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NONHOLONOMIC MAPPING PRINCIPLE FOR CLASSICAL MECHANICS IN SPACETIMES WITH CURVATURE AND TORSION. NEW COVARIANT CONSERVATION LAW FOR ENERGY-MOMENTUM TENSOR*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

The lecture explains the geometric basis for the recently-discovered nonholonomic mapping principle which specifies certain laws of nature in spacetimes with curvature and torsion from those in flat spacetime, thus replacing and extending Einstein's equivalence principle. An important consequence is a new action principle for determining the equation of motion of a free spinless point particle in such spacetimes. Surprisingly, this equation contains a torsion force, although the action involves only the metric. This force changes geodesic into autoparallel trajectories, which are a direct manifestation of inertia. The geometric origin of the torsion force is a closure failure of parallelograms. The torsion force changes the covariant conservation law of the energy-momentum tensor whose new form is derived.

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1. Introduction

According to Einstein's equivalence principle, gravitational forces in a small region of spacetime labeled by coordinates q^μ $(\mu = 0,1,2,3)$ can be removed by going into locally accelerated coordinates x^a $(a = 0,1,2,3)$ by means of a general coordinate transformation $x^a = x^a(q)$, such as a freely falling elevator. A mass point lying at the center of mass of the elevator does not feel any force. It undergoes no acceleration satisfying the

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Newton-Einstein equation of motion $\ddot{x}^a = 0$ $(a = 0, 1, 2, 3)$, where the dot denotes the derivative with respect to the proper time σ. This observation has enabled Einstein to find the physical laws in curved spacetimes knowing those in flat spacetime. He simply transformed equations of motion from the locally Minkowskian coordinates x^a to the original curvilinear coordinates q^μ, and postulated the resulting equations to describe correctly the motion in a spacetime with gravitational forces.

In this procedure, the elimination of forces is not perfect: it holds only at a single point, the center of mass of the elevator, not in its neighborhood, which is affected by tidal forces that cannot be removed in this way.

Mathematically, however, it is possible to remove the tidal forces with the help of a coordinate transformations $x^a = x^a(q)$ whose derivatives $\partial_\mu x^a(q)$ do not satisfy the Schwarz integrability criterion, *i.e.*, they do not possess commuting derivatives. Such transformations cannot be performed in the laboratory, since the corresponding falling elevator would not exist. The spacetime formed by the image coordinates x^a would have defects, as we shall see below, but it could be chosen to be free of tidal forces in the neighborhood of the center of mass. If we admit such mathematical transformations, Einstein's equivalence principle can be formulated as a mapping principle: The correct equations of motion in the presence of gravitational forces can be obtained by finding in an entire small neighborhood of a point q^μ a local set of coordinates x^a with the above non-Schwarzian property and a complete absence of forces, and by simply mapping the force-free trajectories in these coordinates back into the original spacetime q^μ.

This mathematical procedure opens up the possibility for discovering the form of physical laws in more general spacetimes. For instance, we may assume the existence of gravitational forces, which can only be removed by transformations $x^a = x^a(q)$ whose derivatives do not commute. Such forces are outside of Einstein's theory. In fact, such transformations can be used to remove forces in an equation of motion, which appear in an Einstein-Cartan spacetime with torsion. By postulating that the images of the force-free trajectories in x^a-spacetime are the correct trajectories in q^μ-spacetime we obtain the equations for the straightest lines or autoparallels in q-spacetime, thus contradicting present theories which find shortest lines or geodesics for the particle trajectories.

Physically, autoparallel trajectories may be interpreted as a manifestation of inertia, which makes particles run along the straightest lines rather than the shortest ones as generally believed [1–5]. In the absence of torsion, the two lines happen to be the same, but in the presence of torsion it is hard to conceive, how a particle should know where to go to make the trajectory to a distant point the shortest curve. This seems to contradict our concepts of locality.

Nonholonomic mappings have been of great use in the physics of vortices and defects in superfluids and crystals [6–12]. They are also essential for solving the path integral of the hydrogen atom [13] via the Kustaanheimo-Stiefel transformation [14]. In my lecture, I have first shown how multivalued gauge transformations can be used to generate magnetic fields and their minimal coupling to charged particles [15–17]. This part is omitted in these printed notes to comply with page limitations [18]. These multivalued gauge transformations carry over to geometry by introducing multivalued infinitesimal local coordinate transformations producing infinitesimal curvature and torsion in a flat background spacetime.

