

FIRST ORDER PHASE TRANSITIONS IN
GRAVITATIONAL COLLAPSE* **P. BIZOŃ[†]Department of Mathematics, University of Michigan
Ann Arbor, MI 48 109, United States

AND T. CHMAJ

Institute of Nuclear Physics
Radzikowskiego 152, 31–342 Kraków, Poland*(Received January 30, 1998)**Dedicated to Andrzej Trautman in honour of his 64th birthday*

In recent numerical simulations of spherically symmetric gravitational collapse a new type of critical behaviour, dominated by a sphaleron solution, has been found. In contrast to the previously studied models, in this case there is a finite gap in the spectrum of black-hole masses which is reminiscent of a first order phase transition. We briefly summarize the essential features of this phase transition and describe the basic heuristic picture underlying the numerical phenomenology.

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One of the main open problems in classical general relativity is the issue of global dynamics of solutions of Einstein's equations. The presently achievable mathematical techniques appear to be insufficient to address this problem in full generality, so most researchers have focused their attention on a more tractable case of spherical symmetry. In particular, in a remarkable series of papers (see [1, 2] and references therein) Christodoulou has analysed the evolution of regular initial data for the spherically symmetric Einstein–Klein–Gordon equations. He showed that for “weak” initial data there exists a unique global solution which asymptotes to the Minkowski

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[†] On leave of absence from Institute of Physics, Jagiellonian University, Kraków, Poland.

spacetime, whereas “strong” initial data form a singularity, which, in accord with weak cosmic censorship hypothesis, is surrounded by an event horizon (here “weak” and “strong” have well defined meaning in terms of a certain function norm). These results suggested that there is a “critical surface” in the phase space which separates the two kinds of initial data. The initial data lying on this critical surface are at the threshold of black hole formation. A natural question is: what is the mass of a black hole at the threshold? Does it continuously decrease to zero, or is there a finite lower bound (mass gap)? These two possibilities will be referred to as second and first order phase transitions, respectively.

The pioneering numerical investigations of this problem were carried out by Choptuik who analysed the evolution of one-parameter families of initial data crossing the critical surface [3]. For each family, labelled by a parameter p , Choptuik found a critical value p^* such that the data with $p > p^*$ form a black hole, while the data with $p < p^*$ do not. It turned out that the marginally supercritical data form black holes with masses satisfying the power law $M_{BH} \simeq C(p - p^*)^\gamma$ with a universal (*i.e.* family independent) critical exponent γ . Therefore, as $p \rightarrow p^*$, the black hole mass decreases continuously to zero which is reminiscent of the second order phase transition. Moreover, Choptuik discovered that in the intermediate asymptotics (*i.e.* before a solution “decides” whether to collapse or not) all near-critical solutions approach a universal attractor. This attractor, called the critical solution, has an unusual symmetry of discrete self-similarity.

Quickly after Choptuik’s discovery similar critical effects have been observed in other models of gravitational collapse [4–6]. In all cases the overall picture of criticality was qualitatively the same as in the scalar field collapse, possibly with one difference: in certain models the critical solution was continuously (rather than discretely) self-similar. These studies provided convincing evidence that such features as universality, black-hole mass scaling, and self-similarity (discrete or continuous) are the robust properties of second order phase transitions in gravitational collapse. Although none of these properties have been proven rigorously, a substantial progress has been made on a heuristic level. In particular, at present we have a convincing picture of the origin of universality and scaling (see [7] for a recent review).

None of the first studied models had a mass/length scale, in analogy to the vacuum Einstein’s equations. This raises the question: which features of the critical collapse are inherently related to the scale invariance, or, putting this differently, how does the presence of length/mass scale affect the scenario of critical behaviour? It follows from dimensional analysis that, under the assumption of universality, the lack of scale implies that the mass gap must be zero. However, the converse is not true, as was already noticed by

Choptuik in his studies of self-interacting scalar field. It seems that what matters in the evolution is not the scale itself but rather the presence of non-singular stationary solutions (which are of course excluded for scale invariant equations).

In order to understand better the role of scale and stationary solutions in the dynamics of Einstein's equations, we have recently investigated two models: Einstein–Yang–Mills (in collaboration with Matt Choptuik [8]) and Einstein–Skyrme [9]. Both these models possess a mass/length scale (actually the ES model has two scales) and static regular solutions.

