Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 4

GRAVITATIONALLY INTERACTING NONLINEAR SPIN-2 FIELD ARISING FROM NONLINEAR GRAVITY* **

L.M. Sokołowski

Astronomical Observatory, Jagellonian University
Orla 171, Kraków 30-244, Poland
e-mail: uflsokol@thrisc.if.uj.edu.pl

(Received February 6, 1998)

Dedicated to Andrzej Trautman in honour of his 64th birthday

It is well known that fundamental linear higher-spin (≥ 2) fields are unphysical: they cannot be a a source of gravity, *i.e.* their dynamics is *inconsistent* unless they exist as test fields in an empty space. A certain kind of *nonlinear* spin-2 field arises from vacuum nonlinear metric gravity theories (Lagrangian being any smooth scalar function of Ricci tensor) as a component of a multiplet of tensor fields describing gravity. These theories can be reformulated as Einstein gravity theory with gravity described by the metric field alone and the other fields contained in the multiplet acting as a "matter" source in Einstein field equations. This framework provides a consistent gravitational interaction for the spin-2 field. A number of open problems still remains. This paper is a progress report on a joint work done by Guido Magnano and myself.

PACS numbers: 04.20. Cv, 04.50. +h

1. Introduction

From the experimental point of view there is no necessity to construct a theoretical description of a spin-2 field which acts as a source of gravity since up to now we know no other elementary particle carrying spin two besides the hypothetical graviton. Nevertheless there are various reasons for studying linear higher-spin ($s > 1$ integer) fields in classical field theory [1]. Furthermore, in the last decade it was realized that certain spin-0 and spin-2 fields

* Presented at the Workshop on Gauge Theories of Gravitation, Jadwisin, Poland, September 4–10, 1997.
** This work was partially supported by a grant of Polish Committee for Scientific Research no. PB 2 P03B 01113.

minimally coupled to a spacetime metric are generated in a natural way as components of a multiplet of fields describing gravity in some extensions of Einstein's gravity theory. These extensions consisting in replacing Einstein–Hilbert Lagrangian for vacuum gravity by a Lagrangian being any smooth scalar function of Ricci and Weyl tensors are named *metric nonlinear gravity* (NLG) theories or higher-order gravity. Impulses for investigating these theories have recently come from various directions. In the low-energy field-theory limit of superstring effective actions one recovers Einstein–Hilbert Lagrangian plus higher order corrections in Riemann tensor [2]. Moreover quantum theory suggests that a renormalizable quantum gravity necessitates a quadratic Lagrangian having a classical limit more complicated than general relativity. Last but not least, quadratic and higher-order in Riemann curvature Lagrangians are possible candidates for a gravity theory avoiding spacetime singularities [3].

Structure of a spin-2 field theory is quite different in the two cases: of the massive linear field and the massive nonlinear field generated by an NLG theory. In the first case the inconsistency of gravitational interactions of the field is well known and I briefly present it in section 2 merely to stress the difference to the other case. The nonlinear spin-2 field is described in section 3 where I report on progress on the subject made recently by G. Magnano and myself.

2. Linear massive spin-2 field as a test field in empty space

A spin-2 field is usually described by a symmetric tensor field $\psi_{\mu\nu}$. A free field theory in flat spacetime is first constructed for a massless field and the guiding principle is the postulate of a gauge invariance [1]; then an appropriate mass term is added to the Lagrangian. In the next step one introduces an interaction of the field with itself or other matter. The last step, different from the previous one, consists in coupling of the field to gravity.

The free field theory *is* consistent; inconsistency problems appear for interactions. The problems arise since for spin ≥ 1 the number of algebraically independent components of the tensor field is larger than the number of dynamical degrees of freedom (for spin two there are 10 components of $\psi_{\mu\nu}$ versus 5 physical degrees of freedom in the massive case) and some components are redundant (represent a "pure gauge"). Then Lagrange equations of motion form a degenerate system, *i.e.* some of them are first-order constraints on the initial data rather than being hyperbolic propagation equations. The problem already arises for Maxwell electrodynamics and a massive vector (Proca) field; for a free field one deals with it in an analogous way for all spins ≥ 1.

