Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 4

"TRUE DEGREES OF FREEDOM" OF A SPHERICALLY SYMMETRIC, SELF-GRAVITATING DUST SHELL*

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(Received January 9, 1998)

Dedicated to Andrzej Trautman in honour of his 64th birthday

A hamiltonian description of the physical system composed of a dust shell interacting with the gravitational field is considered. In the spherically symmetric case, the phase space of the system is effectively reduced with respect to the Gauss–Codazzi constraints. The Hamiltonian of the system (numerically equal to the value of the A. D. M. mass) is explicitly calculated in terms of the "true degrees of freedom", *i.e.* as a function on the reduced phase space.

PACS numbers: 04.20. Fy

1. Introduction

Matter shell is a singularity of the space-time geometry which arises when two different, smooth 4-geometries are sewed across a (2+1)-dimensional hypersurface Σ in such a way that they induce the same 3-geometry on Σ. Dynamics of the composed "matter shell + gravity" system may be described by the empty-space version of the Einstein equations outside the shell and the Israel condition on the shell (see [3]). In paper [2] the hamiltonian formulation of this dynamics was derived from first principles, for a general fluid-type matter. For this purpose an appropriate Lagrangian $L = L_{\rm grav} + L_{\rm matter}$ was used. For its gravitational part $L_{\rm grav}$, the quantity $\frac{1}{16\pi}\sqrt{|g^{(4)}|}\,\boldsymbol{R}^{(4)}$ was taken, with $\boldsymbol{R}^{(4)} = R^{(4)}_{\rm reg} + R^{(4)}_{\rm sing}$, where $R^{(4)}_{\rm reg}$ is the regular part of the scalar curvature of the (piecewise smooth) spacetime outside of the shell and $R^{(4)}_{\rm sing}$ is its singular part. The δ-like singularity is concentrated

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* Presented at the Workshop on Gauge Theories of Gravitation, Jadwisin, Poland, September 4–10, 1997.

on the shell. It arises due to the fact that the metric, although continuous, is not in general C^1 across the shell and, therefore, the connection coefficients $\Gamma^\lambda_{\mu\nu}$ have a jump on Σ.

As a matter Lagrangian L_{matter}, the usual rest-frame energy density (*cf.* [5]) was taken. It is completely determined by a choice of a single function $e_0 = e_0(\omega)$ describing the dependence of the molar rest-frame energy e_0 of the fluid upon its molar rest-frame, surface density ω. The above function plays role of a *state equation.* It determines completely the mechanical properties of the fluid. The particular example of a dust matter corresponds to the constant function $e_0(\omega) \equiv m$ (the dust carries only the rest-mass energy m which is deformation-independent).

Hence, the phase space of initial data of the above system may be parametrized by the following space of functions:

$$\mathcal{P} := \{(g_{kl}, P^{kl}, y^k, p_k)\} \ .$$

Here g_{kl} is a Riemannian metric on a 3-dimensional, space-like Cauchy surface $\mathcal{C}$ and P^{kl} is the corresponding A. D. M. momentum describing its external curvature (in the present paper we limit ourselves to the topologically trivial case $\mathcal{C} \simeq \boldsymbol{R}^3$ and we assume that its geometry is asymptotically flat at infinity). The remaining objects y^k and p_k describe the configuration and the momentum assigned to all the points of the material. Mathematically, all these points are organized into a 2-dimensional compact manifold Z which we call a *material space* and parameterize typically by coordinates (z^1, z^2). It is equipped with a volume structure, *i.e.* a differential 2-form

$$\chi = h(z)\ dz^1 \wedge dz^2$$

which, when integrated over a domain $D \subset Z$, gives us the amount of matter (calculated *e.g.* in moles) contained in D. Configurations of the material are described by embeddings $F : Z \to \mathcal{C}$. Given a coordinate system (x^k) on $\mathcal{C}$, we may describe such an embedding by three functions y^k, which assign the physical space position $x^k := y^k(z^A)$, $k = 1, 2, 3$; $A = 1, 2$; to each point (z^A) of the material. Functions $p_k = p_k(z)$ describe the amount of mechanical momentum carried by the matter.