2. Nonholonomic mapping principle

Let dq^μ be a small increment of coordinates in the physical spacetime. This is mapped into a coordinate increment dx^a via a transformation [13, 19–21]

$$dx^a \;=\; e^a{}_\lambda(q)\, dq^\lambda\,, \tag{1}$$

whose matrix elements $e^a{}_\lambda(q)$ are *multivalued tetrads*. The transformation can be chosen such that the length of dx^a is measured by the Minkowski metric η_{ab}, so that the metric $g_{\mu\nu}(q)$ in the spacetime q^μ is given by

$$g_{\lambda\mu}(q) = e^a{}_\lambda(q) e^b{}_\mu(q)\eta_{ab}\,, \qquad e^a{}_\lambda(q) \equiv \partial x^a(q)/\partial q^\lambda. \tag{2}$$

Parallel transport in q-spacetime is performed with an affine connection

$$\mathit{\Gamma}_{\mu\nu}{}^\lambda(q) \equiv e_a{}^\lambda(q)\partial_\mu\, e^a{}_\nu(q) \;=\; -e^a{}_\nu(q)\,\partial_\mu\, e_a{}^\lambda(q), \tag{3}$$

where $e_a{}^\lambda(q)$ are reciprocal multivalued tetrads. Its antisymmetric part is the torsion tensor [22]

$$S_{\mu\nu}{}^\lambda(q) = \frac{1}{2}\left[\mathit{\Gamma}_{\mu\nu}{}^\lambda(q) - \mathit{\Gamma}_{\nu\mu}{}^\lambda(q)\right], \tag{4}$$

and its covariant curl the curvature tensor

$$\begin{aligned} R_{\mu\nu\lambda}{}^\kappa(q) \;&=\; e_a{}^\kappa(q)\,(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\, e^a{}_\lambda(q) \\ &=\; \partial_\mu \mathit{\Gamma}_{\nu\lambda}{}^\kappa - \partial_\nu \mathit{\Gamma}_{\mu\lambda}{}^\kappa - \mathit{\Gamma}_{\mu\lambda}{}^\sigma \mathit{\Gamma}_{\sigma\nu}{}^\kappa + \mathit{\Gamma}_{\nu\lambda}{}^\sigma \mathit{\Gamma}_{\sigma\mu}{}^\kappa. \end{aligned} \tag{5}$$

Note that if we were to live in x-spacetime, we would register $S_{\mu\nu}{}^\lambda(q)$ as an object of anholonomity. In the coordinate system q, however, $S_{\mu\nu}{}^\lambda(q)$ is observable as a torsion.

Recall the way in which the affine connection $\mathit{\Gamma}_{\mu\nu}{}^{\kappa}$ serves to define a covariant derivative of vector fields $v_\mu(q),\ v^\mu(q)$:

$$D_\mu v_\nu(q) = \partial_\mu v_\nu(q) - \mathit{\Gamma}_{\mu\nu}{}^{\lambda}(q) v_\lambda(q), \qquad D_\mu v^\lambda(q) = \partial_\mu v^\lambda(q) + \mathit{\Gamma}_{\mu\nu}{}^{\lambda}(q) v^\nu(q). \tag{6}$$

If we lower the last index of the affine connection by a contraction, $\mathit{\Gamma}_{\mu\nu\lambda} \equiv g_{\lambda\kappa}\mathit{\Gamma}_{\mu\nu}{}^{\kappa}$, there exists the decomposition

$$\mathit{\Gamma}_{\mu\nu\kappa} = \bar{\mathit{\Gamma}}_{\mu\nu\kappa} + K_{\mu\nu\kappa}, \tag{7}$$

where $\bar{\mathit{\Gamma}}_{\mu\nu\kappa}$ is the Riemann connection, symmetric in $\mu\nu$,

$$\bar{\mathit{\Gamma}}_{\mu\nu\lambda} \equiv \{\mu\nu, \lambda\} = \frac{1}{2}\left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}\right), \tag{8}$$

and

$$K_{\mu\nu\kappa} = S_{\mu\nu\kappa} - S_{\nu\kappa\mu} + S_{\kappa\mu\nu} \tag{9}$$

is an antisymmetric tensor in $\nu\kappa$, called the contortion tensor [22], formed from the torsion tensor by lowering the last index $S_{\mu\nu\kappa} = g_{\kappa\lambda}S_{\mu\nu}{}^{\lambda}$. With the help of the Riemann connection, we may define another covariant derivative

$$\bar{D}_\mu v_\nu(q) = \partial_\mu v_\nu(q) - \bar{\mathit{\Gamma}}_{\mu\nu}{}^{\lambda}(q) v_\lambda(q), \qquad \bar{D}_\mu v^\lambda(q) = \partial_\mu v^\lambda(q) + \bar{\mathit{\Gamma}}_{\mu\nu}{}^{\lambda}(q) v^\nu(q). \tag{10}$$

3. Application to particle trajectories

As an illustration for nonholonomic mappings to spacetimes with curvature and torsion we use the analogy with defect physics [6–11].

