quad -2 \int\limits_V \mathrm{dip}(\sinh^2\omega p^{3A||B} \varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (q_{3A||B} \varepsilon^{AB}) \\ &\quad + \int\limits_V \frac{\tanh^2\omega}{\Lambda} p^{33} \mathrm{dip}(s^{33}) + \frac{1}{2} \int\limits_{\partial V} \tanh\omega p^{33} \mathrm{dip}(\overset{2}{q}) \\ &\quad + \int\limits_V \mathrm{dip}\left(\frac{p^{33}}{2\cosh^2\omega} + \tanh\omega \overset{\circ}{\Delta}{}^{-1} p^{3A}{}_{||A}\right) \\ &\quad \times \left(\frac{1}{2} \overset{\circ}{\Delta} \overset{2}{q} - 2\sinh\omega\cosh\omega q^{3A}{}_{||A}\right) \,, \end{aligned} \tag{B.2}$$

where invariant ξ is defined as follows

$$\begin{aligned} \xi :&= 2\cosh^2\omega q^{33} + 2\cosh\omega\sinh\omega q^{3A}{}_{||A} + \sinh^2\omega \overset{\circ}{q}{}^{AB}{}_{||AB} \\ &\quad - \cosh\omega\sinh\omega \overset{2}{q}{}_{,3} - \frac{1}{2}(\overset{\circ}{\Delta} + 2) \overset{2}{q} - 2\frac{\sinh^2\omega}{\Lambda} s^{33} \,. \end{aligned}$$

From the vector constraints

$$\sinh\omega p^{33}{}_{,3} + \sinh\omega p^{3A}{}_{||A} - \cosh\omega \overset{2}{p} = 0 \,, \tag{B.3}$$

$$(\sinh^2\omega p^{3A}{}_{||A})_{,3} + (\sinh^2\omega \overset{\circ}{p}{}^{AB}{}_{||AB}) + \frac{1}{2} \overset{\circ}{\Delta} \overset{2}{p} = 0 \,, \tag{B.4}$$

$$(\sinh^2\omega p^{3A||B} \varepsilon_{AB})_{,3} + (\sinh^2\omega \varepsilon^{AC} \overset{\circ}{p}{}_A{}^B{}_{||BC}) = 0 \,, \tag{B.5}$$

we can partially reduce our form

$$\int_V p^{kl} q_{kl} = \int_V p^{33} q_{33} + 2p^{3A} q_{3A} + \frac{1}{2} \overset{2}{p}\overset{2}{q} + \overset{\circ}{p}{}^{AB} \overset{\circ}{q}{}_{AB}$$

$$= \int_V p^{33} q_{33} - 2(\sinh\omega p^{3A}{}_{||A}) \overset{\circ}{\Delta}{}^{-1} (\sinh\omega q^{3A}{}_{||A})$$

$$-2(\sinh\omega p^{3A||B} \varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\sinh\omega q_{3A||B} \varepsilon^{AB})$$

$$+ \int_V \frac{1}{2} \overset{2}{p}\overset{2}{q} + 2(\sinh^2\omega \varepsilon^{AC} \overset{\circ}{p}{}_A{}^B{}_{||BC}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})$$

$$+2 \int_V (\sinh^2\omega \overset{\circ}{p}{}^{AB}{}_{||AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \overset{\circ}{q}{}^{AB}{}_{||AB})$$

$$= \int_V p^{33} q_{33} - 2(\sinh\omega p^{3A}{}_{||A}) \overset{\circ}{\Delta}{}^{-1} (\sinh\omega q^{3A}{}_{||A})$$

$$-2 \int_V (\sinh\omega p^{3A||B} \varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\sinh\omega q_{3A||B} \varepsilon^{AB})$$

$$+ \int_V \frac{1}{2} (\tanh\omega p^{33}{}_{,3} + \tanh\omega p^{3A}{}_{||A}) \overset{2}{q}$$

$$-2 \int_V (\sinh^2\omega p^{3A||B} \varepsilon_{AB})_{,3} \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})$$

$$-2 \int_V \left[(\sinh^2\omega p^{3A}{}_{||A})_{,3} + \frac{1}{2} \overset{\circ}{\Delta} (\tanh\omega p^{33}{}_{,3} + \tanh\omega p^{3A}{}_{||A}) \right] \overset{\circ}{\Delta}{}^{-1}$$

$$\times (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \overset{\circ}{q}{}^{AB}{}_{||AB})$$

