Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 4

ON THE ULTRARELATIVISTIC LIMIT OF GENERAL RELATIVITY*

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(Received January 19, 1998)

Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

As is well-known, Newton's gravitational theory can be formulated as a four-dimensional space-time theory and follows as singular limit from Einstein's theory, if the velocity of light tends to the infinity. Here 'singular' stands for the fact, that the limiting geometrical structure differs from a regular Riemannian space-time. Geometrically, the transition Einstein $\rightarrow$ Newton can be viewed as an 'opening' of the light cones. This picture suggests that there might be other singular limits of Einstein's theory: Let all light cones shrink and ultimately become part of a congruence of singular world lines. The limiting structure may be considered as a nullhypersurface embedded in a five-dimensional spacetime. While the velocity of light tends to zero here, all other velocities tend to the velocity of light. Thus one may speak of an ultrarelativistic limit of General Relativity. The resulting theory is as simple as Newton's gravitational theory, with the basic difference, that Newton's elliptic differential equation is replaced by essentially ordinary differential equations, with derivatives tangent to the generators of the singular congruence. The Galilei group is replaced by the Carroll group introduced by Lévy-Leblond. We suggest to study near ultrarelativistic situations with a perturbational approach starting from the singular structure, similar to post-Newtonian expansions in the $c \rightarrow \infty$ case.

PACS numbers: 04.20. Cv

* Presented at the Workshop on Gauge Theories of Gravitation, Jadwisin, Poland, September 4–10, 1997.

1. Introduction

General Relativity (GR) not only governs the gravitational interactions between bodies, it also dictates the causality structure of spacetime. This latter property is most interesting, if the gravitational fields become strong and develop singularities. In a limit, when the whole spacetime becomes singular or nearly singular, the causality structure should strongly deviate from that of a near-Minkowskian geometry. It is one of the virtues of GR, that the theory covers - if properly interpretet - even such extreme situations.

A well-known example is Newton's theory of gravity. Its four-dimensional formulation requires a spacetime structure, which is singular from the viewpoint of Riemannian geometry [1]. Using Einstein's field equations, this singular structure is obtained, if the velocity of light is taken to tend to infinity [7]. Geometrically, the transition Einstein $\rightarrow$ Newton can be viewed as an opening of the light cones. In the limit $c \rightarrow \infty$, the cones become the spacelike hypersurfaces of Newton's absolute time. In spite of Kuhns claims about the incommensurability of concepts of successive theories, both Newton's and Einstein's theory can be covered by a common spacetime theory [6]. Nevertheless, the causality structure of Newton's theory is radically different from that of Einstein's: Interactions occur simultanously on the hypersurfaces of constant Newtonian time ("action at a distance"). This is reflected by the existence of an *elliptic* differential equation for the Newtonian potential, as compared to *hyperbolic* differential equations for time-dependent situations in GR. Closely related is the fact that the Poincaré group is replaced by the Galilei group.

The visualization of the transition Einstein $\rightarrow$ Newton suggests immediately, that Newton's theory may not be the only singular limit of Einstein's theory. We here discuss a situation, which is in some sense opposite to the Newtonian case. Let all light cones shrink and ultimately become part of a congruence of singular world lines. Geometrically, this limiting structure may be considered as a four-dimensional nullhypersurface $V_4^{(1)}$ embedded in a five-dimensional spacetime. While the light velocity tends to zero here, all other velocities tend to the velocity of light. One may therefore speak of an ultrarelativistic limit of GR (see [4], [9], [10] for previous discussions). Again, the causality structure in the limit is different: Instead of the hyperbolic differential equations of GR and elliptic differential equations of Newton's theory, we have now essentially *ordinary* differential equations, with derivatives tangent to the generators of the singular congruence. There are almost no interactions between spatially separated events, and no true motion occurs in the limit, except for tachyonic motion. However, the isolated and immobile physical objects show evolution. A situation of this type has sometimes been called "Carroll causality" [9], after Louis Carroll's tale

Alice in Wonderland. It may also be characterized as ultralocal approximation, which is perhaps a better notation than ultrarelativistic. While the Newtonian limit is governed by the Galilei group, the invariance group in the ultrarelativistic limit is another degenerate limit of the Poincaré group, the Carroll group, introduced by Lévy–Leblonc, who also first discussed its representations and its Lie algebra [10].

One expects, that an ultrarelativistic approximation procedure, starting from the singular spacetime and similar to the method considered in the opposite Newtonian case [5], might be useful for situations of strong gravity. It is encouraging that the field equations in the ultrarelativistic limit are as simple as in Newton's theory.