We consider two types of special plastic deformations in crystals, by which one produces a single topological defect called a disclination (a defect of rotations) and a dislocation (a defect of translations) (see Fig. 1). Just as crystals with dislocations, spacetimes with torsion have a closure failure, implying that parallelograms do not close. As a consequence, variations $\delta q^\mu(\tau)$ of particle trajectories parametrized arbitrarily by τ which in the absence of torsion form closed paths, cannot be zero at both the initial and the final point, but must be open at the endpoint of the trajectory [23, 24], as illustrated in Fig. 2. As a consequence, the Euler-Lagrange equation of spinless point particles receives a torsion force. This is quite surprising since the Lagrangian of a trajectory $q^\mu(\tau)$,

$$L = -M\sqrt{g_{\mu\nu}(q(\tau))\dot{q}^\mu(\tau)\dot{q}^\nu(\tau)}, \tag{11}$$

(we use natural units with light velocity $c = 1$) contains only the metric [24].

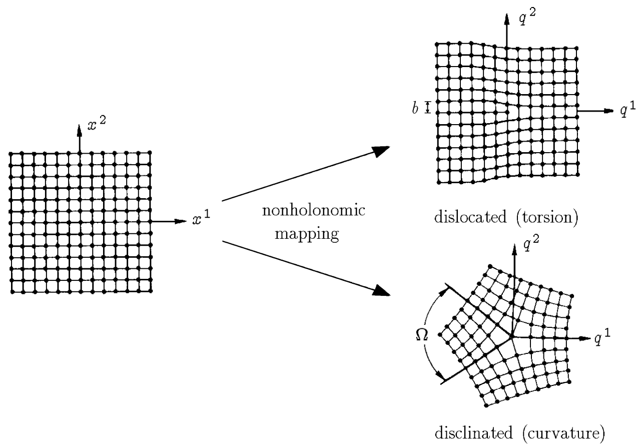

Fig. 1. Crystal with dislocation and disclination generated by nonholonomic coordinate transformations from an ideal crystal. Geometrically, the former transformation introduces torsion and no curvature, the latter curvature and no torsion.

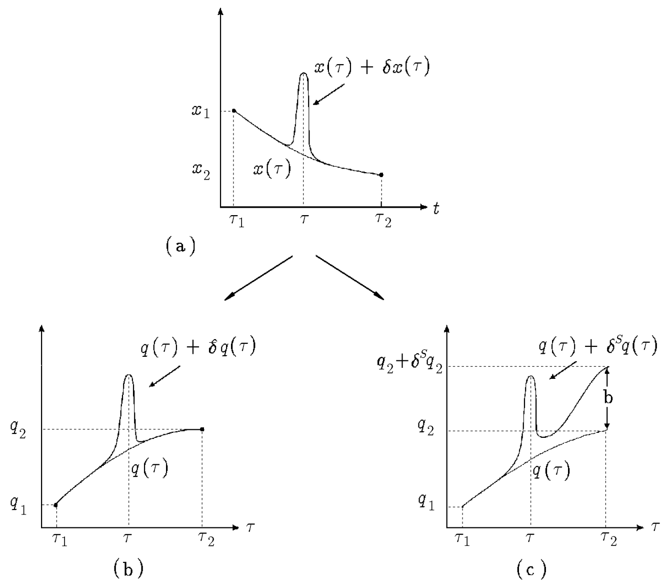

Fig. 2. Images under a holonomic and a nonholonomic mapping of a fundamental path variation. In the holonomic case, the paths $x^a(\tau)$ and $x^a(\tau) + \delta x^a(\tau)$ in (a) turn into the paths $q^\mu(\tau)$ and $q^\mu(\tau) + \delta q^\mu(\tau)$ in (b). In the nonholonomic case with $S_{\mu\nu}{}^\lambda \neq 0$, they go over into $q^\mu(\tau)$ and $q^\mu(\tau) + \delta^S q^\mu(\tau)$ shown in (c) with a closure failure $\delta^S q_2 = b^\mu$ at τ_2 analogous to the Burgers vector b^μ in a solid with dislocations.