$$= \int_{\partial V} \tanh\omega p^{33} \left[\frac{1}{2} \overset{2}{q} - (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \overset{\circ}{q}{}^{AB}{}_{||AB}) \right]$$

$$-2 \int_{\partial V} (\sinh^2\omega p^{3A}{}_{||A}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \overset{\circ}{q}{}^{AB}{}_{||AB})$$

$$-2 \int_{\partial V} (\sinh^2\omega p^{3A||B} \varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})$$

$$-2 \int_V \boxed{(\sinh^2\omega p^{3A||B} \varepsilon_{AB}) \overset{\circ}{\Delta}{}^{-1} \left[q_{3A||B} \varepsilon^{AB} - (\overset{\circ}{\Delta} + 2)^{-1} (\sinh^2\omega \varepsilon^{AC} \overset{\circ}{q}{}_A{}^B{}_{||BC})_{,3} \right]}$$

$$+ \int_V p^{33} \left[q_{33} + (\overset{\circ}{\Delta} + 2)^{-1} \left(\frac{\sinh^3\omega}{\cosh\omega} \overset{\circ}{q}{}^{AB}{}_{||AB} \right)_{,3} - \frac{1}{2} (\tanh\omega \overset{2}{q})_{,3} \right]$$

$$+\int_V \tanh\omega p^{3A}{}_{||A}\left[\frac{1}{2}\overset{2}{q} +2\sinh\omega\cosh\omega\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB}),{}_3\right.$$
$$\left.-(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB})-2\,\overset{\circ}{\Delta}{}^{-1}(\sinh\omega\cosh\omega q^{3A}{}_{||A})\right].$$

The volume term in the framebox is mono-dipole-free and corresponds to the invariants $\mathbf{y}$, $\mathbf{Y}$. The last two terms we can proceed further, but first let us write a scalar constraint in two equivalent forms

$$\left(\frac{\sinh^3\omega}{\cosh\omega}\,\overset{\circ}{q}{}^{AB}{}_{||AB}\right),{}_3+(\overset{\circ}{\Delta}+2)\left[q_{33}-\frac{1}{2}(\tanh r\,\overset{2}{q}),{}_3\right]=(\tanh\omega\xi),{}_3+\xi$$
$$+\frac{2}{\Lambda}\left(\tanh^2\omega s^{33}-\tanh\omega\sinh^2\omega s^{3A}{}_{||A}\right)\,, \tag{B.6}$$
$$\frac{1}{2}\,\overset{\circ}{\Delta}(\overset{\circ}{\Delta}+2)\,\overset{2}{q} +2\sinh\omega\cosh\omega(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB}),{}_3-\overset{\circ}{\Delta}\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB}$$
$$-2(\overset{\circ}{\Delta}+2)(\cosh\omega\sinh\omega q^{3A}{}_{||A})$$
$$=2\cosh\omega(\sinh\omega\xi),{}_3-\overset{\circ}{\Delta}\xi-\frac{2\sinh^2\omega}{\Lambda}\left(\overset{\circ}{\Delta}s^{33}+2\sinh\omega\cosh\omega s^{3A}{}_{||A}\right). \tag{B.7}$$

For the "radiation" part we get the following result:

$$\int_V p^{33}\left[q_{33}+(\overset{\circ}{\Delta}+2)^{-1}(\frac{\sinh^3\omega}{\cosh\omega}\,\overset{\circ}{q}{}^{AB}{}_{||AB}),{}_3-\frac{1}{2}(\tanh\omega\,\overset{2}{q}),{}_3\right]$$
$$+\int_V \tanh\omega p^{3A}{}_{||A}\left[\frac{1}{2}\overset{2}{q} +2\sinh\omega\cosh\omega\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB}),{}_3\right.$$
$$\left.-(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB})-2\,\overset{\circ}{\Delta}{}^{-1}(\sinh\omega\cosh\omega q^{3A}{}_{||A})\right]$$
$$=\int_V p^{33}(\overset{\circ}{\Delta}+2)^{-1}\left[(\tanh\omega\xi),{}_3+\xi+\frac{2}{\Lambda}(\tanh^2\omega s^{33}-\tanh\omega\sinh^2\omega s^{3A}{}_{||A})\right]$$
$$+\int_V \tanh\omega p^{3A}{}_{||A}\,\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\left[2\cosh\omega(\sinh\omega\xi),{}_3-\overset{\circ}{\Delta}\xi\right.$$
$$\left.-\frac{2\sinh^2\omega}{\Lambda}(\overset{\circ}{\Delta}s^{33}+2\sinh\omega\cosh\omega s^{3A}{}_{||A})\right]$$
$$=\int_{\partial V}\left[\tanh\omega p^{33}+2\sinh^2\omega\,\overset{\circ}{\Delta}{}^{-1}p^{3A}{}_{||A}\right](\overset{\circ}{\Delta}+2)^{-1}\xi$$
$$+\int_V\left[\overset{\circ}{\Delta}p^{33}-\overset{\circ}{\Delta}\tanh\omega p^{33},{}_3-2\sinh\omega(\sinh\omega p^{3A}{}_{||A}),{}_3\right.$$
$$\left.-\overset{\circ}{\Delta}\tanh\omega p^{3A}{}_{||A}\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\xi$$