In Section 2 the geometry of a stand-alone singular spacetime $V_4^{(1)}$ is considered, independently of the limiting procedure and of its embedding into higher-dimensional spaces. A Riemannian curvature tensor based on second-order derivatives of the metric is not a genuine geometrical construction here, since no uniquely defined intrinsic connection exists. However, Ricci rotation coefficients can be introduced. The use of adapted coordinates simplifies the relations. Section 3 considers a family $g_{\mu\nu}(x^\mu, \varepsilon)$ of metrics, satisfying the general-relativistic field equations and tending for $\varepsilon\ (= c^2) \to 0$ to the singular spacetime introduced in the previous section. The resulting ultrarelativistic field structures depend on the type and behaviour of matter fields for $\varepsilon \to 0$, which are present together with the gravitational field. In Section 4 some solutions of the ultrarelativistic field equations are discussed, including pure vacuum gravitational fields and dust matter.

Many problems remain open. The limit discussed here can be considered for any general-relativistic field theory. It is straightforward to set up a post-ultrarelativistic expansion and thus to re-introduce the velocities which have disappeared in the limit. Another open question is the relation of solutions of the ultrarelativistic equations to those general-relativistic solutions, which admit an ultrarelativistic limit.

2. Differential geometry of singular Riemannian spaces $V_4^{(1)}$

A degenerate Riemannian space $V_n^{(m)}$ may be defined as a n-dimensional Riemannian space equipped with a covariant metric tensor $\gamma_{\mu\nu}$ of matrix rank $0 < m < n$ [3]. Well-known examples are the usual nullhypersurfaces with $n = 3$ and $m = 1$ (see, *e.g.*, [2]). We are interested in the case $n = 4, m = 1$. At any given point the metric tensor can be reduced to $\gamma_{0\mu} = 0, \gamma_{ik} = \delta_{ik}$ by means of suitable coordinate transformations. The values are preserved under the transformation $x'^0 = g(x^0), x'^i = R^i_k x^k + S^i$, where R^i_k is a 3-rotation and g an arbitrary function of x^0. The linear subgroup of this transformation group is the 10-parameter Carroll group [10]. Returning

to a general coordinate system, at every point there exists a contravariant vector field k^μ ($\mu = 0, ...3$) which is a nonvanishing solution of

$$\gamma_{\mu\nu} k^\nu = 0, \tag{1}$$

defined up to an arbitrary factor. Furthermore, there exists a congruence of curves $x^\mu = x^\mu(\xi^i, v),\ i = 1, 2, 3$, called generators of $V_4^{(1)}$, to which the directions k^μ are tangent, as solutions to the differential equation

$$k^\mu(x^\mu) = \frac{\partial x^\mu}{\partial v}. \tag{2}$$

The three quantities ξ^i fix a generator, and the parameter v along a generator is determined up to a transformation $v' = v'(v, \xi^i), \frac{\partial v'}{\partial v} \neq 0$. k^μ is complemented by three other contravariant vectors $l^\mu_{(i)}$ such that $\gamma_{\mu\nu} l^\nu_i \neq 0$ and

$$\gamma_{\mu\nu} l^\mu_{(i)} l^\nu_{(k)} = \delta_{ik}. \tag{3}$$

The four vectors $(k^\mu, l^\mu_{(i)})$ form a contravariant tetrad, spanning the tangent space at every point of the $V_4^{(1)}$. The cotangent space is spanned by the three vectors

$$l_{\mu(i)} = \gamma_{\mu\nu} l^\nu_{(i)} \neq 0, \tag{4}$$

and

$$k_\mu = \frac{\varepsilon_{\mu\nu\rho\sigma} l^\nu_{(1)} l^\rho_{(2)} l^\sigma_{(3)}}{\varepsilon_{\alpha\beta\gamma\delta} k^\alpha l^\beta_{(1)} l^\gamma_{(2)} l^\delta_{(3)}}, \tag{5}$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita density. Note

$$l^\mu_{(i)} l_{\mu(k)} = \delta_{ik},\ k^\mu l_{\mu(i)} = 0, \tag{6}$$

$$l^\mu_{(i)} k_\mu = 0,\ k^\mu k_\mu = 1. \tag{7}$$

The metric is written as $\gamma_{\mu\nu} = l_{\mu(i)} l_{\nu(i)}$, and the tetrad is determined up to the generalized four-dimensional null rotations