The new Euler-Lagrange equation reads

$$\frac{\partial L}{\partial q^{\mu}} - \frac{d}{d\tau}\frac{\partial L}{\partial \dot{q}^{\mu}} = 2S_{\mu\nu}{}^{\lambda}\dot{q}^{\nu}\frac{\partial L}{\partial \dot{q}^{\lambda}}, \tag{12}$$

differing from the standard equation by the extra force on the right-hand side involving the torsion tensor $S_{\mu\nu}{}^{\lambda}$. This extra force changes geodesic trajectories into autoparallel ones, whose equation of motion is

$$\frac{D}{d\sigma}\dot{q}^{\nu}(\sigma) \equiv \ddot{q}^{\nu}(\sigma) + \Gamma_{\lambda\kappa}{}^{\nu}(q(\sigma))\dot{q}^{\lambda}(\sigma)\dot{q}^{\kappa}(\sigma) = 0, \tag{13}$$

where σ is the proper time defined by $d\sigma = \sqrt{g_{\mu\nu}dq^{\mu}dq^{\nu}}$. The geodesic equation without the extra force would contain only the Riemann part of the connection and reads

$$\frac{\bar{D}}{d\sigma}\dot{q}^{\nu}(\sigma) \equiv \ddot{q}^{\nu}(\sigma) + \bar{\Gamma}_{\lambda\kappa}{}^{\nu}(q(\sigma))\dot{q}^{\lambda}(\sigma)\dot{q}^{\kappa}(\sigma) = 0. \tag{14}$$

A simple variational principle to derive the equation of motion (12) is based on the introduction of an *auxiliary nonholonomic variation* $\bar{\delta} q^{\mu}(\tau)$ of a particle trajectory $q^{\mu}(\tau)$ which has the novel property of not commuting with the τ-derivative $d_{\tau} \equiv d/d\tau$ [24]:

$$\bar{\delta} d_{\tau}q^{\mu}(\tau) - d_{\tau}\,\bar{\delta} q^{\mu}(\tau) = 2S_{\nu\lambda}{}^{\mu}\dot{q}^{\nu}(\tau)\,\bar{\delta} q^{\lambda}(\tau). \tag{15}$$

Then (12) is a direct consequence of the new action principle [23, 24]

$$\bar{\delta}\mathcal{A} = 0. \tag{16}$$

An important consistency check for the correct equations of motion is based on their rederivation from the covariant conservation law for the energy-momentum tensor which, in turn, is a general property of any field theory invariant under arbitrary (singlevalued) coordinate transformations.

4. New covariant conservation law for energy-momentum tensor

To derive this law, we study the behavior of the relativistic action

$$\mathcal{A} = -M\int_{\tau_1}^{\tau_2} d\tau\sqrt{g_{\mu\nu}(q(\tau))\dot{q}^{\mu}(\tau)\dot{q}^{\nu}(\tau)} \tag{17}$$

under the infinitesimal versions of general coordinate transformations

$$dq^{\mu} \to dq'^{\mu} = \alpha^{\mu}{}_{\nu}dq^{\nu}; \qquad \alpha^{\mu}{}_{\nu} \equiv \frac{\partial q'^{\mu}}{\partial q^{\nu}}, \tag{18}$$

$$dq_{\mu} \to dq'_{\mu} = \alpha_{\mu}{}^{\nu}dq_{\nu}; \qquad \alpha_{\mu}{}^{\nu} \equiv \frac{\partial q'_{\mu}}{\partial q_{\nu}}. \tag{19}$$

We shall write them as local translations

$$q^\mu \to q'^\mu(q) \equiv q^\mu - \xi^\mu(q), \tag{20}$$

considering from now on only linear terms in the small quantities ξ^μ.

Inserting (20) into (18) and (19), we have

$$\alpha^\lambda{}_\nu(q) \approx \delta^\lambda{}_\nu - \partial_\nu \xi^\lambda(q), \qquad \alpha_\mu{}^\nu(q) \approx \delta_\mu{}^\nu + \partial_\mu \xi^\nu(q), \tag{21}$$

and find from

$$\begin{aligned} e_a{}^\mu(q) = \frac{\partial q^\mu}{\partial x^a} \to e'_a{}^\mu(q') \equiv \frac{\partial q'^\mu}{\partial x^a} &= \frac{\partial q'^\mu}{\partial q^\nu}\frac{\partial q^\nu}{\partial x^a} = \alpha^\mu{}_\nu(q) e_a{}^\nu(q), \\ e^a{}_\mu(q) = \frac{\partial x^a}{\partial q^\mu} \to e'^a{}_\mu(q') \equiv \frac{\partial x^a}{\partial q'^\mu} &= \frac{\partial q^\nu}{\partial q'^\mu}\frac{\partial x^a}{\partial q^\nu} = \alpha_\mu{}^\nu(q) e^a{}_\nu(q) \end{aligned} \tag{22}$$