$$
\begin{aligned}
&+\int_V \frac{2}{\Lambda}\Big[\tanh\omega p^{33}(\overset{\circ}{\Delta}+2)^{-1}(\tanh\omega s^{33}-\sinh^2\omega s^{3A}{}_{||A}) \\
&-\sinh^2\omega p^{3A}{}_{||A}\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\tanh\omega\overset{\circ}{\Delta}s^{33}+2\sinh^2\omega s^{3A}{}_{||A})\Big] \\
&=\int_{\partial V}\left[\tanh\omega\overset{\circ}{\Delta}p^{33}+2\sinh^2\omega p^{3A}{}_{||A}\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\xi \\
&+\int_V \boxed{\left[\overset{\circ}{\Delta}p^{33}+2\sinh^2\omega\,\overset{\circ}{p}{}^{AB}{}_{||AB}+2\sinh\omega\cosh\omega p^{3A}{}_{||A}\right]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\xi} \\
&+\int_V \frac{2}{\Lambda}\left[\tanh^2\omega p^{33}(\overset{\circ}{\Delta}+2)^{-1}s^{33}-2\sinh^4\omega p^{3A}{}_{||A}\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}s^{3A}{}_{||A}\right] \\
&-\int_V \frac{2}{\Lambda}\tanh\omega\sinh^2\omega\left(p^{33}(\overset{\circ}{\Delta}+2)^{-1}s^{3A}{}_{||A}+p^{3A}{}_{||A}(\overset{\circ}{\Delta}+2)^{-1}s^{33}\right)\,,
\end{aligned}
\tag{B.8}
$$

and again the framebox corresponds to the invariants (here $\mathbf{x}$ and $\mathbf{X}$).

Finally we get in volume integrand the gauge invariant part

$$
\begin{aligned}
\int_V p^{kl}q_{kl} &= \text{monodipole part in } V+\boxed{\text{“radiation” part in } V} \\
&+\int_{\partial V}[\tanh\omega\overset{\circ}{\Delta}p^{33}+2\sinh^2\omega p^{3A}{}_{||A}]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\xi \\
&-2\int_{\partial V}(\sinh^2\omega p^{3A||B}\varepsilon_{AB})\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\varepsilon^{AC}\,\overset{\circ}{q}{}_A{}^B{}_{||BC}) \\
&+\int_{\partial V}\tanh\omega p^{33}\left[\frac{1}{2}\overset{2}{q}-(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB})\right] \\
&-2\int_{\partial V}(\sinh^2\omega p^{3A}{}_{||A})\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\,\overset{\circ}{q}{}^{AB}{}_{||AB})\,,
\end{aligned}
$$

where

$$
\begin{aligned}
\text{monodipole part in V} &= \int_V \frac{1}{2\cosh^2\omega}p^{33}\text{mon}(\xi)+\int_V \frac{\tanh^2\omega}{\Lambda}p^{33}\text{mon}(s^{33}) \\
&+\int_V \frac{1}{2\cosh^2\omega}p^{33}\text{dip}(\xi)-2\int_V \text{dip}(\sinh^2\omega p^{3A||B}\varepsilon_{AB})\overset{\circ}{\Delta}{}^{-1}(q_{3A||B}\varepsilon^{AB}) \\
&+\int_V \frac{\tanh^2\omega}{\Lambda}p^{33}\text{dip}(s^{33})+\int_V \text{dip}\left(\frac{p^{33}}{2\cosh^2\omega}+\tanh\omega\overset{\circ}{\Delta}{}^{-1}p^{3A}{}_{||A}\right)
\end{aligned}
$$