To make this paper self-contained, we first present the general setting for the spherically symmetric evolution. Studying a spherically symmetric Einstein-matter system it is convenient to use the following ansatz for the metric

$$ds^2 = -e^{-2\delta} N dt^2 + N^{-1} dr^2 + r^2 d\Omega^2, \quad (1)$$

where $d\Omega^2$ is the standard metric on the unit 2-sphere and δ, N are functions of (t, r) . Although this coordinate system cannot penetrate an event horizon, this is not a serious disadvantage, as Choptuik emphasized, in studying the *formation* of horizons. The main advantage of the choice (1) is that the Einstein's equations simplify considerably in terms of the metric functions N and δ

$$N' = \frac{1 - N}{r} - 8\pi G r T_{00}, \quad (2)$$

$$\dot{N} = -8\pi G r e^{-\delta} N T_{01}, \quad (3)$$

$$\delta' = -4\pi G r N^{-1} (T_{00} + T_{11}), \quad (4)$$

where overdots and primes denote $\partial/\partial t$ and $\partial/\partial r$, respectively, and the components of the stress-energy tensor of matter T_{ab} are expressed in the orthonormal frame determined by the metric (1) ($e_0 = e^\delta N^{-1/2} \partial_t, e_1 = N^{1/2} \partial_r$).

The full system of evolution equations consists of Eqs. (2)–(4) and evolution equations for a matter field¹. For the sake of simplicity let us assume that matter is described by one scalar function $F(r, t)$ which satisfies a nonlinear wave equation (this is a typical situation). As a consequence of Birkhoff's theorem the essential dynamics of the system resides in $F(r, t)$. In order to solve the evolution equations we need to supplement them with boundary and initial conditions. To ensure regularity we require that the components of the stress-energy tensor are bounded for all (t, r) and

¹ As it is well known, in many cases the evolution equations for matter are implicitly contained in the Einstein equations. In particular, for the two models discussed below the consistency condition for the Eqs. (2) and (3) is equivalent to the wave equation for a matter field.

that $N(0, t) = 1 + O(r^2)$ at the center. Asymptotic flatness requires that $N(r, t) = 1 + O(1/r^2)$ for $r \rightarrow \infty$. The time coordinate is normalized by the boundary condition $\delta(\infty, t) = 0$ (so t is the proper time at spatial infinity). The initial value problem for the above equations is solved as follows. At $t = 0$ one takes asymptotically flat regular initial data for the matter field $F(r, 0)$ and $\dot{F}(r, 0)$. Then the elliptic Eqs. (2) and (4) are solved yielding initial metric functions $N(r, 0)$ and $\delta(r, 0)$. Once a full set of initial data is constructed, it is evolved using Eq. (3) and the hyperbolic evolution equation for F . The function δ is updated at each subsequent moment of time using Eq. (4). This scheme is called the free evolution, as opposed to the alternative scheme of fully constrained evolution in which the hamiltonian constraint (2), rather than Eq. (3), is used to compute the function N .

We are now ready to discuss critical phenomena in the evolution. Here we focus our attention exclusively on the first order phase transitions. In passing, we remark that, from the theoretical perspective, second order phase transitions are much more interesting because of their bearing on the cosmic censorship hypothesis. On the other hand, first order phase transitions might be more important in the astrophysical context.

To make this presentation concise we first describe the basic scenario in a model-independent manner and then illustrate it with concrete models. For a first order transition to occur in a spherically-symmetric Einstein-matter system, we need to make two assumptions²:

Assumption 1. There exists a static regular asymptotically flat solution with *one* linearly unstable eigenmode (in field theory such solution is referred to as a sphaleron). Let us denote this solution by X^u with X standing for (F, N, δ) . The uniqueness of the unstable eigenmode means that the linear evolution of an initially small spherical perturbation about X^u can be decomposed into the sum

$$\delta X^u(r, t) = C e^{\lambda t} \xi_\lambda(r) + \sum_{i=1}^{\infty} C_i e^{\mu_i t} \xi_{\mu_i}(r) \quad (5)$$

of the single growing mode with a positive eigenvalue λ and the decaying modes with $Re(\mu_i) < 0$. Physically the damping of non-growing modes is due to the loss of energy by radiation. Mathematically this is reflected in non-self-adjointness of the eigenvalue problem.

Assumption 2. The ultimate fate of the perturbation (5) depends only on the sign of the amplitude C . For one sign of C , say $C > 0$, a black hole forms, while for $C < 0$ there exists a global in time regular evolution. In the

² The importance and consequences of the analogous assumptions in the context of second order phase transitions in critical collapse were first spelt out by Koike, Hara, and Adachi [10].

latter case the final state depends on the details of a model — it might be the Minkowski spacetime (this case is usually referred to as dispersion), or some stable regular solution.

Assumption 1 means that the stable manifold W_S of the solution X^u has codimension one. Assumption 2 means that W_S is a “critical” surface in the sense of dividing (locally) the phase space into collapsing and non-collapsing initial data.

