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TWO COMPONENT THEORY AND ELECTRON MAGNETIC MOMENT

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The two-component formulation of quantum electrodynamics is studied. The relation with the usual Dirac formulation is exhibited, and the Feynman rules for the two-component form of the theory are presented in terms of familiar objects. The transformation from the Dirac theory to the two-component theory is quite amusing, involving Faddeev-Popov ghost loops of a fermion type with bose statistics. The introduction of an anomalous magnetic moment in the two-component formalism is simple; it is not equivalent to a Pauli term in the Dirac formulation. Such an anomalous magnetic moment appears not to destroy the renormalizability of the theory but violates unitarity.

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1. Introduction

In 1958, Feynman and Gell-Mann [1] revived the two-component fermion theory in the context of their work on the V-A form of the weak interactions. This was studied extensively by several authors [2-4], who reformulated quantum electrodynamics in the form of a two-component theory, and also applied the formalism to the then current theory of weak interactions, the V-A theory. While nobody really followed up on this development, there are nonetheless certain advantages to the two-component theory, as it shows quite clearly and separately the electric and magnetic interactions of the electron with the photon field.

We review the two-component formulation in a somewhat more familiar form, that is in terms of ordinary Dirac spinors. This also exhibits very clearly the relationship between the Dirac form and the two-component form of the theory.

In the case of Compton scattering one may compute the cross-section using that two-component formalism, in the expectation that it then becomes clear which part of the cross-section is due to the magnetic moment of the electron. This indeed can be done. Varying the electron magnetic moment (*i.e.* adding an anomalous part) in the two-component formalism can be done quite easily; one might think that this amounts to the addition of a Pauli term in the Dirac formulation of the theory, but this is not the case. Some remarks to this effect have been made already by Brown [2]. The addition of an anomalous magnetic moment in the two-component theory appears not to destroy the property of renormalizability, but on the other hand unitarity is no longer maintained. Translating the so introduced moment into the Dirac form one finds a complicated non-polynomial interaction; in lowest order it gives the same contribution as the Pauli term.

We use the Pauli metric, *i.e.* $g_{\mu\nu}$ is the unit matrix. For conventions concerning spinors, gamma matrices, etc. see Ref. [5].

2. The two-component Lagrangian

The Lagrangian of quantum electrodynamics (QED) in the Dirac formulation is

$$\mathcal{L}_{\rm D}^{qed} = -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \overline{\psi}(\gamma^\mu D_\mu + m)\psi\,,$$

with as usual $D_\mu = \partial_\mu + ieA_\mu$. We now substitute

$$\psi \to (-\gamma^\nu D_\nu + m)\psi$$

without changing $\overline{\psi}$. Thus we obtain

$$\begin{aligned}
\mathcal{L}_2^{\rm QED} = \;& -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu \\
& -\overline{\psi}\left(-\partial^2 - ie\gamma^\mu\gamma^\nu\partial_\mu A_\nu - 2ieA_\nu\partial_\nu + e^2 A_\mu A_\mu + m^2\right)\psi \\
= \;& -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu \\
& -\overline{\psi}\left(-\partial^2 - 2ie\sigma^{\mu\nu}\partial_\mu A_\nu - ie\partial_\mu A_\mu - 2ieA_\nu\partial_\nu + e^2 A^2 + m^2\right)\psi \\
= \;& -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu \\
& -(\partial_\mu - ieA_\mu)\overline{\psi}(\partial_\mu + ieA_\mu)\psi - m^2\overline{\psi}\psi + 2ie\left(\overline{\psi}\sigma^{\mu\nu}\psi\right)\partial_\mu A_\nu\,.
\end{aligned}$$

Here $\sigma^{\mu\nu} = \frac{1}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Except for the last term this Lagrangian is precisely that of a charged scalar particle of mass m interacting with the e.m. field in the standard way; the last term, containing $\sigma^{\mu\nu}$, is the magnetic part of the interaction. Since in this Lagrangian any term has either no gamma matrix or two, one may rewrite the above in terms of right- and left-handed spinors. With

and
$$\psi_L = \tfrac{1+\gamma^5}{2}\,\psi, \qquad \overline{\psi}_L = \overline{\psi}\,\tfrac{1-\gamma^5}{2}$$
$$\psi_R = \tfrac{1-\gamma^5}{2}\,\psi, \qquad \overline{\psi}_R = \overline{\psi}\,\tfrac{1+\gamma^5}{2}$$

the Lagrangian will only have terms containing $\overline{\psi}_L\psi_R$ or $\overline{\psi}_R\psi_L$. The Lagrangian of the two-component theory is obtained by keeping only the terms of the form $\overline{\psi}_R\psi_L$, thus only terms containing $1+\gamma^5$, which makes the Lagrangian non-hermitean. That Lagrangian can be found, in a slightly different notation, in Ref. [2]. Most of the work on the two-component theory can be viewed, from our perspective, as demonstrating that the omitted piece of the Lagrangian (the terms of the form $\overline{\psi}_L\psi_R$) gives the same contribution to the S-matrix as the part kept. In the two-component formulation unitarity of the S-matrix is not obvious, as also emphasized in Ref. [3-4].