the infinitesimal changes of the multivalued tetrads $e_a{}^\mu(q)$ and $e^a{}_\mu(q)$:

$$\delta_E e_a{}^\mu(q) \equiv e'_a{}^\mu(q) - e_a{}^\lambda(q) = \xi^\lambda(q)\partial_\lambda e_a{}^\mu(q) - \partial_\lambda \xi^\mu(q) e_a{}^\mu(q), \tag{23}$$

$$\delta_E e^a{}_\mu(q) \equiv e'^a{}_\mu(q) - e^a{}_\mu(q) = \xi^\lambda(q)\partial_\lambda e^a{}_\mu(q) + \partial_\mu \xi^\lambda(q) e^a{}_\lambda(q). \tag{24}$$

The subscript of δ_E indicates that these changes are caused by the infinitesimal versions of the general coordinate transformations introduced by Einstein.

To save parentheses, differential operators are supposed to act only on the expression after it. Inserting (24) into (2), we obtain the corresponding transformation law for the metric tensor

$$\delta_E g_{\mu\nu}(q) = \xi^\lambda(q)\partial_\lambda g_{\mu\nu}(q) + \partial_\mu \xi^\lambda(q) g_{\lambda\nu}(q) + \partial_\nu \xi^\lambda(q) g_{\mu\lambda}(q). \tag{25}$$

With the help of the covariant derivative (10), this can be rewritten as

$$\delta_E g_{\mu\nu}(q) = \bar{D}_\mu \xi_\nu(q) + \bar{D}_\nu \xi_\mu(q). \tag{26}$$

For the coordinates q^μ themselves, the infinitesimal transformation is

$$\delta_E q^\mu = -\xi^\mu(q), \tag{27}$$

which is just the initial transformation (20) in this notation.

We now calculate the change of the action (17) under infinitesimal Einstein transformations:

$$\delta_E \mathcal{A} = \int d^4 q \frac{\delta \mathcal{A}}{\delta g_{\mu\nu}(q)} \delta_E g_{\mu\nu}(q) + \int d\tau \frac{\delta \mathcal{A}}{\delta q^\mu(\tau)} \delta_E q^\mu(\tau). \tag{28}$$

The functional derivative $\delta\mathcal{A}/\delta g_{\mu\nu}(q)$ is the general definition of the energy-momentum tensor of a system:

$$\frac{\delta\mathcal{A}}{\delta g_{\mu\nu}(q)} \equiv -\frac{1}{2}\sqrt{-g(q)}\,T^{\mu\nu}(q), \tag{29}$$

where $-g$ is the determinant of $-g_{\mu\nu}$. For the spinless particle at hand, the energy-momentum tensor becomes

$$T^{\mu\nu}(q) = \frac{1}{\sqrt{-g}}M\int d\sigma\, \dot{q}^\mu(\sigma)\dot{q}^\nu(\sigma)\,\delta^{(4)}(q-q(\sigma)), \tag{30}$$

where σ is the proper time. Equation (30) and the explicit variations (26) and (27), bring (28) to the form

$$\delta_E\mathcal{A} = -\frac{1}{2}\int d^4q\sqrt{-g}T^{\mu\nu}(q)[\bar{D}_\mu\xi_\nu(q) + \bar{D}_\nu\xi_\mu(q)] - \int d\tau\frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\xi^\mu(q(\tau)). \tag{31}$$

A partial integration of the derivatives yields (neglecting boundary terms at infinity and using the symmetry of $T^{\mu\nu}$)

$$\begin{aligned}\delta_E\mathcal{A} \;&=\; \int d^4q\,\left\{\partial_\nu[\sqrt{-g}T^{\mu\nu}(q)]+\sqrt{-g}\bar{\mathit{\Gamma}}_{\nu\lambda}{}^\mu(q)T^{\lambda\nu}(q)\right\}\xi_\mu(q)\\ &\;-\; \int d\tau\frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\xi^\mu(\tau).\end{aligned} \tag{32}$$

Because of the manifest invariance of the action under general coordinate transformations, the left-hand side has to vanish for arbitrary (infinitesimal) functions $\xi^\mu(\tau)$. We therefore obtain

$$\begin{aligned}&\left\{\partial_\nu[\sqrt{-g}T^{\mu\nu}(q)] + \sqrt{-g}\bar{\mathit{\Gamma}}_{\nu\lambda}{}^\mu T^{\lambda\nu}(q)\right\}\xi_\mu(q)\\ &\qquad -\int d\tau\frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\delta^{(4)}(q-q(\tau))\xi^\mu(\tau) = 0.\end{aligned} \tag{33}$$