The phase space is, therefore, described by 12 functions (g_{kl}, P^{kl}) depending on 3 independent variables x^k and 6 functions (y^k, p_k) depending on 2 independent variables z^A.

Dynamical equations of the system impose the following constraints on the above data:

$$\mathcal{H} := \frac{1}{16\pi}\left\{\frac{1}{\sqrt{g}}\left(P^{kl}P_{kl} - \frac{1}{2}(P^k_k)^2\right) - \sqrt{g}\ \boldsymbol{R}\right\} + E(x, p, g) \equiv 0 \ , \qquad (1)$$

$$\mathcal{H}_l := -\frac{1}{8\pi} P^k{}_{l|k} + P_l \equiv 0 \ , \qquad (2)$$

where the matter energy-density E and momentum-density P_l are defined as components of the canonical energy-momentum tensor density $\mathcal{T}^{\lambda}{}_{\mu}$ of matter: $E := \mathcal{T}^0{}_{\mu} n^{\mu}$ and n^{μ} is a future oriented, space-time vector, orthogonal to $\mathcal{C}$ and $P_l := \mathcal{T}^0{}_l$. They are derived in a usual way from the matter Lagrangian L_{matter}. Since the latter vanishes outside the shell, E and P_l are δ-like, singular objects concentrated on the 2-dimensional shell's configuration $\mathcal{S} := F(Z) \subset \mathcal{C}$. More precisely, we have

$$P_l = \Delta_{\mathcal{S}} F_*(\chi) p_l \ , \tag{3}$$

$$E = \Delta_{\mathcal{S}} F_*(\chi) e(x, p, g) \ , \tag{4}$$

where by $\Delta_{\mathcal{S}}$ we denote the invariant Dirac "delta" concentrated on $\mathcal{S}$ (we have chosen this unusual notation in order to keep the lower case δ for denoting "variations", *i.e.* differential forms in infinite-dimensional, functional spaces). By $F_*(\chi)$ we denote the physical-space matter density defined by the push-forward of χ from the material space Z to $\mathcal{S}$. Every state equation $e_0 = e_0(\omega)$ implies a specific form of the fluid energy e as a function of momenta and the metric. For the dust matter *e.g.* we have

$$e = \sqrt{m^2 + g^{kl} p_k p_l} \ . \tag{5}$$

The constraints may actually be formulated in terms of equations (1) and (2), if we understand them in the sense of distributions. These equations contain, in fact, the usual vacuum constraints outside of the shell (*i.e.* where E and P_l vanish), together with information about necessary discontinuities of the connection coefficients and of P^{kl} along $\mathcal{S}$. These discontinuities are contained in a singular part of the 3-dimensional curvature $\boldsymbol{R}$ (equation (1)) and in a singular part of the divergence $P^k{}_{l|k}$ (equation (2)). They *must* arise in order to match the singular matter energy E and momentum P_l, respectively.

Hence, we assume from the very beginning that g_{kl} and P^{kl} are piecewise smooth outside of the shell and that g_{kl} is continuous also across the shell.

The above phase space is equipped with the canonical pre-symplectic structure

$$\Omega := \frac{1}{16\pi} \int\limits_{\mathcal{C}} \left\{ \delta P^{kl}(x) \wedge \delta g_{kl}(x) \right\} d^3x + \int\limits_{Z} \left\{ \delta p_k(z) \wedge \delta y^k(z) \right\} h d^2 z \ , \tag{6}$$

which means that – without constraints – the Poisson brackets between P^{kl} and g_{kl} together with Poisson brackets between p_k and y^k would have been δ-like, whereas remaining Poisson brackets would have vanish. Taking into account constraints, the above quantities are no longer independent, which gives rise to the degeneracy of Ω.

The total Hamiltonian of the system is equal to the A. D. M. mass (see [2] and [4]) and may be calculated in terms of the standard 2-dimensional surface integral "at infinity" of $\mathcal{C}$ (the usual volume term $N\mathcal{H} + N^k\mathcal{H}_k$ is skipped here because we always work *on shell*).