In this article we will use the term two-component theory in different ways. The precise two-component theory is the theory in terms of $\overline{\psi}_R$ and ψ_L, with a non-hermitean Lagrangian. The theory that we obtained from the Dirac theory by the transformation shown, with the Lagrangian in terms of the familiar Dirac spinors without any γ^5, will be called the 'two-component' theory, with quotation marks. It is really a theory in terms of four-component spinors.

The transformation shown is actually far from trivial from a field theoretical point of view. While it may seem to be a local transformation, thus not giving rise to a Faddeev-Popov ghost, a slightly more precise consideration shows that the transformation involves a non-local part, and therefore Faddeev-Popov ghost loops must be included. We will exhibit this on a purely formal level and also directly in terms of diagrams.

3. Faddeev-Popov ghost loops

It is quite instructive to consider first a very simplified theory, involving one fermion and one scalar field with the simplest possible interaction:

$$\mathcal{L} = \mathcal{L}_{\rm sc} - \overline{\psi}(\gamma^\mu\partial_\mu + m)\psi + g(\overline{\psi}\psi)\varphi\,, \quad \text{with} \quad \mathcal{L}_{\rm sc} = -\tfrac{1}{2}\varphi(-\partial^2+m^2)\varphi\,.$$

This has a fermion propagator as usual and a very simple vertex:

$$\text{Propagator}: \quad \frac{-i\gamma p + m}{p^2+m^2-i\epsilon} \qquad\qquad \text{Vertex}: \quad g$$

Now we do the transformation $\psi \to (-\gamma^\mu\partial_\mu + m)\psi$. The Lagrangian becomes:

$$\mathcal{L} = \mathcal{L}_{\rm scalar} - \overline{\psi}(-\partial^2+m^2)\psi + g\left(\overline{\psi}(-\gamma^\mu\partial_\mu + m)\psi\right)\varphi\,.$$

We then have the following fermion propagator and vertex:

$$\text{Propagator}: \quad \frac{1}{p^2 + m^2 - i\varepsilon} \qquad\qquad \text{Vertex}: \quad g(-i\gamma p + m)$$

where in the vertex the (incoming) momentum is that of the ψ line. Obviously, nothing could be more trivial: the numerator of the propagator has simply been shifted to the vertex. The theory is the same as before. It looks very strange though; the fermion appears to have a boson type propagator.

Concerning external lines, we may note that spinors corresponding to incoming particles or outgoing anti-particles now have the energy projection operator $-i\gamma p + m$ in front (p is the momentum of the external incoming fermion, thus flowing into the vertex). Because such spinors obey the Dirac equation that gives just a factor $2m$. One may either keep the energy projection operator explicitly, at the same time providing spinors with a factor $1/\sqrt{2m}$, or else keep the spinors as usual but omitting the energy projection operator at the external lines.

Since the diagrams are literally the same as in the original theory there is no question that the theory is still unitary after the transformation, using at the external lines the same spinors that satisfy the Dirac equation. Can this be seen directly?

If we consider the S-matrix generated by the new Feynman rules, and if we restrict ourselves to spinors at the outgoing lines that satisfy the Dirac equation then it is not immediately obvious that the S-matrix is unitary, because a cut propagator has a residue of 1, rather then $-i\gamma p+m$. However, this cut propagator has a factor $-i\gamma p + m$ on one side, and since

$$-i\gamma p + m = \frac{(-i\gamma p + m)^2}{2m} + \frac{p^2 + m^2}{2m}\,,$$

which is equal to $(-i\gamma p+m)^2/2m$ on mass shell, we see that we could equally well have taken the residue $(-i\gamma p+m)/2m$ for the cut propagator. In other words, the S-matrix is unitary if we restrict ourselves to external lines with spinors that obey the Dirac equation. Those spinors must be provided with a factor $1/\sqrt{2m}$, as already noted earlier.

It is noteworthy that the new Lagrangian is not hermitean; the hermitean conjugate differs by an interaction of the form

$$-g\left(\overline{\psi}\gamma^\mu\psi\right)\partial_\mu\varphi\,.$$

Such an extra interaction gives zero only if the external line spinors obey the Dirac equation. To see that requires a derivation having some similarity to the work needed when demonstrating gauge invariance of the diagrams in quantum electrodynamics.