$$l'^\mu_{(i)} = A^k_i l^\mu_{(k)} + B_i k^\mu, \tag{8}$$

$$k'^\mu = C k^\mu, \tag{9}$$

which form a 7-parameter group, the coefficients A^k_i represent a 3-rotation. There exists no contravariant metric tensor $\varepsilon^{\mu\nu}$ satisfying $\varepsilon^{\mu\rho}\gamma_{\nu\rho} = \delta^\mu_\nu$, however $\gamma_{\mu\rho}\gamma_{\nu\sigma}\varepsilon^{\rho\sigma} = \gamma_{\mu\nu}$ has solutions. The simplest one is given by

$$\varepsilon^{\mu\nu} = l^\mu_{(i)} l^\nu_{(i)}, \tag{10}$$

but depends on the choice of the tetrad. One can also easily show, that in general there exists no connection $\Gamma^{\rho}_{\mu\nu}$, satisfying the Ricci lemma $\gamma_{\mu\nu;\rho} = 0$ and depending only on the metric and its *first* derivatives. Instead, one may define tetrad-dependent affine connections by

$$\Gamma^{\rho}_{\mu\nu} = \tfrac{1}{2}k^{\rho}(k_{\mu,\nu} + k_{\nu,\mu}) + \varepsilon^{\rho\sigma}\Gamma_{\mu\nu\sigma} \tag{11}$$

with

$$\Gamma_{\mu\nu\rho} = \tfrac{1}{2}(\gamma_{\rho\mu,\nu} + \gamma_{\rho\nu,\mu} - \gamma_{\mu\nu,\rho}). \tag{12}$$

The affine connection (11) also does not in general satisfy the Ricci lemma $\gamma_{\mu\nu;\rho} = 0$, one obtains instead

$$\gamma_{\mu\nu;\rho} = h_{\mu\rho}k_{\nu} + h_{\nu\rho}k_{\mu}, \tag{13}$$

where the tensor

$$h_{\mu\nu} = \Gamma_{\mu\nu\rho}k^{\rho} \tag{14}$$

is (up to a factor $\frac{1}{2}$) the Lie derivative of the metric in the direction k^{ρ}. The $\Gamma^{\rho}_{\mu\nu}$ transform as an affine connection with respect to coordinate transformations, thus the four-dimensional Ricci and Riemann tensors formed with the connection are indeed tensors. They have nevertheless no geometrical meaning in general, since they depend on the choice of tetrad, and are therefore to an large extent arbitrary. The situation can be different for nullhypersurfaces $V_3^{(1)}$ embedded in a Riemannian V_4. Here a rigging of the surfaces would allow us to fix the affine connection and to introduce tensorial curvature measures (for different geometries on nullhypersurfaces, see [2] and [11]). A way to obtain true geometrical statements in $V_4^{(1)}$ is the introduction of Ricci rotation coefficients, which are obtained by expressing the derivatives of the tetrad $(k^{\mu}, l^{\mu}_{(i)})$ in terms of tetrad [3]. The rotation coefficients are scalars with respect to coordinate transformations, but transform under a tetrad change. Geometrically relevant propositions may be formulated as tetrad-invariant statements on the rotation coefficients. For instance, differential invariants of the $V_4^{(1)}$ may be given as suitable functions of the rotation coefficients and their derivatives. Contrary to nonsingular Riemannian geometries, invariants depending only on the metric and its *first* derivatives exist here, *six* for the $V_4^{(1)}$, *one* for the $V_3^{(1)}$. It is possible to write down the invariants and field equations in the $V_4^{(1)}$ in a *manifestly covariant* manner. However, as in Newtonian theory, the existence of intrinsic geometrical structures allows the introduction of adapted coordinates, here ξ^i and v. Since adapted coordinates simplify many relations considerably, they will be used throughout subsequently. Note that they are determined up to the transformations $v' = v'(v, \xi^i),\ \xi'^i = \xi'^i(\xi^i)$.

3. The transition $c \to 0$

We assume that the singular space $V_4^{(1)}$ arises as the limit $\varepsilon \to 0$ of a family of normal-hyperbolic Riemannian spacetimes, with the metric $g_{\mu\nu}(x^\mu, \varepsilon)$, satisfying the Einstein field equations for $\varepsilon > 0$. ε is taken as the *square* of the velocity of light, $\varepsilon = c^2$. A mathematically rigourous approach should employ Geroch's technique of embedding in a 5-manifold [8]. Instead, we use for simplicity asympotic representations for the metric, writing down an expansion of the type

$$g_{\mu\nu}(x^\mu, \varepsilon) = \gamma_{\mu\nu}(x^\mu) + g_{(1)\mu\nu}(x^\mu)\varepsilon + g_{(2)\mu\nu}(x^\mu)\varepsilon^2 + o(\varepsilon^3)\,. \tag{15}$$