The phase space $\mathcal{P}$ may be quotiented with respect to the degeneracy of Ω. Cauchy data (points of $\mathcal{P}$) which may be joint be a one-parameter family of data (a curve) whose tangent vector belongs to the degeneracy of Ω are treated as physically equivalent. The goal of the present paper is to describe explicitely this quotient space $\tilde{\mathcal{P}}$ (*i.e.* the space of classes of gauge-equivalent Cauchy data) and to express the hamiltonian in terms of the "true degrees of freedom" in $\tilde{\mathcal{P}}$. We are able to accomplish this task in the spherically symmetric case. The resulting reduced phase space $\tilde{\mathcal{P}}$ is 2-dimensional and may be nicely parametrized by two geometric quantities characterizing the shell's space-like configuration.

Two strategies are possible when dealing with the reduction problem in gauge theories. The first strategy consists in looking for the complete system of gauge-invariants which may be used to parameterize classes of gauge-equivalent physical states. Unfortunately, this *royal road* is not always possible and we must use the other strategy, based on gauge-fixing. It consists in choosing particular representatives in each class, selected by imposing various gauge conditions. In this treatment, the quotient space may be parameterized by the space of those special representatives. This description depends *a priori* upon the gauge conditions used. It turns out that in our case the purely spatial gauge, generated by the momentum constraint (2) may be treated in a gauge invariant (*royal*) way, whereas time-like gauge generated by the energy constraint (1) needs a gauge-fixing approach (or, at least, we are not able to do it without any gauge-fixing). Finally, we show how to reconstruct the total space-time geometry from the reduced dynamics.

2. Spherically symmetric case

Assuming the S^2 topology of the shell, it is always possible to find a "spherical parameterization" of Z, where $z^1 = \Theta$, $z^2 = \Phi$ and the density h is proportional to the volume form on the unit sphere. Hence, without any loss of generality we may always assume that

$$\chi = \sin\Theta \; d\Theta \wedge d\Phi \; . \tag{7}$$

Assuming spherical symmetry of the data $(g_{kl}, P^{kl}, y^k, p_k)$ means that there is a coordinate system $(x^1, x^2, x^3) = (\theta, \varphi, r)$ in $\mathcal{C}$ in which we have

$$
\begin{aligned}
y^1(\Theta, \Phi) &= \theta \ , \\
y^2(\Theta, \Phi) &= \varphi \ , \\
y^3(\Theta, \Phi) &= \zeta = \text{const} \ , \\
p_3 &= p = \text{const} \ , \\
p_A &= 0 \ , \\
g_{AB} &= l(r) \ \gamma_{AB} \ , \\
g_{33} &= n^2(r) \ , \\
g_{3A} &= 0 \ , \\
P^{AB} &= \tfrac{1}{2} s(r) \ \gamma^{AB} \sqrt{\det \gamma} \ , \\
P^{33} &= \frac{f(r)}{n(r)} \ \sqrt{\det \gamma} \ , \\
P^{3A} &= 0 \ ,
\end{aligned}
$$

where by γ_{AB} we denote the standard metric on the unit sphere

$$
\gamma_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta \end{pmatrix} \ . \tag{8}
$$

The subspace $\mathcal{P}_{\text{sym}}$ of spherically symmetric data may, therefore, be parameterized by two numbers (ζ and p) and by four functions (l, n, s, f) depending on a single variable r. The functions are piecewise smooth outside of the shell position $r = \zeta$, whereas the metric coefficients n and l must be also continuous at $r = \zeta$.