To find the physical content of this equation we consider first a space without torsion. On a particle trajectory, the action is extremal, so that the second term vanishes, and we obtain the covariant conservation law:

$$\partial_\nu[\sqrt{-g}T^{\mu\nu}(q)] + \sqrt{-g}\bar{\mathit{\Gamma}}_{\nu\lambda}{}^\mu(q)T^{\lambda\nu}(q) = 0. \tag{34}$$

Inserting (30), this becomes

$$M\int ds\,[\dot{q}^\mu(\sigma)\dot{q}^\nu(\sigma)\partial_\nu\delta^{(4)}(q-q(\sigma)) + \bar{\mathit{\Gamma}}_{\nu\lambda}{}^\mu(q)\dot{q}^\nu(\sigma)\dot{q}^\lambda(\sigma)\,\delta^{(4)}(q-q(\sigma))] = 0. \tag{35}$$

A partial integration turns this into

$$M\int d\sigma\,[\ddot{q}^{\mu}(\sigma)+\bar{\Gamma}_{\nu\lambda}{}^{\mu}(q)\dot{q}^{\nu}(\sigma)\dot{q}^{\lambda}(\sigma)]\,\delta^{(4)}(q-q(\sigma))=0. \tag{36}$$

Integrating this over a small volume around any trajectory point $q^{\mu}(\sigma)$, we obtain Eq. (14) for a geodesic trajectory.

A similar calculation was used by Hehl [5] in his derivation of particle trajectories in the presence of torsion. Since torsion does not appear in the action, he found that the trajectories to be geodesic.

The conservation law (34) can be written more covariantly as

$$\bar{D}_{\nu}T^{\mu\nu}(q)=0. \tag{37}$$

This follow directly from the identity

$$\frac{1}{\sqrt{-g}}\partial_{\nu}\sqrt{-g}=\frac{1}{2}g^{\lambda\kappa}\partial_{\nu}g_{\lambda\kappa}=\bar{\Gamma}_{\nu\lambda}{}^{\lambda}, \tag{38}$$

and is a consequence of the rule of partial integration applied to (31), according to which a covariant derivative can be treated in a volume integral $\int d^4q\sqrt{-g}f(q)\bar{D}g(q)$ just like an ordinary derivative in an euclidean integral $\int d^4x f(x)\partial_a g(x)$ After a partial integration neglecting surface terms, Eq. (31) goes over into

$$\delta_E\mathcal{A}=\frac{1}{2}\int d^4q\sqrt{-g}\,[\bar{D}_{\nu}T^{\mu\nu}(q)\xi_{\mu}(q)+\bar{D}_{\mu}T^{\mu\nu}(q)\xi_{\nu}(q)]-\int d\tau\frac{\delta\mathcal{A}}{\delta q^{\mu}(\tau)}\xi^{\mu}(q(\tau)). \tag{39}$$

whose vanishing for all $\xi^{\mu}(q)$ yields directly (37), if the action is extremal under ordinary variations of the orbit.

Our theory does not lead to this conservation law. In the presence of torsion, the particle trajectory does not satisfy $\delta\mathcal{A}/\delta q^{\mu}(\tau)=0$, but according to (12):

$$\frac{\delta\mathcal{A}}{\delta q^{\mu}(\tau)}=\frac{\partial L}{\partial q^{\mu}}-\frac{d}{d\tau}\frac{\partial L}{\partial\dot{q}^{\mu}}=2S_{\mu\nu}{}^{\lambda}\dot{q}^{\nu}\frac{\partial L}{\partial\dot{q}^{\lambda}}, \tag{40}$$

the right-hand side being equal to $-M\,2S_{\mu\nu\lambda}\dot{q}^{\nu}(\sigma)\dot{q}^{\lambda}(\sigma)$ if we choose τ to be the proper time σ. Inserting this into (33), equation (36) receives an extra term and becomes

$$M\int d\sigma\,\left\{\ddot{q}^{\mu}(\sigma)+[\bar{\Gamma}_{\nu\lambda}{}^{\mu}(q)+2S^{\mu}{}_{\nu\lambda}(q)]\dot{q}^{\nu}(\sigma)\dot{q}^{\lambda}(\sigma)\right\}\,\delta^{(4)}(q-q(\sigma))=0, \tag{41}$$

thus yielding the correct autoparallel trajectories (13) for spinless point particles.