The transformation that we wish to perform in the case of quantum electrodynamics may be written as the product of two transformations:

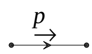
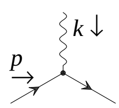
$$\psi \to (-\gamma\partial + m)\left(1 - \frac{1}{-\gamma\partial + m}\, ie\gamma^\mu A_\mu\right)\psi\,.$$

The first transformation is the one we discussed above. The second is now in a form that shows clearly that we are dealing with a non-local transformation. The structure is very suggestive: a fermion propagator times the usual Dirac e.m. interaction. It will give rise to Faddeev-Popov ghost loops. The Jacobian of the transformation can be established without any problem; alternatively one can use combinatorial methods to deduce the precise form of the ghost loops (see Ref. [6], in particular section 10.4). That latter approach has the advantage that it results in a detailed understanding of the structure of the loops, and the associated sign. The result is very simple: the Faddeev-Popov loops are precisely as those of the Dirac theory, however with a plus sign. That plus sign can be understood as a combination of the minus sign for a fermion loop combined with a minus sign for the Faddeev-Popov ghost loop. This may be re-arranged in an elegant manner. Consider as an example any diagram with one closed loop in the Dirac theory. Such a diagram has a minus sign. In the new theory we will have a loop generated by the new rules (which must be provided with the usual minus sign for a fermion loop), plus a loop precisely equal to the original loop of the Dirac theory, however now with a plus sign. Thus, − Dirac loop = − New loop + Dirac loop. Or, we can simply take the Faddeev-Popov loops into account by giving the new loops a factor $\frac{1}{2}$.

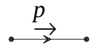
Clearly, all this is quite complicated and not very transparent. However, the results of this section can be reproduced quite simply for the diagrams of the theory, without any considerations concerning Faddeev-Popov ghosts. In that way we can verify the rules for the new theory without leaving any lingering doubts.

4. Diagrams

The Feynman rules for the Dirac theory are as usual:

Electron propagator	$\dfrac{-i\gamma p + m}{p^2 + m^2 - i\varepsilon}$	
Photon propagator	$\dfrac{\delta_{\mu\nu}}{k^2 - i\varepsilon}$	
Vertex	$-ie\gamma^\mu\,.$	

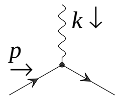
The Feynman rules for the 'two-component' theory are:

Electron propagator $\dfrac{1}{p^2+m^2-i\varepsilon}$ 

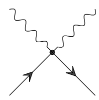
Photon propagator $\dfrac{\delta_{\mu\nu}}{k^2-i\varepsilon}$

and furthermore a three-point and a four-point vertex:

$$-ieV_\mu(p,k) = ie(2ip_\mu + ik_\mu + 2i\sigma^{\nu\mu}k_\nu)$$



$$-2e^2\delta_{\mu\nu}\,.$$



As will be made clear, every string of gamma matrices terminated by spinors must be provided with a factor $1/2m$. Those spinors are the usual ones, precisely as in the Dirac theory. Closed fermion loops have a minus sign and furthermore a factor $\frac{1}{2}$ must be provided.

The convention for the sign of the momenta in the three-point vertex is as usual, *i.e.* the momenta p and k are in going.

It should be noted that the propagator must still be drawn with an arrow, which is of importance in the vertices to which the propagators connect.

There are two basic identities needed to perform the change-over from the Dirac theory to the 'two-component' theory on the diagram level. Consider the vertex of the Dirac theory as occurring in some diagram, together with an electron propagator attached to the incoming electron line at that vertex. The momenta of the incoming electron and the photon will be called p and k respectively. There will be a factor $-i\gamma p+m$ associated with the propagator of the line with momentum p, and we want to move this factor to the other line through the gamma matrix in the vertex. The momentum of that line (the outgoing electron) will be denoted by $p_1 = p+k$:

$$\begin{aligned}
\gamma^\mu(-i\gamma p+m) &= (i\gamma p+m)\gamma^\mu - 2ip_\mu \\
&= (i\gamma p_1+m)\gamma^\mu - i\gamma k - 2ip_\mu \\
&= (i\gamma p_1+m)\gamma^\mu - \tfrac{i}{2}(\gamma^\nu\gamma^\mu - \gamma^\mu\gamma_\mu)k_\nu - ip_{1\mu} - ip_\mu \\
&= (i\gamma p_1+m)\gamma^\mu - 2i\sigma^{\nu\mu}k_\nu - ip_{1\mu} - ip_\mu\,.
\end{aligned}$$

The reader may recognize the three-point vertex of the 'two-component' theory in the last three terms of this expression. If we abbreviate

$$V_\mu(p,k) = -2i\sigma^{\nu\mu}k_\nu - i(p+k)_\mu - ip_\mu$$

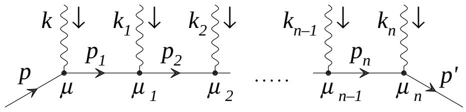
then our first basic identity is

$$\gamma^\mu(-i\gamma p+m) = (i\gamma p_1+m)\gamma^\mu + V_\mu(p,k)\,.$$

We now use this identity on a string such as encountered in diagrams. In a diagram, following a fermion line, we encounter in the Dirac theory vertex factors γ^μ separated by propagators. Let us denote such a string, beginning and ending with a vertex (a γ matrix) by $\boldsymbol{S}$; it is of the form:

$$\boldsymbol{S} = \gamma^{\mu_n} \frac{-i\gamma p_n + m}{p_n^2 + m^2 - i\varepsilon} \gamma^{\mu_{n-1}} \cdots \frac{-i\gamma p_2 + m}{p_2^2 + m^2 - i\varepsilon} \gamma^{\mu_1} \frac{-i\gamma p_1 + m}{p_1^2 + m^2 - i\varepsilon} \gamma^\mu .$$

The figure shows the type of diagram substructure corresponding to such an expression. We have also indicated momenta as used below.