As we are interested in gravitational interactions of a massive spin-2 field, we omit the free field theory and construct a Lagrangian for $\psi_{\mu\nu}$ in a curved spacetime. To this aim we use a gravitational perturbation analogy [4]. One takes any spacetime metric and perturbs it, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$. The second variation of Einstein–Hilbert action $S[g] = \int d^4x\sqrt{-g}R$ evaluated at the "background" metric $g_{\mu\nu}$ is a functional quadratic in the metric fluctuations and if one identifies $\delta g_{\mu\nu}$ with $\psi_{\mu\nu}$ the functional provides an action giving rise to linear Lagrange equations for the field. Actually $\psi_{\mu\nu}$ can be defined as a linear function of $\delta g_{\mu\nu}$, *e.g.* as $\delta(\sqrt{-g}g_{\mu\nu})$. Here we make the simplest choice $\psi_{\mu\nu} = \delta g_{\mu\nu}$ (which however leads to field equations which are not the simplest possible; other choices provide equivalent theories) and the action is defined as $S[\psi] \equiv -2\delta^2 S[g]$. Next one assumes that $\psi_{\mu\nu}$ is a non-geometric massive tensor field which interacts with gravity. In other terms from now on $g_{\mu\nu}$ is not regarded as a fixed background metric but rather as a dynamical field coupled to $\psi_{\mu\nu}$. (This means that the equations of motion for the fields are *not* perturbation equations of a given solution for pure gravity.) To assign a mass to $\psi_{\mu\nu}$ one puts in the Lagrangian by hand a mass term which is appropriately chosen to avoid any additional scalar field. Finally the second order action for $\psi_{\mu\nu}$ reads (the action is linear in second derivatives)

$$S[\psi] = \int d^4x\sqrt{-g}\left[\psi^{\mu\nu}G^L_{\mu\nu} + \frac{1}{2}\psi\psi^{\mu\nu}G_{\mu\nu} + \frac{m^2}{2}(\psi_{\mu\nu}\psi^{\mu\nu} - \psi^2)\right] \qquad (1)\,,$$

where $\psi = g^{\mu\nu}\psi_{\mu\nu}$ and $G^L_{\mu\nu}$ is the linear in $\psi_{\mu\nu}$ part of Einstein tensor,

$$G_{\mu\nu}(g+\psi) = G_{\mu\nu}(g) + G^L_{\mu\nu}(g,\psi) + \ldots,$$

$$\begin{aligned} G^L_{\mu\nu}(g,\psi) \;\equiv\; & \frac{1}{2}(-\Box\psi_{\mu\nu} + \psi_{\mu\alpha;\nu}{}^{;\alpha} + \psi_{\nu\alpha;\mu}{}^{;\alpha} - \psi_{;\mu\nu} - g_{\mu\nu}\psi^{\alpha\beta}{}_{;\alpha\beta} + g_{\mu\nu}\Box\psi \\ & + g_{\mu\nu}\psi^{\alpha\beta}R_{\alpha\beta} - \psi_{\mu\nu}R)\,. \end{aligned} \qquad (2)$$

Adding Einstein–Hilbert action $S[g]$ to $S[\psi]$ one derives the full system of equations of motion consisting of Einstein's field equations $G_{\mu\nu} = T_{\mu\nu}(g,\psi)$ (we set $8\pi G = c = \hbar = 1$) where $T_{\mu\nu}$ is the variational energy-momentum tensor following from (1) and Lagrange equations

$$E_{\mu\nu} \equiv G^L_{\mu\nu}(g,\psi) + \frac{1}{2}G_{\mu\nu}(g)\psi - \frac{m^2}{2}(g_{\mu\nu}\psi - \psi_{\mu\nu}) = 0. \qquad (3)$$