Observe that the extra term in (39) can be expressed via (40) in terms of the energy-momentum tensor (30) as

$$\int d^4q\sqrt{-g}\,2S^{\mu}{}_{\nu\lambda}(q)T^{\lambda\nu}(q)\xi_{\mu}(q). \tag{42}$$

We may therefore rewrite the change of the action (31) as

$$\delta_E\mathcal{A} = -\frac{1}{2}\int d^4q\sqrt{-g}\,T^{\mu\nu}(q)[\bar{D}_\mu\xi_\nu(q) + \bar{D}_\nu\xi_\mu(q) - 4S^{\lambda}{}_{\mu\nu}\xi_\lambda(q)]. \tag{43}$$

The quantity in brackets will be denoted by $\bar{\delta}_E g_{\mu\nu}(q)$, and is equal to

$$\bar{\delta}_E g_{\mu\nu}(q) = D_\mu\xi_\nu(q) + D_\nu\xi_\mu(q), \tag{44}$$

where D_μ is the covariant derivative (6) involving the full affine connection. Thus we have

$$\delta_E\mathcal{A} = -\int d^4q\sqrt{-g}\,T^{\mu\nu}(q)D_\nu\xi_\mu(q). \tag{45}$$

Integrals over invariant expressions containing the covariant derivative D_μ can be integrated by parts according to a rule very similar to that for the Riemann covariant derivative $\bar{D}_\mu$ (which is derived in Appendix A of [18]). After neglecting surface terms we find

$$\delta_E\mathcal{A} = \int d^4q\sqrt{-g}\,D^*_\nu T^{\mu\nu}(q)\xi_\mu(q), \tag{46}$$

where $D^*_\nu = D_\nu + 2S_{\nu\lambda}{}^{\lambda}$. Thus, due to the closure failure in spacetimes with torsion, the energy-momentum tensor of a free spinless point particle satisfies the conservation law

$$D^*_\nu T^{\mu\nu}(q) = 0. \tag{47}$$

This is to be contrasted with the conservation law (37). The difference between the two laws can best be seen by rewriting (37) as

$$D^*_\nu T^{\mu\nu}(q) + 2S_{\kappa}{}^{\mu}{}_{\lambda}(q)T^{\kappa\lambda}(q) = 0. \tag{48}$$

This is the form in which the conservation law has usually been stated in the literature [1–4, 7]. When written in the form (37) it is obvious that it is satisfied only by geodesic trajectories.

Note that the variation $\bar{\delta}_E g_{\mu\nu}(q)$ plays a similar role in deriving the new conservation law (48) as the nonholonomic variation $\bar{\delta}q^\mu(\tau)$ with the noncommutative property (15) does in deriving equations of motion for point particles. Indeed, we may rewrite the transformation (28) formally as

$$\bar{\delta}_E\mathcal{A} = \int d^4q\frac{\delta\mathcal{A}}{\delta g_{\mu\nu}(q)}\,\bar{\delta}_E g_{\mu\nu}(q) + \int d\tau\frac{\delta\mathcal{A}}{\delta q^\mu(\tau)}\,\bar{\delta}_E q^\mu(\tau). \tag{49}$$

Now the last term vanishes according to the new action principle (16), from which we derived the autoparallel trajectory (13).

5. Consequences for field theory of gravitation with torsion

The question arises whether the new conservation law (47) allows for the construction of an extension of Einstein's field equation

$$\bar{G}^{\mu\nu} = \kappa T^{\mu\nu} \tag{50}$$

to spacetimes with torsion, where $\bar{G}^{\mu\nu} \equiv \bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \bar{R}_{\sigma}{}^{\sigma}$ is the Einstein tensor formed from the Ricci tensor $\bar{R}_{\mu\nu} \equiv \bar{R}_{\lambda\mu\nu}{}^{\lambda}$ in Riemannian spacetime [$\bar{R}_{\mu\nu\lambda}{}^{\kappa}$ being the same covariant curl of $\bar{\Gamma}_{\mu\nu}{}^{\lambda}$ as $R_{\mu\nu\lambda}{}^{\kappa}$ is of $\Gamma_{\mu\nu}{}^{\lambda}$ in Eq. (5)].