These data are subject to constraints. It is a matter of simple calculations to show that the radial momentum constraint $\mathcal{H}_3 = 0$ may be written as

$$
-8\pi \frac{p}{n} \ \Delta(r - \zeta) = f' - \frac{1}{2} \frac{s}{n} \ l' \ , \tag{9}
$$

where by "$'$" we denote radial derivative $\partial / \partial r$ and Δ stands for the usual, one-dimensional Dirac delta distribution. It is obvious that remaining momentum constraints $\mathcal{H}_A = 0$, $A = 1, 2$, are automatically satisfied. Similarly, the energy constraint may be written as follows:

$$
-8\pi e(p, l, n) \ \Delta(r - \zeta) = \left(\frac{l'}{n} \right)' - n - \frac{1}{4} \frac{(l')^2}{ln} + \frac{1}{4} \frac{n}{l} \ f^2 - \frac{1}{2} f s \ . \tag{10}
$$

Different materials correspond to different state equations $e = e(p, l, n)$. For the incoherent (dust) matter, equation (5) implies:

$$
e = \sqrt{m^2 + \left(\frac{p}{n(\zeta)} \right)^2} \ . \tag{11}
$$

The above 2 (per each point "r") constraints generate 2-dimensional (per point) group of spacetime reparameterizations, where variables (t, r) may be replaced with any other variables $(\tilde{t}, \tilde{r})$, respecting the spherical symmetry of our problem. Gauge transformations arise as degeneracy directions of the symplectic structure of $\mathcal{P}_{\rm sym}$, obtained by restricting the form Ω from $\mathcal{P}$ to $\mathcal{P}_{\rm sym}$. To calculate this restriction we insert our spherical symmetric *Ansatz* into formula (6) and integrate over angles. This yields us the following symplectic structure in $\mathcal{P}_{\rm sym}$:

$$\Omega = \delta(4\pi p) \wedge \delta\zeta + \int_0^\infty \left(\frac{1}{4} \delta s(r) \wedge \delta l(r) + \frac{1}{2} \delta f(r) \wedge \delta n(r) \right) dr \ . \tag{12}$$

For computational reasons, it will be useful to represent the 2-form Ω as an exterior derivative $\Omega = \delta \Lambda$ of the following 1-form:

$$\Lambda = P\delta\zeta + \int_0^\infty \left(\frac{1}{4} s(r) \delta l(r) - \frac{1}{2} n(r) \delta f(r) \right) dr \ , \tag{13}$$

where we have denoted $P := 4\pi p$.

Every spherically symmetric solution of the vacuum Einstein equations must be isomorphic to a Schwarzschild solution, and the only one between them which satisfies regularity condition at $r = 0$ is the flat Minkowski space. This implies that the geometry of our $\mathcal{C}$ must be a smooth 3-dimensional subspace of the Minkowski space for $r < \zeta$ and a subspace of a Schwarzschild space-time for $r > 0$. Both the value of the mass of the external Schwarzschild space-time and the shape of the cut between it and the internal Minkowski spacetime have to be determined from the dynamics.

3. Solving constraint equations

To be able to solve the constraints we are going to impose a gauge condition which enables us to fix uniquely the time variable. For technical reasons we start with a family of gauge conditions which have been used in [7] to prove the positivity of the A. D. M. energy. We will show elsewhere how to pass to any other gauge condition.

Consider the following gauge condition: $\beta P^{33} g_{33} + P^{AB} g_{AB} = 0$, where β is a fixed costant (as a particular case we may obtain this way the "maximal surface" condition for $\beta = 1$). In terms of our parameterization, this gauge condition reads:

$$\frac{s}{n} = -\beta \frac{f}{l} \ . \tag{14}$$

In the present paper only values $\beta < -1$ are considered. Inserting the above relation into momentum constraint (9) we see that outside of the shell our data must fulfill the following equation

$$f' + \frac{1}{2}\beta\, \frac{l'}{l} f = 0\ . \tag{15}$$

This implies that the function $\log(f\, l^{\frac{\beta}{2}})$ is constant outside of the shell. We conclude that

$$f = \begin{cases} A_+ l^{-\frac{\beta}{2}} & \text{for} \quad r > \zeta\ , \\ A_- l^{-\frac{\beta}{2}} & \text{for} \quad r < \zeta\ . \end{cases} \tag{16}$$