The latter form a degenerate system: only 6 out of 10 equations $E_{\mu\nu} = 0$ are hyperbolic propagation ones for $\psi_{\mu\nu}$, four equations $E_{0\mu}(g,\psi) = 0$ do not contain second time derivatives of the field and form so-called *primary constraints* on the initial Cauchy data. To get a consistent dynamics one should replace the primary constraints by other constraints which allow to transform the primary ones in four missing propagation equations. To this aim one proceeds as in the massive case in flat spacetime. By taking divergence of (3) one gets four *secondary constraints* on the initial data [5],

$$\nabla^{\nu} E_{\mu\nu}(g,\psi) \equiv Q_{\mu}(g,\psi) = \nabla_{\nu}(\psi^{\nu\alpha} G_{\mu\alpha}) + \frac{1}{2}\psi^{\alpha\beta}{}_{;\mu} G_{\alpha\beta} - \frac{m^2}{2}(\psi_{;\mu} - \psi_{\mu\alpha}{}^{;\alpha}) = 0, \tag{4}$$

where one has applied the linearized Bianchi identity

$$\nabla^{\nu} G^{L}_{\mu\nu}(g,\psi) \equiv \nabla_{\nu}(\psi^{\nu\alpha} G_{\mu\alpha}) + \frac{1}{2}\psi^{\alpha\beta}{}_{;\mu} G_{\alpha\beta} - \frac{1}{2}\psi_{;\alpha} G^{\alpha}{}_{\mu}. \tag{5}$$

However the constraints $Q_{\mu} = 0$ are defective in the following sense. It is natural to view them as first order differential constraints on the initial data for $\psi_{\mu\nu}$ at $t = 0$. But then replacing $G_{\mu\nu}$ by $T_{\mu\nu}$ one arrives at the following equations of motion:

$$E'_{\mu\nu} = G^{L}_{\mu\nu}(g,\psi) + \frac{1}{2}T_{\mu\nu}(g)\psi - \frac{m^2}{2}(g_{\mu\nu}\psi - \psi_{\mu\nu}) = 0\,, \tag{6a}$$

$$Q'_{\mu} = \nabla_{\nu}(\psi^{\nu\alpha} T_{\mu\alpha}) + \frac{1}{2}\psi^{\alpha\beta}{}_{;\mu} T_{\alpha\beta} - \frac{m^2}{2}(\psi_{;\mu} - \psi_{\mu\alpha}{}^{;\alpha}) = 0\,. \tag{6b}$$

The expression for $T_{\mu\nu}$ is extremely complicated and contains $\psi_{\mu\nu;\alpha\beta}$ [4], hence (6b) are four nonlinear third order propagation equations. The components $T_{0\mu}$ contain $\psi_{\alpha\beta,00}$ what implies that $E'_{0\mu} = 0$ are no more constraints. As a consequence there are no constraints imposed on $\psi_{\mu\nu}$ which are preserved in time and which decouple the unphysical modes ensuring the existence of the correct number (five) of degrees of freedom.

The opposite possibility is to consider $\psi_{\mu\nu}$ as a test field on a fixed background determined by Einstein's equations $G_{\mu\nu} = t_{\mu\nu}(\phi)$ with $t_{\mu\nu}$ being the stress tensor for some matter ϕ. Then $E_{0\mu} = 0$ and $Q_{\mu} = 0$ are constraints on initial values of $\psi_{\mu\nu}$. The necessary condition for having a consistent dynamics for $\psi_{\mu\nu}$ is that both primary and secondary constraints are preserved in time. This can be shown only for very special cases [6]. In general there is no consistent dynamical description of spin-2 field on a given curved spacetime.

The third possibility is to regard $Q_{\mu} = 0$ as restrictions on the spacetime metric. Writing them as

$$Q_{\mu} = \left[\left(G_{\mu\alpha}\delta^{\sigma}_{\beta} + \frac{1}{2}G_{\alpha\beta}\delta^{\sigma}_{\mu}\right)\nabla_{\sigma} + G_{\mu\alpha;\beta}\right]\psi^{\alpha\beta} - \frac{m^2}{2}(\psi_{;\mu} - \psi_{\mu\alpha}{}^{;\alpha}) = 0 \tag{7}$$

one sees that they contain third time derivatives of the metric and thus restrict it in the whole spacetime. An admissible solution (though there is no rigorous proof) is $G_{\mu\nu} = 0$. Then the secondary constraints reduce to

$$\psi_{\mu\nu}{}^{;\nu} - \psi_{;\mu} = 0. \tag{8}$$

Their divergence is

$$\psi^{\mu\nu}{}_{;\mu\nu} - \psi_{;\mu}{}^{;\mu} = 0. \tag{9}$$

On the other hand the trace of (3) is

$$g^{\mu\nu}E_{\mu\nu} = \psi_{;\mu}{}^{;\mu} - \psi^{\mu\nu}{}_{;\mu\nu} - \frac{3}{2}m^2\psi = 0. \tag{10}$$