The standard extension of (50) to spacetimes with torsion [1–4,7] replaces the left-hand side by the Einstein-Cartan tensor $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R_{\sigma}{}^{\sigma}$ and becomes

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{51}$$

The Einstein-Cartan tensor $G^{\mu\nu}$ satisfies a Bianchi identity

$$D^{*}_{\nu} G_{\mu}{}^{\nu} + 2 S_{\lambda\mu}{}^{\kappa} G_{\kappa}{}^{\lambda} - \frac{1}{2} S^{\lambda}{}_{\kappa}{}^{;\nu} R_{\mu\nu\lambda}{}^{\kappa} = 0, \tag{52}$$

where $S^{\lambda}{}_{\kappa}{}^{;\nu}$ is the Palatini tensor defined by

$$S_{\lambda\kappa}{}^{;\nu} \equiv 2(S_{\lambda\kappa}{}^{\nu} + \delta_{\lambda}{}^{\nu} S_{\kappa\sigma}{}^{\sigma} - \delta_{\kappa}{}^{\nu} S_{\lambda\sigma}{}^{\sigma}). \tag{53}$$

It is then concluded that the energy-momentum tensor satisfies the conservation law

$$D^{*}_{\nu} T_{\mu}{}^{\nu} + 2 S_{\lambda\mu}{}^{\kappa} T_{\kappa}{}^{\lambda} - \frac{1}{2\kappa} S^{\lambda}{}_{\kappa}{}^{;\nu} R_{\mu\nu\lambda}{}^{\kappa} = 0. \tag{54}$$

For standard field theories of matter, this is indeed true if the Palatini tensor satisfies the second Einstein-Cartan field equation

$$S^{\lambda\kappa;\nu} = \kappa \Sigma^{\lambda\kappa;\nu}, \tag{55}$$

where $\Sigma^{\lambda\kappa;\nu}$ is the canonical spin density of the matter fields. A spinless point particle contributes only to the first two terms in (54), in accordance with (48).

Which tensor will stand on the left-hand side of the field equation (51) if the energy-momentum tensor satisfies the conservation law (47) instead of (48)? At present, we can give an answer only for the case of a pure gradient torsion [25]

$$S_{\mu\nu}{}^{\lambda} = \frac{1}{2}[\delta_{\mu}{}^{\lambda} \partial_{\nu} \sigma - \delta_{\nu}{}^{\lambda} \partial_{\mu} \sigma]. \tag{56}$$

Then we may simply replace (51) by

$$e^{\sigma} G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{57}$$

Note that for gradient torsion, $G^{\mu\nu}$ is symmetric as can be deduced from the fundamental identity (which expresses merely the fact that the Einstein-Cartan curvature tensor $R_{\mu\nu\lambda}{}^{\kappa}$ is the covariant curl of the affine connection)

$$D^*{}_{\lambda} S_{\mu\nu}{}^{;\lambda} = G_{\mu\nu} - G_{\nu\mu}. \tag{58}$$

Indeed, inserting (56) into (53), we find the Palatini tensor

$$S_{\lambda\mu}{}^{;\kappa} \equiv -2[\delta_{\lambda}{}^{\kappa}\partial_{\mu}\sigma - \delta_{\mu}{}^{\kappa}\partial_{\lambda}\sigma]. \tag{59}$$

This has a vanishing covariant derivative

$$D^*_{\lambda} S_{\mu\nu}{}^{;\lambda} = -2[D^*_{\mu}\partial_{\nu}\sigma - D^*_{\nu}\partial_{\mu}\sigma] = 2[S_{\mu\nu}{}^{\lambda}\partial_{\lambda}\sigma - 2S_{\mu\lambda}{}^{\lambda}\partial_{\nu}\sigma + 2S_{\nu\lambda}{}^{\lambda}\partial_{\mu}\sigma], \tag{60}$$

since the terms on the right-hand side cancel after using (56) and $S_{\mu\lambda}{}^{\lambda} \equiv S_{\mu} = -\frac{3}{2}\partial_{\mu}\sigma$. Now we insert (56) into the Bianchi identity (52), with the result

$$\bar{D}^*_{\nu} G_{\lambda}{}^{\nu} + \partial_{\lambda}\sigma G_{\kappa}{}^{\kappa} - \partial_{\nu}\sigma G_{\lambda}{}^{\nu} + 2\partial_{\nu}\sigma R_{\lambda}{}^{\nu} = 0. \tag{61}$$

Inserting here $R_{\lambda\kappa} = G_{\lambda\kappa} - \frac{1}{2} g_{\lambda\kappa} G_{\nu}{}^{\nu}$, this becomes

$$D^*_{\nu} G_{\lambda}{}^{\nu} + \partial_{\nu}\sigma G_{\lambda}{}^{\nu} = 0. \tag{62}$$

Thus we find for the gradient torsion (56) the Bianchi identity

$$D^*_{\nu}(e^{\sigma} G_{\lambda}{}^{\nu}) = 0. \tag{63}$$

This makes the left-hand side of the new field equation (57) compatible with the covariant new conservation law (47), just as in Einstein's theory.

The field equation for the σ-field and thus for the torsion is still unknown.

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