Assuming $g_{(1)\mu\nu}k^\mu k^\nu \neq 0$, the contravariant components of the metric may be represented asymptotically as

$$g^{\mu\nu}(x^\mu, \varepsilon) = \frac{1}{\varepsilon} f k^\mu k^\nu + g^{\mu\nu}_{(0)}(x^\mu) + g^{\mu\nu}_{(1)}(x^\mu)\varepsilon + o(\varepsilon^2), \tag{16}$$

where $f(x^\mu)$ is a scalar function. In adapted coordinates we put $k^\mu = \delta^\mu_0$. The relations between the co- and contravariant metric components give

$$g_{(1)00} = 1/f,\ g_{(1)0k} = -\gamma_{ik} g^{0i}_{(0)}/f,\ g^{ik}_{(0)} = \gamma^{ik} \tag{17}$$

(γ^{ik} is inverse to γ_{ik}). It is useful to expand also the coordinate transformation as a series in ε:

$$x'^\mu(x^\rho) = x^\mu_{(0)}(x^\rho) + \varepsilon x^\mu_{(1)}(x^\rho) + o(\varepsilon^2). \tag{18}$$

Putting $x^\mu_{(0)} = x^\mu$, the next order $x^\mu_{(1)}$ can be considered as gauge. From

$$g'^{0i}_{(0)} = g^{0i}_{(0)} + f\frac{\partial x'^i_1}{\partial v}, \tag{19}$$

$$g'^{00}_{(0)} = g^{00}_{(0)} + 2f\frac{\partial x'^0_1}{\partial v}, \tag{20}$$

$$g'^{ik}_{(0)} = \gamma^{ik} \tag{21}$$

it is evident, that $g^{0i}_{(0)}$ and $g^{00}_{(0)}$ may be transformed to zero. This simplifies the field equations considerably. If we calculate the Ricci tensor with the series (15),(16), singular terms proportional to ε^{-2} and ε^{-1} arise. A closer inspection shows that $R_{(-)2\mu\nu} = \lim_{\varepsilon\to 0}(\varepsilon^2 R_{\mu\nu})$ reduces to zero. We use the field equation with a matter tensor $T_{\mu\nu}$. The product of $T_{\mu\nu}$ with $\kappa = \frac{8\pi G}{c^2}$ is assumed to have for $\varepsilon \to 0$ a finite limit $t_{0\mu\nu}$, which allows us to write

$$\kappa T_{\mu\nu}(x^\mu, \varepsilon) = t_{(0)\mu\nu} + t_{(1)\mu\nu}\varepsilon + o(\varepsilon^2). \tag{22}$$

It is not difficult to show that these assumptions are compatible with, *e.g.*, dust matter. Then the field equations start with

$$R_{(-1)\mu\nu} = -\tfrac{1}{2} g_{(0)\mu\nu} t_{(0)00} f, \tag{23}$$

$$R_{(0)\mu\nu} = t_{(0)\mu\nu} - \tfrac{1}{2} g_{(1)\mu\nu} t_{(0)00} f - \tfrac{1}{2} g_{(0)\mu\nu} (t_{(0)\alpha\beta} g^{(0)\alpha\beta} + t_{(1)00} f)\,. \tag{24}$$

From equation (23) only the pure spatial components survive. The ultrarelativistic field equations are obtained from (23) (see below (27) and (32)) and from the time-time and time-space components of (24) (below (25),(26) and (33)). The space-space components of (24) introduce already post-ultrarelativistic corrections, which are not discussed here.

4. Some solutions

The vacuum field equations can be written with $f = -e^{-H}$

$$\dot{\gamma}_{ik}\dot{\gamma}^{ik} + (\dot{\gamma}_{ik}\gamma^{ik})^2 = 0, \tag{25}$$

$$\gamma^{kl}\dot{\gamma}_{kl|i} - \gamma^{kl}\dot{\gamma}_{il|k} - \tfrac{1}{2} H_{,i}\gamma^{kl}\dot{\gamma}_{kl} = 0 \tag{26}$$

and

$$\ddot{\gamma}_{ik} + \tfrac{1}{2}\dot{\gamma}_{ik}(\gamma^{lm}\dot{\gamma}_{lm} - \dot{H}) - \dot{\gamma}_{il}\dot{\gamma}_{km}\gamma^{lm} = 0, \tag{27}$$

where a stroke denotes the covariant derivative with respect to the 3-metric γ_{ik}, a dot means $\partial/\partial v$ ('time'-derivative). These equations should describe the motion of gravitational waves in the ultrarelativistic limit. One recognizes an initial value problem with (25),(26) as initial conditions and (27) as propagation equation. Equation (25) (which signifies the vanishing of one of the first-order invariants of the $V_4^{(1)}$ mentioned above) is preserved under (27), but (26) leads to the additional constraint