Adding (9) to (10) one finds that $\psi_{\mu\nu}$ is traceless, $g^{\mu\nu}\psi_{\mu\nu} \equiv \psi = 0$, which in turn reduces (8) to $\psi^{\mu\nu}{}_{;\nu} = 0$. These are five secondary constraints ensuring the purely spin-2 nature of the field. The constraints considerably simplify Lagrange equations (3) to

$$(\Box - m^2)\psi_{\mu\nu} + 2\psi^{\alpha\beta}R_{\mu\alpha\nu\beta} = 0, \tag{11}$$

where $R_{\mu\nu} = 0$ has been used. These form a nondegenerate system of 10 hyperbolic propagation equations. As a final step in constructing a consistent dynamics one proves the proposition [6]:
if the equations of motion (11) hold throughout an empty spacetime and the following constraints restrict the initial data at $t = 0$: $E_{0\mu} = 0$ and $\psi = \psi_{;0} = \psi^{\mu\nu}{}_{;\nu} = 0$, then all the constraints, $\psi^{\mu\nu}{}_{;\nu} = 0 = \psi$ and $E_{0\mu} = 0$ are preserved in time.
Here one must assume an additional initial data constraint $\psi_{;0} = 0$ at $t = 0$ to ensure vanishing of ψ in the spacetime. The final conclusion is [4, 5]: *massive linear spin-2 field is consistent only if it is a test field in an empty spacetime; then in the limit of vanishing mass it coincides with small gravitational fluctuations*. The same argument applies to the massless field. Inclusion of any non-minimal coupling to gravity cannot help [5]. A linear spin-2 field cannot be a source of gravity and in this sense it is unphysical.

3. Consistent nonlinear spin-2 field in a curved spacetime

Dynamical evolution of a Lorentzian manifold $(M, \psi_{\mu\nu})$ is determined in the framework of a generic NLG theory by the Lagrangian
$L = \sqrt{-\psi} f(\psi_{\mu\nu}, \bar{R}_{\alpha\beta}(\psi_{\mu\nu}))$ where a scalar function f may also depend on the conformal tensor (in this section $\psi = det(\psi_{\mu\nu})$). In general Lagrange equations of motion are of fourth order. Such a theory gives rise to a massive nonlinear spin-2 field (and a massive scalar field) in two ways. Firstly,

one can both lower the order of the equations of motion and generate additional fields describing gravity in a way analogous to replacing Lagrange formalism by canonical one in classical mechanics. One introduces "canonical momenta" conjugate to Christoffel connection for $\psi_{\mu\nu}$ using Legendre transformations with respect to the irreducible parts of Ricci tensor [7]:

$$\sqrt{-\psi}\pi^{\mu\nu} = \frac{\partial L}{\partial S_{\mu\nu}} \qquad \text{and} \qquad \sqrt{-\psi}\phi = \frac{\partial L}{\partial \bar{R}}\,, \tag{12}$$

where $S_{\mu\nu} \equiv \bar{R}_{\mu\nu} - \frac{1}{4}\bar{R}\psi_{\mu\nu}$. The fields $\pi^{\mu\nu}$ and ϕ turn out to be massive and carry spin two and zero respectively. The original Lagrangian L is then replaced by a Helmholtz Lagrangian L_H generating second order field equations for the triplet of the fields [7]. It is remarkable that for $\psi_{\mu\nu}$ one gets exactly Einstein's field equations $G_{\mu\nu}(\psi) = T_{\mu\nu}(\psi,\pi,\phi)$ with a stress tensor for the nongeometric part of the triplet which is however indefinite [6]. In a weak-field limit one recovers the well known Stelle's results for a quadratic L [8].