The difference $(A_+ - A_-)$ is also determined by the constraint (9). Indeed, the only singular term on the right hand side is produced by the jump of f and is equal to $(A_+ - A_-)(l(\zeta))^{-\frac{\beta}{2}}\Delta(r-\zeta)$. Let us denote by

$$R := \sqrt{l(\zeta)}$$

the physical radius of the shell and by

$$U := \frac{P}{n(\zeta)\sqrt{l(\zeta)}} = \frac{4\pi p}{n(\zeta) R} \tag{17}$$

the normalized radial momentum. Equation (9) implies:

$$A_+ - A_- = -2UR^{1+\beta}\ . \tag{18}$$

The constraint equation is not yet solved, because we still have to fulfill the boundary conditions: for $r \to \infty$ our data must be asymptotically flat and for $r \to 0$ must be regular. Taking into account that l must behave like r^2 at 0 and at infinity, the above condition implies that for $\beta < -1$ we have $A_+ = 0$ because of the necessary condition $f(\infty) = 0$. Hence, we have:

$$f = \begin{cases} 0 & \text{for} \quad r > \zeta\ , \\ 2UR^{1+\beta} l^{-\frac{\beta}{2}} & \text{for} \quad r < \zeta\ . \end{cases} \tag{19}$$

Now we are going to solve the hamiltonian constraint (10). We first do it piecewise, outside the shell. Using our gauge condition (14) and the previous result (16) we obtain the following equation:

$$0 = \left(\frac{l'}{n}\right)' - n - \frac{1}{4}\frac{(l')^2}{ln} + \frac{1+2\beta}{4l^{1+\beta}} n A_\pm^2\ , \tag{20}$$

where $A_\pm$ stands for A_+ outside of the shell and for A_- inside the shell. Let us introduce the following quantity:

$$k := \frac{l'}{n\sqrt[4]{l}} \ . \tag{21}$$

It is easy to check that equation (20) may be rewritten as follows:

$$0 = \frac{\sqrt[4]{l}}{n}\ k' - 1 + \frac{1+2\beta}{4l^{1+\beta}} A_\pm^2 \ . \tag{22}$$

This, in turn, is equivalent to

$$0 = \left\{ k^2 - 4\sqrt{l}\left(1 + \frac{A_\pm^2}{4l^{1+\beta}}\right)\right\}' \ . \tag{23}$$

This means that the function under differentiation sign must be piecewise constant. However, the *internal* constant (*i.e. inside* the shell) must vanish due to the regularity condition for the metric at $r = 0$. Indeed, for small r the function l must behave like n^2r^2. This implies that k^2 behaves like $4\sqrt{l}$ and there is no room for any non-vanishing constant (we see that for $-1 < \beta < 0$ we would have got a singularity a $r \to 0$). Denoting the remaining *external* constant by "$-8H$" we conclude that

$$\frac{l'}{n} = \begin{cases} \pm 2\sqrt{l}\sqrt{1 - \frac{2H}{\sqrt{l}} + \frac{A_+^2}{4l^{1+\beta}}} & \text{for} \quad r > \zeta \ , \\ 2\sqrt{l}\sqrt{1 + \frac{A_-^2}{4l^{1+\beta}}} & \text{for} \quad r < \zeta \ , \end{cases} \tag{24}$$

where the sign in the first line is $+1$ or -1, according to whether l' is positive or negative. Outside of the shell the sign may even change at those points where the expression under the square root vanishes. On the contrary, inside the shell the sign is unambiguously positive because for a space-like hypersurface in the Minkowski space l is *always* increasing.