Of course, one must for a given process consider all possible diagrams; of relevance here are the strings obtained by permuting the photon lines. In order to keep things transparent we will not indicate that explicitly, but discuss the consequence of that where relevant.

Consider such a string multiplied on the right with the energy projection operator:

$$\boldsymbol{S}(-i\gamma p + m) .$$

Here p is the initial momentum, as shown in the figure above. Now use the basic identity once, involving γ^μ. Writing only the last part of the string we get:

$$\begin{aligned}
\boldsymbol{S}(-i\gamma p + m) &= \cdots \frac{-i\gamma p_2 + m}{p_2^2 + m^2 - i\varepsilon} \gamma^{\mu_1} \frac{-i\gamma p_1 + m}{p_1^2 + m^2 - i\varepsilon} \left((i\gamma p_1 + m)\gamma^\mu + V_\mu(p,k) \right) \\
&= \cdots \frac{-i\gamma p_2 + m}{p_2^2 + m^2 - i\varepsilon} \gamma^{\mu_1} \gamma^\mu \\
&\quad + \cdots \frac{-i\gamma p_2 + m}{p_2^2 + m^2 - i\varepsilon} \gamma^{\mu_1} \frac{-i\gamma p_1 + m}{p_1^2 + m^2 - i\varepsilon} V_\mu(p,k) .
\end{aligned}$$

The first term involves the product of two γ's, namely $\gamma^{\mu_1}\gamma^\mu$. It is here that it is important to include diagrams with the photon lines permuted. One of them will give precisely the same string as above, but with the indices μ_1 and μ interchanged. We therefore, without explicitly including the diagrams with permuted photon lines, replace this product by $\delta_{\mu_1 \mu}$. Together with the permuted diagram, and including the factors $-ie$ of the Dirac vertices we so obtain the four-point vertex of the 'two-component' theory.

The steps taken so far may be repeated, in the first term using the energy projection operator $-i\gamma p_2 + m$, and in the second term $-i\gamma p_1 + m$. In this way, moving the energy projection operator to the left, we arrive at the final result, to be called the string equation:

$$\boldsymbol{S}(-i\gamma p + m) = \boldsymbol{S}_2 + (i\gamma p' + m)\gamma^{\mu_n} \frac{1}{p_n^2 + m^2} \boldsymbol{S}_2' ,$$

where $\boldsymbol{S}_2$ and $\boldsymbol{S}_2'$ denote strings of propagators and vertices of the 'two-component' theory, always beginning and ending with a three- or four-point vertex. More precisely, they are sums of strings, with the three- and four-point vertices distributed in all possible ways.

We are ready to consider the S-matrix in detail. There are two types of fermion lines, namely through-going lines terminated by spinors on both sides and closed loops. First a fermion line in the Dirac theory with spinors at the beginning and end. The spinors obey the Dirac equation:

$$(i\gamma p + m)u(p) = 0 \qquad \text{and} \qquad \overline{u}(p')(i\gamma p' + m) = 0\,.$$

Consider an expression of the form

$$\overline{u}(p')\,\boldsymbol{S}\,u(p)\,.$$

Since $u(p)$ obeys the Dirac equation we may write:

$$u(p) = \frac{1}{2m}(-i\gamma p + m)u(p)\,.$$

Using our string equation for the combination $\boldsymbol{S}(-i\gamma p + m)$ we find, using the Dirac equation for $\overline{u}(p')$:

$$\overline{u}(p')\,\boldsymbol{S}\,u(p) = \frac{1}{2m}\overline{u}(p')\,\boldsymbol{S}\,(-i\gamma p + m)u(p) = \frac{1}{2m}\overline{u}(p')\,\boldsymbol{S}_2\,u(p)\,.$$

Thus we find for those through-going lines the strings corresponding to the 'two-component' theory, including the factor $1/2m$.