$$\dot{\gamma}_{ik}\gamma^{kl}\dot{H}_{,l} = 0. \tag{28}$$

Hence, considering only the general case $det\ |\dot{\gamma}_{ik}\gamma^{kl}| \neq 0$, H is given by

$$H = h(v) + hh(\xi^i), \tag{29}$$

where $hh(\xi^i)$ enters the initial conditions (26) and $h(v)$ influences the propagation equation (27). As an example consider the vacuum solution

$$\gamma_{ik} = \mathrm{diag}(\gamma_1, \gamma_2, \gamma_2),\ H = -4ln(a + bv) + hh(\xi^1) \tag{30}$$

with

$$\gamma_1 = a + bv, \gamma_2\gamma_3 = c(\xi^2, \xi^3)e^{-hh(\xi^1)}. \tag{31}$$

Time-dependent solutions of this type may be considered as the ultrarelativistic limit of gravitational waves. It is seen that caustics of the congruence may occur at $v = -a/b$ in the example.

The case of dust matter is only slightly more complicated. The propagation equation for the 3-metric attains a source term (writing ρ for the $c \to 0$ limit of the matter density)

$$\ddot{\gamma}_{ik} + \tfrac{1}{2}\dot{\gamma}_{ik}(\gamma^{lm}\dot{\gamma}_{lm} - \dot{H}) - \dot{\gamma}_{il}\dot{\gamma}_{km}\gamma^{lm} = 8\pi G\rho e^{H}\gamma_{ik}. \tag{32}$$

A source term is also present in the first (scalar) initial equation

$$\dot{\gamma}_{ik}\dot{\gamma}^{ik} + (\dot{\gamma}_{ik}\gamma^{ik})^2 = 64\pi G\rho e^{H}. \tag{33}$$

The second (vectorial) initial equation (26) remains unchanged. The scalar constraint (33) is preserved in time, if the matter density ρ changes as

$$\frac{\dot{\rho}}{\rho} = -\tfrac{1}{2}\dot{\gamma}_{ik}\gamma^{ik}. \tag{34}$$

The latter relation is equivalent to $\rho \sim (det|\gamma_{ik}|)^{-1/2}$, which corresponds to matter conservation in comoving coordinates (note that ρ can vary arbitrarily as a function of ξ^i). The vectorial constraint equation (26) is not preserved in time, but leads to the relation

$$\dot{\gamma}_{ik}\gamma^{kl}\dot{H}_{,l} + \tfrac{3}{8}H_{,i}(\dot{\gamma}_{kl}\dot{\gamma}^{kl} + (\dot{\gamma}_{kl}\gamma^{kl})^2) = 0. \tag{35}$$

Again, the integration of these equations is much simpler than the corresponding general relativistic equations. Consider the general case of an isotropic expansion, defined by

$$\dot{\gamma}_{ik} = \phi(\xi^i, v)\gamma_{ik}. \tag{36}$$

The solution can be written in terms of an arbitrary function $\lambda(v)$ as

$$\phi(\xi^i, v) = \frac{\lambda}{r_0^3(1 + \frac{9}{4r_0^3}\int_{v_0}^{v}\lambda\, dv)}. \tag{37}$$

The arbitrariness of $\lambda(v)$ reflects the fact, that no affine parameter along the generators is singled out, all parameters $v' = v'(v)$ are at the same footing. r_0 is a spatially varying length scale: Assuming an initial density $\rho_0(\xi^i)$ at $v = v_0$, we define this scale by

$$r_0(\xi^i) = (\frac{3}{32\pi G\rho_0(\xi^i)})^{1/2}. \tag{38}$$

The matter density as a function of v is then given by

$$\rho(\xi^i, v) = \frac{\rho_0(\xi^i)}{(1 + \frac{9}{4r_0^3}\int_{v_0}^{v} \lambda\, dv)^{2/3}}, \tag{39}$$

and H can be found from $e^H = 3\phi^2/(32\pi G\rho)$. It is interesting, that a singular origin of the expanding matter distribution is as inevitable as in GR.

I appreciate useful remarks from and discussions with many participants of the Jadwisin workshop. In particular I wish to thank P. Aichelburg, C. Bär, S. Bażański, W. Drechsler, J. Ehlers, J. Goldberg, F. Hehl, H. Kleinert, C. Lämmerzahl, Y. Ne'eman, J. Stachel, A. Trautman and H. Urbantke.

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