Now, consider a one-parameter family of initial data $\Phi(r, p)$, where p is a parameter, which intersects W_S at some parameter value $p = p^*$. Let $X_p(r, t)$ denote the solution corresponding to these initial data. The critical initial data are attracted along W_S towards the solution X^u . A near-critical solution, by continuity, remains close to W_S and, once it gets close to X^u , can be approximated by the linearization (5) around X^u with the amplitudes C and C_i depending on the initial data. Since by definition $C(p^*) = 0$, we have

$$X_p(r, t) = X^u(r) + A(p - p^*) e^{\lambda t} \xi_\lambda(r) + \text{decaying modes}, \quad (6)$$

where $A = \frac{dC}{dp}(p^*)$. The range of t for which this linear approximation is valid is called the intermediate asymptotics. In this asymptotics, the solution $X_p(r, t)$ initially approaches X^u but later the growing mode becomes dominant and the solution is repelled from W_S along the unstable manifold of X^u . Because of this behaviour the solution X^u is sometimes called the intermediate attractor. The duration of the intermediate asymptotics is determined by the time T in which the unstable mode grows to a finite size $|p - p^*| e^{\lambda T} \sim O(1)$, which gives $T \sim -\lambda^{-1} \ln |p - p^*|$. Thus, the larger λ , the better fine-tuning is required to see the solution X^u clearly pronounced as the intermediate attractor.

The scenario of critical collapse summarized above naturally explains the universality (that is family-independence) of this phenomenon — it simply follows from the fact that the evolution of *all* near-critical data is governed by the *same* unstable mode around the intermediate attractor. Within this framework we also see why it is essential that the solution X^u has exactly one unstable mode. If it was linearly stable (that is if it had no unstable modes), then it would be an attractor of an open set of initial data and it would not be related to any critical behaviour. On the other hand if X^u had two or more unstable modes then a generic one-parameter family of initial data would not intersect its stable manifold and *eo ipso* the critical behaviour would not be generic.

By Assumption 2 all solutions starting with initial data $X^u(r) + \varepsilon \xi_\lambda(r)$ with some small positive amplitude ε form black holes. As follows from (6), for any ε , such initial data can be extracted from the evolution of super-critical solutions with sufficiently small $p - p^*$ at some time t_p satisfying

$A(p - p^*)e^{\lambda t_p} = \varepsilon$. Although the time t_p depends on p , the evolution for $t > t_p$ is independent of p [7]. Denoting the mass of a resulting black hole by $m_{BH}(\varepsilon)$, we can define the mass gap as $m^* = \lim_{\varepsilon \rightarrow 0} m_{BH}(\varepsilon)$. The mass gap m^* is bounded from above by the mass m_u of the solution X^u . The difference $m_u - m^*$ can be interpreted as the total energy radiated away to infinity during the critical collapse.

Now we substantiate the general picture presented above with two models in which the first order phase transitions were observed: Einstein–Yang–Mills (EYM) and Einstein–Skyrme (ES).

EYM: We assume the following ansatz for the SU(2)-YM field

$$eF = dw \wedge (\tau_1 d\vartheta + \tau_2 \sin \vartheta d\varphi) - (1 - w^2)\tau_3 d\vartheta \wedge \sin \vartheta d\varphi, \quad (7)$$

where e is the coupling constant, τ_i are the Pauli matrices, and the YM potential w is a function of (r, t) . The evolution equation for $w(r, t)$ is

$$-(e^\delta N^{-1}\dot{w})' + (e^{-\delta} N w')' + \frac{1}{r^2} e^{-\delta} w(1 - w^2) = 0, \quad (8)$$

while the Einstein equations have the form (2)-(4) with the stress-energy tensor

$$T_{00} = \frac{1}{4\pi e^2 r^2} \left(N w'^2 + e^{2\delta} N^{-1} \dot{w}^2 + \frac{(1 - w^2)^2}{2r^2} \right), \quad (9)$$

$$T_{11} = \frac{1}{4\pi e^2 r^2} \left(N w'^2 + e^{2\delta} N^{-1} \dot{w}^2 - \frac{(1 - w^2)^2}{2r^2} \right), \quad (10)$$

$$T_{01} = \frac{1}{2\pi e^2 r^2} e^\delta \dot{w} w'. \quad (11)$$

The EYM equations have a countable family of static asymptotically flat regular solutions X_n ($n \in \mathbb{N}$) discovered numerically by Bartnik and McKinnon [11] and later proven rigorously to exist by Smoller and Wassermann [12]. Within the ansatz (7) the solution X_n has n unstable modes, hence the first Bartnik-McKinnon solution X_1 satisfies the Assumption 1 [13]. Moreover, the nonlinear instability analysis of this solution performed by Zhou and Straumann [14] strongly suggested that the Assumption 2 is also true. In fact, Choptuik and the present authors showed that for some initial data the solution X_1 acts as the intermediate attractor and controls the first order phase transition [8]. Since vacuum is the only stable solution, the subcritical solutions disperse. For supercritical solutions the mass gap was found to be equal (up to 0.1%) to the mass of X_1 (the mass scale is given by $1/(e\sqrt{G})$).