Now closed loops. That is ever so slightly more complicated. Consider a closed loop and choose a propagator. Exhibiting this propagator explicitly the closed loop takes the form:

$$\mathrm{Tr}\left[\boldsymbol{S}\,\frac{-i\gamma p + m}{p^2 + m^2 - i\varepsilon}\right]\,.$$

Using the identity

$$(-i\gamma p + m)^2 = 2m(-i\gamma p + m) - (p^2 + m^2)$$

we substitute

$$\frac{-i\gamma p + m}{p^2 + m^2 - i\varepsilon} = \frac{1}{2m}(-i\gamma p + m)\frac{-i\gamma p + m}{p^2 + m^2 - i\varepsilon} + \frac{1}{2m}\,.$$

Let us concentrate on the first term. We have:

$$\begin{aligned}
&\frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}\,(-i\gamma p+m)\frac{-i\gamma p+m}{p^2+m^2-i\varepsilon}\right]=\frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}_2\,\frac{-i\gamma p+m}{p^2+m^2-i\varepsilon}\right]\\
&\quad+\frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}'_2\,\frac{-i\gamma p+m}{p^2+m^2-i\varepsilon}(i\gamma p+m)\gamma^{\mu_n}\frac{1}{p_n^2+m^2-i\varepsilon}\right]\\
&=\ \frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}_2\,\frac{-i\gamma p+m}{p^2+m^2-i\varepsilon}\right]+\frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}'_2\,\gamma^{\mu_n}\frac{1}{p_n^2+m^2-i\varepsilon}\right].
\end{aligned}$$

At this point it must be remembered that the strings $\boldsymbol{S}_2$ and $\boldsymbol{S}'_2$ involve an even number of γ's only. Since the trace of an odd number of γ's is zero the second term is zero and from the first term only the part with m in the numerator survives. We thus find:

$$\mathrm{Tr}\left[\boldsymbol{S}\,\frac{-i\gamma p+m}{p^2+m^2-i\varepsilon}\right]=\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{S}_2\frac{1}{p^2+m^2}\right]+\frac{1}{2m}\mathrm{Tr}\,[\boldsymbol{S}]\ .$$

Remember that $\boldsymbol{S}_2$ starts and ends with a vertex. All together this is indeed a trace as encountered in the 'two-component' theory, and we also see the factor two, understood as due to a Faddeev-Popov ghost. However, note that the first term has an explicit propagator at the location of the propagator that we started from, and we thus still miss a piece, namely a 'two-component' type diagram with a four-vertex at that location. That however, is precisely what the last term provides, but to see that we must still do some work. Peeling off one vertex and one propagator from the Dirac string $\boldsymbol{S}$ we find:

$$\begin{aligned}
\frac{1}{2m}\mathrm{Tr}\,[\boldsymbol{S}]\ &=\ \frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}'\frac{-i\gamma p_1+m}{p_1^2+m^2-i\varepsilon}\gamma^{\mu}\right]\\
&=\ \frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}'_2\,\frac{1}{p_1^2+m^2}\gamma^{\mu}\right]\\
&\quad+\frac{1}{2m}\mathrm{Tr}\left[(i\gamma p'+m)\gamma^{\mu_n}\frac{1}{p_n^2+m^2}\boldsymbol{S}''_2\,\frac{1}{p_1^2+m^2}\gamma^{\mu}\right]\\
&=\ \frac{1}{2}\mathrm{Tr}\left[\gamma^{\mu_n}\frac{1}{p_n^2+m^2}\boldsymbol{S}''_2\,\frac{1}{p_1^2+m^2}\gamma^{\mu}\right]\\
&=\ \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{S}''_2\,\frac{1}{p_1^2+m^2}\gamma^{\mu}\gamma^{\mu_n}\frac{1}{p_n^2+m^2}\right].
\end{aligned}$$

We used that the trace over a product of an odd number of γ's is zero. Only the part containing m survives. We note the product $\gamma^{\mu}\gamma^{\mu_n}$, and (remembering the argument about including diagrams with permuted photon lines) we recognize a four-point vertex with its surrounding propagators. Note that the propagator of momentum p has disappeared. In other words, this term generates a diagram with a four-point vertex at the location of the propagator that we started from. That is indeed the missing piece. Taking all together we thus find:

$$\mathrm{Tr}\,[\boldsymbol{S}] = \tfrac{1}{2}\mathrm{Tr}\,[\boldsymbol{S}_2]\ .$$

5. The two-component theory

The transition from the 'two-component' to the true two-component theory relies on a remarkable identity. Inserting γ^5 in either a through going fermion line or a loop in the 'two-component' theory makes those diagrams zero. Consider a through going line, and insert γ^5 in front of the spinor u:

$$\overline{u}(p')\boldsymbol{S}_2\,\gamma^5 u(p)\,.$$

Using our string identity backwards this is equal to:

$$\begin{aligned}&\big(\overline{u}(p')\boldsymbol{S}\,(-i\gamma p + m)\gamma^5 u(p)\big)\\ &\quad -\left(\overline{u}(p')(i\gamma p' + m)\gamma^{\mu_n}\frac{1}{p_2^2 + m^2 - i\varepsilon}\boldsymbol{S}_2'\,\gamma^5 u(p)\right) = 0\,.\end{aligned}$$

Using $(-i\gamma p + m)\gamma^5 = \gamma^5(i\gamma p + m)$ in the first term we see that it is zero because $u(p)$ obeys the Dirac equation; similarly the second term is zero for the same reason with respect to $\overline{u}(p')$.