The second approach is more sophisticated. One assumes that the field $\psi_{\mu\nu}$ appearing in L is a kind of a unifying field and does *not* coincide with the physical spacetime metric (the Lagrangian may possibly come from a more fundamental theory and $\psi_{\mu\nu}$ is not a geometric quantity). The genuine measurable metric should be recovered from L via a Legendre transformation [9, 10]

$$g^{\mu\nu} \equiv \left|\det\left(\frac{\partial L}{\partial \bar{R}_{\alpha\beta}}\right)\right|^{-1/2} \frac{\partial L}{\partial \bar{R}_{\mu\nu}}\,. \tag{13}$$

If this transformation can be inverted one expresses the canonical "velocity" $\bar{R}_{\mu\nu}$ in terms of the "positions and momenta", $\bar{R}_{\mu\nu}(\psi) = r_{\mu\nu}(g^{\alpha\beta},\psi_{\alpha\beta})$. To view $g^{\mu\nu}$ as a spacetime metric one assumes that it is nonsingular, *i.e.* $\det(\partial f/\partial \bar{R}_{\mu\nu}) \neq 0$ and $g_{\mu\nu}$ is its inverse. In other terms one maps $(M,\psi_{\mu\nu})$ onto $(M,g_{\mu\nu})$ and from now on one treats $\psi_{\mu\nu}$ as some tensor field on the spacetime $(M,g_{\mu\nu})$. As in classical mechanics one replaces the Lagrangian by the Hamiltonian,

$$H(g,\psi) \equiv \sqrt{-g}g^{\mu\nu}r_{\mu\nu}(g,\psi) - (-\det\psi)^{\frac{1}{2}} f(\psi_{\mu\nu}, r_{\mu\nu}) \tag{14}$$

and then the latter by a Helmholtz Lagrangian

$$L_H(g,\psi) \equiv \sqrt{-g}g^{\mu\nu}\bar{R}_{\mu\nu}(\psi) - H(g,\psi). \tag{15}$$

The Helmholtz action $S_H = \int d^4x L_H$ generates Hamilton equations for $g_{\mu\nu}$ and $\psi_{\mu\nu}$ as variational Lagrange equations and these are of *second order*. Introducing a tensor being the difference of the Christoffel connections for the two tensors, $Q^{\alpha}_{\mu\nu} \equiv \bar{\Gamma}^{\alpha}_{\mu\nu}(\psi) - \Gamma^{\alpha}_{\mu\nu}(g)$, and applying the following identity valid for *any* two nonsingular tensor fields [9],

$$\bar{R}_{\mu\nu}(\psi) - R_{\mu\nu}(g) = \nabla_{\alpha} Q^{\alpha}_{\mu\nu} - \nabla_{(\mu} Q^{\alpha}_{\nu)\alpha} + Q^{\alpha}_{\mu\nu} Q^{\beta}_{\alpha\beta} - Q^{\alpha}_{\mu\beta} Q^{\beta}_{\nu\alpha}, \tag{16}$$

where $\nabla_{\alpha} T \equiv T_{;\alpha}$ is the covariant derivative with respect to $g_{\mu\nu}$, one can finally express L_H in the form (up to a divergence term)

$$\begin{aligned} L_H \;=\; & \sqrt{-g}\Big[R(g) + K(g,\psi) - 2V(g,\psi)\Big] \equiv \sqrt{-g}\Big[R(g) + g^{\mu\nu}\Big(Q^{\alpha}_{\mu\nu} Q^{\beta}_{\alpha\beta} \\ & - Q^{\alpha}_{\mu\beta} Q^{\beta}_{\nu\alpha}\Big)\Big] - \sqrt{-g}\Big[g^{\mu\nu} r_{\mu\nu}(g,\psi) - \left(\frac{\det\psi}{g}\right)^{\frac{1}{2}} f(\psi_{\mu\nu}, r_{\mu\nu})\Big] . \end{aligned} \tag{17}$$

Here K being the quadratic polynomial in $Q^{\alpha}_{\mu\nu}$ is a kinetic Lagrangian for $\psi_{\mu\nu}$ and is *universal* (is independent of the form of f) while the potential V is determined by the original Lagrangian. It is a straightforward calculation to show that the theory based on L_H is dynamically equivalent to that based on $L = \sqrt{-\psi} f$ [9, 10]. It is far from being obvious that it is possible to define the genuine metric $g_{\mu\nu}$ in such a way that the gravitational part of the Helmholtz Lagrangian is exactly equal to the curvature scalar. In this sense *Einstein general relativity is a universal Hamiltonian image (under a Legendre transformation) of any NLG theory*.