$$\times\left(\frac{1}{2}\overset{\circ}{\Delta}\overset{2}{q}-2\sinh\omega\cosh\omega q^{3A}{}_{||A}\right),$$

$$\boxed{\text{“radiation” part in } V}=\int_V\Big[\overset{\circ}{\Delta}p^{33}+2\sinh^2\omega\,\overset{\circ}{p}{}^{AB}{}_{||AB}$$

$$+2\sinh\omega\cosh\omega p^{3A}{}_{||A}\Big]\overset{\circ}{\Delta}{}^{-1}(\overset{\circ}{\Delta}+2)^{-1}\xi$$

$$-2\int_V(\sinh^2\omega p^{3A||B}\varepsilon_{AB})\overset{\circ}{\Delta}{}^{-1}\Big[q_{3A||B}\varepsilon^{AB}$$

$$-(\overset{\circ}{\Delta}+2)^{-1}(\sinh^2\omega\varepsilon^{AC}\,\overset{\circ}{q}{}_A{}^B{}_{||BC}),_3\Big].$$

Appendix C

List of symbols

V – three-dimensional volume or function in Bondi-Sachs type metric

L – Lagrangian density

Σ – hyperboloid

φ – scalar field

ψ – rescaled scalar field

δ – “variational” derivative

∂_μ – partial derivative

$T^\mu{}_\nu$ – symmetric energy-momentum tensor

$\eta_{\mu\nu}$ – flat Minkowski metric

η – $\det\eta_{\mu\nu}$

$\mathcal{T}^\mu{}_\nu$ – canonical energy-momentum density

$\delta^\mu{}_\nu$ – Kronecker's delta

p^μ – canonical field momenta

π – canonical momenta

$\mathcal{H}$ – Hamiltonian, energy generator

H – density of a Hamiltonian or two-dim. trace of h_{AB}

x^μ, y^ν – coordinates on M

t – time coordinate on M_2

r – radial coordinate on M_2

ω – related radial coordinate on M_2, $r=\sinh\omega$

ρ – “inverse” radial coordinate on M_2, $r=\rho^{-1}$

s – “hyperboloidal time” coordinate on M_2, $s=t-\sqrt{1+r^2}$

u, v – null coordinates on M_2, $u=t-r$, $v=t+r$

$\overline{u}$ – coordinate on M_2, $\overline{u} = -2r$

θ, ϕ – spherical coordinates on S^2

d – exterior derivative

$\mu, \nu, \ldots$ – four-dimensional indices running $0, \ldots, 3$

$k, l, \ldots$ – three-dimensional indices running $1, \ldots, 3$

$A, B, \ldots$ – two-dimensional indices on a sphere

$a, b, \ldots$ – two-dimensional "null" indices on M_2

$\Box$ – d'Alambertian, wave operator

$\bar{\Box}$ – conformally related wave operator

$\overline{\eta}_{\mu\nu}$ – conformally related metric

R – scalar curvature

X – vector field

i^0 – spatial infinity

$\mathcal{J}$ – null infinity

$\mathcal{J}^+$ – future null infinity

$\mathcal{J}^-$ – past null infinity

N – null surface "parallel" to $\mathcal{J}^+$ or a piece of $\mathcal{J}^+$

m_{ADM} – ADM mass

S^2 – sphere parameterized by θ, ϕ

$S(s, \rho)$ – sphere in M corresponding to coordinates s, ρ

$S_s(\omega)$ – sphere in M corresponding to coordinates s, ω

$S(s, 0)$ – sphere on $\mathcal{J}^+$

$S(1)$ – unit sphere

$\overset{\circ}{\gamma}_{AB}$ – metric on a unit sphere

$\overset{\circ}{\Delta}$ – two-dimensional laplacian on a unit sphere

$\overset{\circ}{\varepsilon}{}^{AB}$ – skew-symmetric tensor on a unit sphere, $\sin\theta \, \overset{\circ}{\varepsilon}{}^{\theta\phi} = 1$

ε^{AB} – two-dimensional skew-symmetric tensor, $r^2 \sin\theta \varepsilon^{\theta\phi} = 1$

$||$ – two-dimensional covariant derivative on a sphere

$\hat{\partial}_A$ – dual of ∂_A, $\hat{\partial}_A = \varepsilon_A{}^B \partial_B$