For loops the reasoning is slightly more complex due to the complications relating to the four-point vertex. Let us consider a closed loop (or rather the sum of closed loops of a given order with all possible distributions of the three- and four-point vertices), and select a propagator. There will be a diagram were that propagator is missing, having a four-point vertex at that spot. If again $\boldsymbol{S}_2$ (and $\boldsymbol{S}_2'$) denotes strings with a vertex at beginning and end we have, introducing a γ^5 as well, an expression of the form:

$$\mathrm{Tr}\left[\boldsymbol{S}_2\,\frac{1}{p^2 + m^2}\gamma^5\right] + \mathrm{Tr}\left[\boldsymbol{S}_2'\,\frac{1}{p_1^2 + m^2}\gamma^{\mu}\gamma^{\mu_n}\frac{1}{p_n^2 + m^2}\gamma^5\right]\,.$$

In the last term we move γ^5 one place to the left, getting a crucial minus sign:

$$\frac{1}{2}\mathrm{Tr}\left[\boldsymbol{S}_2\,\frac{1}{p^2 + m^2}\gamma^5\right] - \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{S}_2'\,\frac{1}{p_1^2 + m^2}\gamma^{\mu}\gamma^5\gamma^{\mu_n}\frac{1}{p_n^2 + m^2}\right]\,.$$

Now doing the reverse work, going back to Dirac strings we find as result:

$$\frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}\,(-i\gamma p + m)\gamma^5\frac{-i\gamma p + m}{p^2 + m^2 - i\varepsilon}\right] - \frac{1}{2m}\mathrm{Tr}\left[\boldsymbol{S}\,\gamma^5\right]\,.$$

The reader will have no difficulty verifying that the two terms cancel, and we find again zero.

The transition to the true two-component theory is now very simple: we simply insert a helicity projection operator $\frac{1+\gamma^5}{2}$ in loops as well as in through going lines. That takes care of the factor $\frac{1}{2}$ associated with closed loops; our spinor normalization must now include a factor $1/\sqrt{m}$. Since the square of such a projection operator is that operator itself one may actually put one at every vertex. Note that we have no need for $\frac{1-\gamma^5}{2}$. In a representation were γ^5 is diagonal (see appendix) one obtains the two-component theory in all detail, because a helicity projection operator applied to a four-component spinor makes two of its components equal to zero.

Doing calculations in the two-component theory one needs the expression for the sum over spins of the product $u(p)\overline{u}(p)$. Our spinors are still the usual Dirac spinors, so that is as usual:

$$\sum_{\text{spins}} u(p)\overline{u}(p) = \frac{1}{2E}(-i\gamma p + m)\,.$$

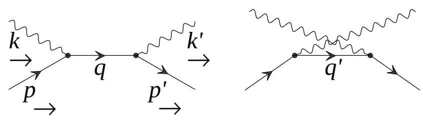
This expression will have factors $\frac{1+\gamma^5}{2}$ on both sides, and therefore the piece $-i\gamma p$ will not contribute. Including a factor $1/m$ in the spinor normalization we find a very simple result, namely just $1/2E$. In the literature the factor $1/2E$ is often put somewhere else, and with that convention the result is simply one (or rather the two-by-two unit matrix σ^0), as stated for example by Brown [2].

6. Anomalous magnetic moment: Compton scattering

Having understood the details of the 'two-component' theory it is now easy to study the consequences of an anomalous magnetic moment of the electron. For this purpose we provide the magnetic term in the interaction (in the 'two-component' formulation) with a factor κ. If $\kappa \neq 1$ there is an anomalous magnetic moment; here however we do not intend to study an anomalous moment, but rather use κ to label the terms due to the magnetic part of the interaction. The Feynman rules are as shown before, except that we provide the term $\sigma^{\mu\nu}k_\nu$ with a factor κ.

We consider here a very simple case, namely Compton scattering. The known expression for that process is remarkable for its simplicity, and it might be interesting to separate out the electric from the magnetic part. This process was also considered by Brown [2], although with specially chosen polarization vectors and no explicit separation of the magnetic part.

In lowest non-vanishing order there are two diagrams contributing to the amplitude, see figure. As variables we use the two dot-products pk and pk' for which we substitute $-m\omega$ and $-m\omega'$. In the rest system of the initial electron, ω and ω' are then the energies of the initial and final photon respectively. The result is (we have not bothered to work out the overall factor)



$$\sum_{\text{el. spins}} |\mathcal{A}|^2 \propto \frac{e^4}{\omega^2\omega'^2}\Bigg[\kappa^4\left(-2\omega\omega'(\omega-\omega')^2 + 8\frac{\omega^2\omega'^2}{m}(\omega-\omega')\right)$$
$$+\kappa^3\left(8\omega\omega'(\omega-\omega')^2 - 16\frac{\omega^2\omega'^2}{m}(\omega-\omega')\right)$$
$$+\kappa^2\left(2m(\omega-\omega')^3 - 10\omega\omega'(\omega-\omega')^2 + 12\frac{\omega^2\omega'^2}{m}(\omega-\omega')\right)$$
$$+\kappa\left(-4m(\omega-\omega')^3 + 12\omega\omega'(\omega-\omega')^2 - 8\frac{\omega^2\omega'^2}{m}(\omega-\omega')\right)$$
$$+4\omega\omega'(\omega^2+\omega'^2) + 2m(\omega-\omega')^3 + (4m^2 - 8\omega\omega')(\omega-\omega')^2$$
$$+\left(4\frac{\omega^2\omega'^2}{m} - 8m\omega\omega'\right)(\omega-\omega')\Bigg] .$$