We claim that the value of H is equal to the A. D. M. energy calculated at $r \to \infty$. The easiest way to verify this statement consists in choosing the coordinate r in such a way that, for big values of r, we have $l(r) = r^2$. In this gauge equation (24) implies:

$$g_{33} = n^2 = \frac{1}{1 - \frac{2H}{r} + \frac{A_+^2}{4r^{2+2\beta}}} \ , \tag{25}$$

where $A_+ = 0$. We conclude that H plays role of the Hamiltonian of our physical system. Its value may be calculated from the singular part of equation (10). Indeed, the singular part of the right hand side of (10) is equal to the jump of the function $\frac{l'}{n}$. This way we obtain

$$-8\pi e = 2R\left(\epsilon\sqrt{1-\frac{2H}{R}+\frac{A_+^2}{4R^{2+2\beta}}}-\sqrt{1+\frac{A_-^2}{4R^{2+2\beta}}}\right)\ , \tag{26}$$

where by ϵ we denote the sign of l' on the external face of the shell. Using our previous results and denoting by $M := 4\pi m$ the total rest mass of the shell we obtain

$$-\sqrt{\left(\frac{M}{R}\right)^2+U^2} = \epsilon\sqrt{1-\frac{2H}{R}}-\sqrt{1+U^2}\ . \tag{27}$$

Consequently, we obtain an unambiguous expression for the Hamiltonian:

$$H(R,U) = \frac{R}{2}\left\{1-\left(\sqrt{1+U^2}-\sqrt{\left(\frac{M}{R}\right)^2+U^2}\right)^2\right\} \tag{28}$$

Also the value of ϵ may be obtained unambiguously from equation (27):

$$\epsilon = \mathrm{sgn}\ \left(\sqrt{1+U^2}-\sqrt{\left(\frac{M}{R}\right)^2+U^2}\right)\ . \tag{29}$$

We see that our reduced space $\tilde{\mathcal{P}}$ may be globally parameterized by two variables: R and U, with the function l and the constants ζ and β playing role of pure gauge parameters. For a given point (R, U), each choice of these gauge parameters enables us to reconstruct the entire Cauchy data. In this reconstruction we use the values of the constants $A_\pm$ and H, given uniquely by (19) and (28). The gauge parameters are not, however, completely free. The first condition is obvious: $l(\zeta) = R^2$. Moreover, l must be monotonically increasing from 0 to R^2 for $r < \zeta$. Whether or not l remains monotonic also outside of the shell depends upon the sign "ϵ" given by (29). If it is positive, l must be everywhere increasing (and – because of the asymptotic flatness – it must behave like r^2 at infinity). If it is negative, the function l must first be decreasing outside of the shell till it arrives to its minimal value $l_{\min}$ and then it must increase again up to infinity. The value $l_{\min}$ corresponds to the point in which we have $l' = 0$. Due to equation (24), it may be obtained as the root of the following equation:

$$1-\frac{2H}{\sqrt{l_{\min}}}+\frac{A_+^2}{4l_{\min}^{1+\beta}} = 0\ . \tag{30}$$

The sphere corresponding to $l_{\min}$ has the *smallest* area and may be called a *throat* of our space (for $\beta < -1$ the above equation gives us the standard value $\sqrt{l_{\min}} = 2H$). Any function l which satisfies these conditions and which has a non-degenerate minimum at the throat (*i.e.* such that the second derivative of l does not vanish) may be chosen at will. Then, equations (24), (19) and, finally, (14) enable us to reconstruct completely the data (n, l, f, s). Within our gauge subspace given by (14), all states may be obtained this way. We will show that states obtained for different choices of ζ and l are gauge equivalent.

Equation (19) implies that for negative β's the entire external curvature of our surface vanishes outside of the shell: $P^{kl} = 0$. Because the spacetime outside of the shell must be Schwarzschild with mass H, the only surfaces satisfying this condition are the standard Schwarzschild $\{t = \text{const}\}$ subspaces. Our Cauchy surfaces corresponding to different *negative* values of β coincide, therefore, outside of the shell and differ only inside the shell.

4. Canonical structure of the reduced phase space

We will be able to derive the dynamics of our system from the Hamiltonian (28) once we know the reduces symplectic structure in terms of the variables (R, U). For this purpose we restrict the canonical form Ω to the gauge space (14) and express it in terms of the parameters (R, U, l, ζ). It turns out that the form does not depend upon l and ζ which proves that the latter are really gauge parameters. Technically, it is simpler to work with the 1-form Λ given by formula (13) because the reduction commutes with external derivative.