ES: In this model matter is described an SU(2)-valued scalar function U (called a chiral field). In spherical symmetry $U = \exp(i\vec{\tau} \cdot \hat{r}F(r, t))$ with the

dynamics of $F(r, t)$ governed by the equation

$$\begin{aligned}
 & - (ue^\delta N^{-1} \dot{F}) + (ue^{-\delta} NF')' = \sin(2F)e^{-\delta} \\
 & \times \left(f^2 + \frac{1}{e^2} \left(\frac{\sin^2 F}{e^2 r^2} + NF'^2 - e^{2\delta} N^{-1} \dot{F}^2 \right) \right), \quad (12)
 \end{aligned}$$

where $u = f^2 r^2 + 2e^2 \sin^2 F$, and f and e are coupling constants. The components of stress-energy tensor in the Einstein equations (2)–(4) are

$$T_{00} = \frac{u}{2r^2} (NF'^2 + N^{-1} e^{-2\delta} \dot{F}^2) + \frac{\sin^2 F}{r^2} \left(f^2 + \frac{\sin^2 F}{2e^2 r^2} \right), \quad (13)$$

$$T_{11} = \frac{u}{2r^2} (NF'^2 + N^{-1} e^{-2\delta} \dot{F}^2) - \frac{\sin^2 F}{r^2} \left(f^2 + \frac{\sin^2 F}{2e^2 r^2} \right), \quad (14)$$

$$T_{01} = \frac{u}{r^2} e^\delta \dot{F} F'. \quad (15)$$

Regularity at the center requires that $F(0, t) = 0$, while asymptotic flatness requires that $F(\infty, t) = B\pi$, where an integer B , called the baryon number, is equal to the topological degree of the chiral field. As long as no horizon forms, the baryon number is conserved during the evolution, so we have topological selection rules for the possible end states of given initial data. The number of static regular solutions of the ES equations depends on the dimensionless parameter $\alpha = 4\pi G f^2$ (this parameter is the square of the ratio of two length scales \sqrt{G}/e and $1/\sqrt{4\pi e f}$). For large values of α there are no regular static solutions. As α decreases, in each topological sector there is a countable sequence of bifurcations at which there appear pairs of static regular solutions [15]. Here we briefly summarize our results in the topological sectors of baryon number zero and one (see [9] for the details).

In the $B = 0$ sector for $\alpha < \alpha_0 \simeq 0.00147$ there exists a static regular solution satisfying the Assumptions 1 and 2 which plays the role of an intermediate attractor in the critical collapse of specially prepared initial data. Since the vacuum (*i.e.* the Minkowski spacetime) is the only regular stable $B = 0$ solution, the subcritical solutions disperse, as in the EYM case. The case $B = 1$ is more interesting. Here, for $\alpha < \alpha_1 \simeq 0.040378$, there is a pair of regular static solutions X^s and X^u . The solution X^s is linearly stable while X^u has one unstable mode [15, 16]. In the limit $\alpha \rightarrow 0$ the solution X^s tends to the flat space skyrmion while X^u has no regular limit. Again, the solution X^u satisfies the Assumption 2 and plays the role of an intermediate attractor. However, now the dispersion is topologically forbidden, and instead the subcritical solutions decay into the stable solution X^s (this was observed previously in [17]). The solution X^u has larger mass than X^s so during its decay the excess energy has to be radiated away to infinity.

The process of settling down to X^s has the form of damped oscillations (quasinormal ringing). The relaxation time increases with α and tends to infinity as $\alpha \rightarrow \alpha_1$, where the solutions X^s and X^u coalesce. The evolution of supercritical solutions also depends on the baryon number. For $B = 1$ almost no energy is lost during the critical collapse, hence the mass gap is equal to the mass of X^u , in analogy to the EYM case. In contrast, for $B = 0$ supercritical data a substantial amount of energy is radiated away, and consequently the mass gap is smaller than the mass of the unstable solution. For example, $m^* \simeq 0.76m_u$ when $\alpha = 0.00145$.

In conclusion, the numerical results in the EYM and ES models (and also the results of [18]) give strong evidence that our understanding of first order phase transitions in gravitational collapse based on the Assumptions 1 and 2 is correct. In particular, in both cases the formula (6) was shown to approximate very well the evolution in the intermediate asymptotics. Actually, this formula was used to reproduce with good accuracy the results of linear stability analysis. Let us close with the discouraging remark that a rigorous description of the phenomenon described in this paper does not seem feasible to us, because it would require to overcome a number of mathematical problems which have not been solved even for much simpler systems.

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