For $\kappa = 1$ the result reduces to the standard result:

$$\sum_{\text{el. spins}} |\mathcal{A}|^2 \propto \frac{e^4}{\omega^2\omega'^2}\left[4\omega\omega'(\omega^2+\omega'^2) + 4m^2(\omega-\omega')^2 - 8m\omega\omega'(\omega-\omega')\right] .$$

The final expression is surprisingly simple, but that simplicity is lost if the magnetic moment terms are kept separately. It is hard to see the physics of it. While one might expect that magnetic moment terms would contain higher powers of the photon momenta, this turns out not to be the case. The electron propagators are inversely proportional to these same momenta, and nothing special survives. The physics picture remains complicated.

7. Anomalous magnetic moment, renormalizabilty and unitarity

Finally we will discuss briefly the more theoretical aspects of an anomalous magnetic moment. The Lagrangian in the 'two-component' formulation including an anomalous moment is

$$\begin{aligned}\mathcal{L}_2^{\text{anom}} = \quad & -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - (\partial_\mu - ieA_\mu)\overline{\psi}(\partial_\mu + ieA_\mu)\psi - m^2\overline{\psi}\psi \\ & +2ie(1+\lambda)\left(\overline{\psi}\sigma^{\mu\nu}\psi\right)\partial_\mu A_\nu .\end{aligned}$$

This Lagrangian appears to be of a renormalizable type, despite the addition of an anomalous magnetic moment. It is simple to see how to modify the Dirac formulation in order to generate the extra term:

$$\begin{aligned}\mathcal{L}_{\mathrm{D}}^{\mathrm{anom}} = \quad & -\tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \overline{\psi}(\gamma^\mu D_\mu + m)\psi \\ & +2ie\lambda\Big(\overline{\psi}\sigma^{\mu\nu}\frac{1}{-\gamma^\alpha D_\alpha + m}\psi\Big)\partial_\mu A_\nu\,.\end{aligned}$$

This is a non-local interaction that will in general violate unitarity. The extra term can be expanded:

$$\frac{1}{-\gamma\partial + m - ie\gamma^\alpha A_\alpha} = \frac{1}{-\gamma\partial + m}\left(1 + \frac{1}{-\gamma\partial + m}\,ie\gamma^\alpha A_\alpha + \ldots\right)\,.$$

It is clear that this term interferes with our transformation, shifting the factor $-\gamma\partial + m$ from the propagator to the vertex. We therefore loose unitarity, even if the interaction appears local in the 'two-component' theory. The theory is not acceptable for $\lambda \neq 0$.

It is interesting though that an anomalous magnetic moment term in the two component theory is not equivalent to a Pauli term. There is another observation that can be made here. In the 'two-component' formulation with an anomalous moment the string equation will not be valid anymore. Inserting a γ^5 will not give zero. Therefore the true two-component Lagrangian (with only $\overline{\psi}_R\psi_L$ terms) will no more produce a unitary S-matrix, because the hermitean conjugate of that Lagrangian (with $\overline{\psi}_L\psi_R$ terms) produces a different S-matrix.

8. Conclusions

The two-component theory provides for an interesting exercise in field theory and diagrammatic methods. Also, it neatly separates out the electric and magnetic parts of the interaction of photons with electrons, which may be of interest in certain situations. Unitarity however is not automatic in that formalism, and needs separate verification. An anomalous magnetic moment in the two-component theory violates unitarity, as also noted explicitly in Ref. [3].

The original motivation to re-introduce the two-component theory, namely as a formalism appropriate to the study of the theory of weak interactions, has somehow been obscured. We may ask whether the transformation introduced in this paper may with advantage be applied to the Standard Model. We have not investigated that.

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Note added in proof. Some recent work on the two-component formalism has been brought to my attention. See Ref. [7].