Using (14) we rewrite (13) in the following way:

$$\begin{aligned}
\Lambda &= P\delta\zeta + \int_0^\infty \left[\frac{1}{4}s\delta l - \frac{1}{2}n\delta\left(l^{-\frac{\beta}{2}}\ f\ l^{\frac{\beta}{2}}\right)\right] , \\
&= P\delta\zeta + \int_0^\infty \left[\frac{1}{4}n\left(\frac{s}{n} + \beta\frac{f}{l}\right)\delta l - \frac{1}{2}nl^{-\frac{\beta}{2}}\ \delta\left(fl^{\frac{\beta}{2}}\right)\right] . \qquad (31)
\end{aligned}$$

Reducing the form to the subspace of data fulfilling our gauge condition (14) consists in dropping out the first term under the above integral.

Equation (19) implies:

$$fl^{\frac{\beta}{2}} = 2UR^{1+\beta}\ B(\zeta - r) , \qquad (32)$$

where by B we denote the Heaviside function. Hence

$$\left[\delta\left(fl^{\frac{\beta}{2}}\right)\right](r) = 2B(\zeta - r)\delta(UR^{1+\beta}) + 2UR^{1+\beta}\Delta(\zeta - r)\delta\zeta , \qquad (33)$$

because the derivative of $B(\zeta - r)$ with respect to ζ produces the Dirac distribution. We obtain this way

$$-\frac{1}{2}\int_0^\infty nl^{-\frac{\beta}{2}}\left[\delta\left(fl^{\frac{\beta}{2}}\right)\right](r)dr = -\left\{\int_0^\zeta \left(nl^{-\frac{\beta}{2}}\right)(r)dr\right\}\delta(UR^{1+\beta}) - URn(\zeta)\delta\zeta\ , \tag{34}$$

because $l(\zeta) = R^2$. But $URn(\zeta) = P$ and we see that the last terms kills the term $P\delta\zeta$ in Λ. Finally, we obtain:

$$\Lambda = -w_-\delta(UR^{1+\beta})\ , \tag{35}$$

where we have denoted by w_- the following quantity:

$$w_- := \int_0^\zeta \left(nl^{-\frac{\beta}{2}}\right)(r)dr\,. \tag{36}$$

Using equation (24) we get:

$$w_- := \int_0^\zeta \frac{n}{l'}\ l^{-\frac{\beta}{2}}l'dr = \int_0^\zeta \frac{l^{-\frac{1+\beta}{2}}}{2\sqrt{1+\left[U\left(\frac{\sqrt{l}}{R}\right)^{-(1+\beta)}\right]^2}}\ l'dr\ . \tag{37}$$

Because l is a monotonic function in the interval $[0,\zeta]$ and $-(1+\beta) > 0$, we may calculate the integral with respect to the following variable

$$\xi := U\left(\frac{\sqrt{l}}{R}\right)^{-(1+\beta)}\ , \tag{38}$$

over the interval $[0,U]$. Hence

$$w_- = -\frac{1}{1+\beta}\left(UR^{1+\beta}\right)^{\frac{2}{1+\beta}-1}\int_0^U \frac{\xi^{-\frac{2}{1+\beta}}}{\sqrt{1+\xi^2}}\ d\xi\ . \tag{39}$$

The last integral may be denoted as $F(U)$, where F is the indefinite integral. It turns out, that the exact form of the function F will not be necessary for our purposes. Indeed, we have:

$$\begin{aligned}\Lambda &= \frac{1}{1+\beta}F(U)\left(UR^{1+\beta}\right)^{\frac{2}{1+\beta}-1}\delta(UR^{1+\beta}) = \frac{1}{2}F(U)\delta\left[U^{\frac{2}{1+\beta}}R^2\right]\ ,\\ &= \frac{1}{2}\delta\left[F(U)U^{\frac{2}{1+\beta}}\ R^2\right] - \frac{1}{2}\ R^2U^{\frac{2}{1+\beta}}F'(U)\delta(U)\,,\\ &= \frac{1}{2}\delta\left[F(U)U^{\frac{2}{1+\beta}}\ R^2\right] - \frac{1}{2}\ R^2\ \frac{1}{\sqrt{1+U^2}}\ \delta(U)\,.\end{aligned} \tag{40}$$