The field equations $\delta S_H/\delta g^{\mu\nu} = 0$ are just Einstein ones,

$$\begin{aligned} G_{\mu\nu}(g) \;=\; & T_{\mu\nu}(g,\psi) \equiv Q^{\alpha}_{\alpha(\mu;\nu)} - Q^{\alpha}_{\mu\nu;\alpha} - Q^{\alpha}_{\mu\nu} Q^{\beta}_{\alpha\beta} + Q^{\alpha}_{\mu\beta} Q^{\beta}_{\alpha\nu} + r_{\mu\nu} \\ & + \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} (Q^{\lambda}_{\alpha\beta;\lambda} - Q^{\lambda}_{\lambda\alpha;\beta} + Q^{\sigma}_{\alpha\beta} Q^{\lambda}_{\sigma\lambda} - Q^{\sigma}_{\alpha\lambda} Q^{\lambda}_{\sigma\beta} - r_{\alpha\beta}) . \end{aligned} \tag{18}$$

This $T_{\mu\nu}$ is the variational energy-momentum tensor following from (17); the quickest way of deriving it is to apply the identity (16). In general the stress tensor is indefinite and the energy density is not determined by initial data since it depends on $\psi_{\mu\nu;\alpha\beta}$. The kinetic part of it (made up of $Q^{\alpha}_{\mu\nu}$) is universal while the potential part is determined by $r_{\mu\nu}(g,\psi)$ and does not depend explicitly on f. This stress tensor is rather complicated, nevertheless it is considerably simpler than that for the linear inconsistent field discussed in previous section [4].

Lagrange equations $\delta S_H/\delta\psi_{\mu\nu} = 0$ are too complicated to be presented here [6]. These are quasi-linear second order equations whose "kinetic" part is universal (due to (17)). This universality shows that there is no need to study a generic NLG theory — to find out the physical content of all these theories it is sufficient to investigate the simplest case: the original L being a quadratic function of the curvature. Furthermore, if L is a cubic or higher order polynomial in $\bar{R}_{\mu\nu}$ there are technical problems with inverting the definition (13) to get $\bar{R}_{\mu\nu}(\psi) = r_{\mu\nu}(g^{\alpha\beta}, \psi_{\alpha\beta})$. The particle content is the same for all cases: in a weak-field limit of (17) we find that $g_{\mu\nu}$ describes the massless graviton (helicity 2) while $\psi_{\mu\nu}$ is a mixture of massive spin-2 and spin-0 fields. (If Weyl tensor is present in L this picture may become more complex.) Our results are in agreement with Stelle [8] who studied the quadratic Lagrangian $\bar{R} + a\bar{R}^2 + b\bar{R}_{\mu\nu}\bar{R}^{\mu\nu}$.

We choose the Lagrangian $L = \sqrt{-\psi}(\bar{R} + a\bar{R}^2 - 3a\bar{R}_{\mu\nu}\bar{R}^{\mu\nu})$ with $a = \text{const} > 0$ of dimension (length)^2. The fourth-order field equations imply $\bar{R} = 0$. This corresponds to absence of the scalar canonical momentum ϕ defined in (12), $\phi = \text{const}$. The field $\pi^{\mu\nu}$ (being now proportional to $\bar{R}^{\mu\nu}$) is traceless and divergenceless (with respect to $\psi_{\mu\nu}$), thus it describes purely spin-2 particles; its equations of motion are

$$\Box\pi_{\mu\nu} - \frac{1}{3a}\pi_{\mu\nu} + \frac{1}{12a}g_{\mu\nu}\pi^{\alpha\beta}\pi_{\alpha\beta} - 2R_{\alpha(\mu\nu)\beta}\pi^{\alpha\beta} = 0. \tag{19}$$

The field cannot be massless, $m_\pi^2 = \frac{1}{3a}$.