Appendix A

Spinor representations

The most common representation has diagonal γ^4:

$$\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix} \qquad \gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \qquad \gamma^4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\gamma^5 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

where $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$. Using the σ matrices including the identity σ^0:

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

this can also written as

$$\gamma^j = \begin{pmatrix} 0 & -i\sigma^j \\ i\sigma^j & 0 \end{pmatrix} \qquad \gamma^4 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & -\sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}.$$

In this representation the spinors are ($E = \sqrt{\vec{p}^2 + m^2}$):

$$\times\sqrt{\frac{m+E}{2E}}\quad
\begin{array}{|c|c|c|c|}
\hline
u^1(p)\uparrow & u^2(p)\downarrow & u^3(p)\downarrow & u^4(p)\uparrow \\
\hline
1 & 0 & \frac{-p_3}{m+E} & \frac{p_1-ip_2}{m+E} \\
0 & 1 & -\frac{p_1+ip_2}{m+E} & \frac{-p_3}{m+E} \\
\frac{p_3}{m+E} & \frac{p_1-ip_2}{m+E} & -1 & 0 \\
\frac{p_1+ip_2}{m+E} & \frac{-p_3}{m+E} & 0 & 1 \\
\hline
\end{array}$$

The first two columns are the particle spinors; they are solutions of the equation $(i\gamma p + m)u(p) = 0$. The last two columns are the anti-particle spinors, solutions of the equation $(-i\gamma p + m)u(p) = 0$. One may verify:

$$u^{1,2}(p) = \frac{-i\gamma p + m}{\sqrt{2E(m+E)}}\; v^{1,2} \qquad u^{3,4}(p) = \frac{i\gamma p + m}{\sqrt{2E(m+E)}}\; v^{3,4}\,,$$

where the v^j are equal to the u^j for zero momentum (and thus also $E = m$): $v^j = u^j(0)$.

Concerning the sum over spins one has:

$$\begin{aligned}
\sum_{j=1}^{2} u^j\,(\vec{p})\,\bar{u}^j\,(\vec{p}) &= \tfrac{1}{2E}\,(-i\gamma p + m) \qquad \text{and} \qquad \textstyle\sum_{j=1}^{2} v^j \bar{v}^j = \tfrac{1}{2}(\gamma^4 + 1) \\
\sum_{j=3}^{4} u^j\,(\vec{p})\,\bar{u}^j\,(\vec{p}) &= \tfrac{1}{2E}\,(-i\gamma p - m) \qquad \text{and} \qquad \textstyle\sum_{j=1}^{2} v^j \bar{v}^j = \tfrac{1}{2}(\gamma^4 - 1).
\end{aligned}$$

The relation between the spin sums for the u and v can be made explicit:

$$\begin{aligned}
\sum_{j=1}^{2} u^j\,(\vec{p})\,\bar{u}^j(\vec{p}) &= \frac{-i\gamma p + m}{\sqrt{2E(m+E)}}\,\tfrac{1}{2}(\gamma^4 + 1)\,\frac{-i\gamma p + m}{\sqrt{2E(m+E)}} \\
&= \frac{-i\gamma p + m}{4E(m+E)}\,(2m - 2ip_4) = \frac{-i\gamma p + m}{2E}\,.
\end{aligned}$$

A representation in which γ^5 is diagonal can be obtained by means of the transformation

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & -\sigma^0 \\ \sigma^0 & \sigma^0 \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sigma^0 & \sigma^0 \\ -\sigma^0 & \sigma^0 \end{pmatrix}.$$

The γ^j, $j = 1, 2, 3$ are invariant under this transformation. For $T\gamma^4 T^{-1}$ and $T\gamma^5 T^{-1}$ one finds:

$$\gamma^4 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}.$$

Thus simply $\gamma^5 \to \gamma^4$, $\gamma^4 \to -\gamma^5$. The combination $\frac{1+\gamma^5}{2}$ when applied to a spinor projects out the upper two components. The transformed spinors $u^j(p) \to Tu^j(p)$ are:

$\times\sqrt{\dfrac{m+E}{4E}}$

$u^1(p)\uparrow$	$u^2(p)\downarrow$	$u^3(p)\downarrow$	$u^4(p)\uparrow$
$1-\dfrac{p_3}{m+E}$	$-\dfrac{p_1-ip_2}{m+E}$	$1-\dfrac{p_3}{m+E}$	$\dfrac{p_1-ip_2}{m+E}$
$-\dfrac{p_1+ip_2}{m+E}$	$1+\dfrac{p_3}{m+E}$	$-\dfrac{p_1+ip_2}{m+E}$	$-1-\dfrac{p_3}{m+E}$
$1+\dfrac{p_3}{m+E}$	$\dfrac{p_1-ip_2}{m+E}$	$-1-\dfrac{p_3}{m+E}$	$\dfrac{p_1-ip_2}{m+E}$
$\dfrac{p_1+ip_2}{m+E}$	$1-\dfrac{p_3}{m+E}$	$-\dfrac{p_1+ip_2}{m+E}$	$1-\dfrac{p_3}{m+E}$

and the $v \to Tv$:

$\times\dfrac{1}{\sqrt{2}}$

$v^1\uparrow$	$v^2\downarrow$	$v^3\downarrow$	$v^4\uparrow$
1	0	1	0
0	1	0	−1
1	0	−1	0
0	1	0	1

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