The first term is a complete (variational) derivative. Hence, it may by skipped out when we calculate the external derivative $\Omega = \delta\Lambda$. The second term may be rewritten as follows:

$$\tilde{\Lambda} = -\tfrac{1}{2}R^2\ \delta\mu\ , \tag{41}$$

where the quantity μ has been defined as $\mu :=$ arcsinh U or $U = \sinh\mu$. It plays role of a momentum canonically conjugate to the "area" variable $\rho := R^2$, because of the formula

$$\Omega = \delta\tilde{\Lambda} = -\tfrac{1}{2}\delta(R^2) \wedge \delta\mu = \tfrac{1}{2}\delta\mu \wedge \delta\rho\ . \tag{42}$$

Rewriting the Hamiltonian (28) in terms of the canonical variables (μ, ρ), we finally get:

$$H(\mu,\rho) = \frac{1}{2}\sqrt{\rho}\left\{1 - \left(\cosh\mu - \sqrt{\frac{M^2}{\rho} + \sinh^2\mu}\right)^2\right\} \tag{43}$$

We are going to prove that the quantity μ may be interpreted as the *hyperbolic angle* between the vector normal to Schwarzschild surface $\{t = \text{const}\}$ (calculated on the external face of the shell) and the vector normal to the Minkowski surface $\{t = \text{const}\}$ (calculated on the internal face of the shell). The angle $\alpha(\boldsymbol{u},\boldsymbol{v})$ between two normalized vectors $\boldsymbol{u},\boldsymbol{v}$ is defined by their (hyperbolic) scalar product: $\cosh\alpha(\boldsymbol{u},\boldsymbol{v}) := (\boldsymbol{u}|\boldsymbol{v})$. Similarly as in Euclidean geometry, we may call this quantity the angle between the two surfaces: the Schwarzschild one and the Minkowski one. To prove the above interpretation of μ it is sufficient to use formula (24). On the internal side of the shell we use the second formula and put $l = R^2$. We get as a result:

$$\frac{l'}{2\sqrt{l}n} = \sqrt{1 + U^2}\ . \tag{44}$$

But inside of the shell the geometry of $\mathcal{C}$ is given by a 3-dimensional, spherically symmetric surface in the Minkowski space. It is easy to show that in the Minkowski space the quantity on the left hand side is equal to $\cosh\alpha$, where α is *precisely* the angle between such a subspace and the Minkowskian flat $\{t = \text{const}\}$ surface. This implies that $U = \sinh\alpha$. But our surface is just the smooth continuation (across the shell) of the external Schwarzshild $\{t = \text{const}\}$ surface. This finally proves that $\mu = \alpha$ is the angle between the Schwarzshild and the Minkowski 3-dimensional leaves.

5. Dynamics. Reconstruction of the spacetime geometry

The Hamiltonian (43) generates uniquely the dynamical equations for the canonical variables (ρ, μ). Take any solution of these equations. Next, for each time separately let us choose the gauge variables ζ and l. This enables us to reconstruct the complete set of Cauchy data at each instant of time, separately. To reconstruct the entire space-time geometry we have to find also the lapse and the shift functions. For this purpose we use Einstein equations, where canonical data g_{kl} and P^{kl} (together with their time derivatives) are already known. The resulting equations are elliptic equations for the lapse and the shift. For the lapse function we obtain this way a second order equation in variable r as a condition to preserve the β-gauge in time. This equation has to be solved with boundary conditions: $N = 1$ at infinity and $\frac{dN}{dr} = 0$ at $r = 0$. To calculate the shift function we use the equation for the time derivative of the 3-dimensional metric. This is a first order equation with respect to the shift and enables us to reconstruct it uniquely.

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