$\mathcal{F}^{\mu\nu}$ – electromagnetic induction density

$f_{\mu\nu}$ – electromagnetic field

A_μ – electromagnetic potential

$\psi, {*\psi}$ – gauge-invariant positions for electromagnetism

$\pi, {*\pi}$ – gauge-invariant momenta for electromagnetism

$\tilde{J}_z$ – angular momentum

g_{kl} – three-dimensional riemannian metric
P^{kl} – ADM momentum
K^{kl} – extrinsic curvature
$R^{\mu}{}_{\nu\lambda\sigma}$ – curvature tensor
$R_{\mu\nu}$ – Ricci tensor
$\Gamma^{\lambda}{}_{\mu\nu}$ – Christoffel symbol
n^{μ} – normal unit future directed vector
$\mathcal{R}$ – three-dimensional scalar curvature
h_{kl} – linearized metric
$\mathcal{P}^{kl}$ – linearized momentum
p^{kl} – "new" linearized momentum
g – $\det g_{kl}$
Λ – volume element, $\Lambda = r^2 \sin\theta$
ξ_{μ} – gauge in linearized gravity
κ – $\coth\omega$
$\mathbf{x}, \mathbf{X}, \mathbf{y}, \mathbf{Y}$ – invariants
χ_{AB} – traceless part of h_{AB}
S_{AB} – traceless part of p_{AB}
S – trace of p_{AB}
$\triangle_{\Sigma}$ – laplacian on a hyperboloid
J_z – angular momentum generator
P_z – linear momentum generator
Ψ_x, Ψ_y – "asymptotic position" on a hyperboloid
Π_x, Π_y – "asymptotic momenta" on a hyperboloid
ι – abstract index, $\iota = x, y$
E_{ab} – $\frac{1}{2}E_{ab}\mathrm{d}x^a \wedge \mathrm{d}x^b = \mathrm{d}u \wedge \mathrm{d}v$
$\mathbf{y}_a$ – invariant in null coordinates
$\mathbf{x}_{ab}$ – invariant in null coordinates
$\beta, V, U^A, \gamma_{AB}$ – parameters describing Bondi-Sachs type metric
C – null cone or van der Burg asymptotics
m_{TB} – Trautman-Bondi mass
Ψ^{AB} – nonlinear asymptotic position on a null cone
Π_{AB} – nonlinear asymptotic momenta on a null cone
$\mathcal{L}_X$ – Lie derivative with respect to vector field X
$s^z, s^l, \mathbf{s}$ – spin charge
$\mathbf{m}$ – mass charge

$p^z, p^l, \mathbf{p}$ – linear momentum charge

$j^{l0}, \mathbf{k}_0$ – static momentum charge (center of mass)

M – Minkowski space or asymptotics of function V in van der Burg notation

γ, δ – van der Burg parameterization of γ_{AB}

U, W – van der Burg parameterization of U^A

N, P – van der Burg parameterization of N^A

N^A – asymptotics of U^A

c, C, D – van der Burg notation for the asymptotics of γ

d, H, K – van der Burg notation for the asymptotics of δ

$\overset{\circ}{\chi}_{AB}, C_{AB}, D_{AB}$ – asymptotics of γ_{AB}

σ_x, σ_z – Pauli matrices

S_{ABC} – traceless symmetric tensor appearing in eq. (8.8)

$\hat{S}_{ABC}$ – "dual" of S_{ABC}, $\hat{S}_{ABC} = \overset{\circ}{\varepsilon}{}_A{}^D S_{DBC}$

$\hat{C}_{AB}$ – "dual" of C_{AB}, $\hat{C}_{AB} = \overset{\circ}{\varepsilon}{}_A{}^D C_{DB}$

$\hat{\chi}_{AB}$ – "dual" of $\overset{\circ}{\chi}_{AB}$, $\hat{\chi}_{AB} = \overset{\circ}{\varepsilon}{}_A{}^D \overset{\circ}{\chi}_{DB}$

$\hat{D}_{AB}$ – "dual" of D_{AB}, $\hat{D}_{AB} = \overset{\circ}{\varepsilon}{}_A{}^C D_{CB}$

$\mathrm{mon}(F)$ – monopole part of F

$\mathrm{dip}(F)$ – dipole part of F

$\underline{F}$ – mono-dipole-free part of F

$\overline{F}$ – supertranslation of F

Y_l – spherical harmonics with eigenvalue $-l(l+1)$ of the laplacian $\overset{\circ}{\Delta}$

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