The Legendre transformation (13) (usually named a transition to Einstein conformal frame (ECF)) clearly shows that Einstein–Hilbert term in L is essential. In fact, one gets

$$g^{\mu\nu} = A(g,\psi)[(1 + 2a\bar{R})\gamma^{\mu\nu} - 6a\bar{R}^{\mu\nu}] \tag{20}$$

with $A \equiv |\det(\psi_{\mu\nu})/g|^{1/2}$ and $\gamma^{\mu\nu}$ being the inverse matrix to $\psi_{\mu\nu}$. From now on we raise and lower tensor indices with the aid of $g_{\mu\nu}$, *e.g.*

$$\psi^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\psi_{\alpha\beta} \quad \text{and} \quad \gamma^{\mu\alpha}\psi_{\alpha\nu} = \delta^\mu_\nu = \psi^{\mu\alpha}\gamma_{\alpha\nu}.$$

Due to presence of the linear term in L, for spacetimes close to flat space ($\bar{R}_{\mu\nu} \approx 0$) the tensor $g^{\mu\nu}$ is nonsingular and has the correct signature; for Minkowski space the transformation is the identity one (see also [11, 12]). To invert (20) one first expresses $\bar{R}$ in terms of A and the trace $t \equiv g^{\mu\nu}\psi_{\mu\nu}$, then one finds

$$\bar{R}_{\mu\nu}(\psi) = r_{\mu\nu}(g,\psi) = \frac{1}{6aA}[(t - 3A)\psi_{\mu\nu} - \psi_{\mu\alpha}\psi^\alpha_\nu]. \tag{21}$$

The Helmholtz Lagrangian in ECF takes now the form

$$L_H = \sqrt{-g}\left[R(g) + K(g,\psi) + \frac{1}{12a}\left(-\frac{B}{A} + 6t - 12A\right)\right] \tag{22}$$

with $B \equiv t^2 - \psi^{\mu\nu}\psi_{\mu\nu}$. The potential contains a term linear in $\psi_{\mu\nu}$.

The tensor $\psi_{\mu\nu}$ should satisfy a number of constraints to represent a spin-2 field on the spacetime $(M, g_{\mu\nu})$. When viewed as a metric field it satisfies Bianchi identity $\bar{\nabla}_\nu \bar{G}^\nu_\mu(\psi) \equiv 0$ which gives rise to 4 differential constraints $\gamma^{\alpha\beta}(r_{\mu\alpha;\beta} - \frac{1}{2}r_{\alpha\beta;\mu} - Q^\lambda_{\alpha\beta} r_{\mu\lambda}) = 0$. Inserting here $r_{\mu\nu}$ from (21) one finds after some manipulations that

$$\frac{1}{4}t\gamma^{\alpha\beta}\psi_{\alpha\beta;\mu} - \psi_{\mu\alpha}{}^{;\alpha} = 0\,. \tag{23}$$

Denoting by

$$P^{\mu\nu}(g,\psi) \equiv \frac{\partial}{\partial\psi_{\mu\nu}}\left(\frac{-2}{\sqrt{-g}}V\right)$$

the potential part of Lagrange equations one may write these equations as $E^{\mu\nu} = L^{\mu\nu}(Q) - P^{\mu\nu} = 0$, $L^{\mu\nu}(Q)$ being the universal kinetic part. Using the explicit (and very intricate) form of $L^{\mu\nu}$ one can show that $L^{\mu\nu}\psi_{\mu\nu} = 0$ when (23) holds. This in turn implies $P^{\mu\nu}\psi_{\mu\nu} = 0$ and for the potential as in (22) the latter reads $P^{\mu\nu}\psi_{\mu\nu} = A\gamma^{\mu\nu}r_{\mu\nu} = 0$. In other terms one recovers in ECF that $\bar{R} = 0$ and from (21) one finds that it is equivalent to $4A = t$. Using then a differential identity for the scalar A one can simplify the constraints (23). The final outcome is that $\psi_{\mu\nu}$ is subject to 5 constraints: $4A = t$ and $t_{;\mu} = 2\psi_{\mu\alpha}{}^{;\alpha}$. With their aid one gets

$$r_{\mu\nu} = \frac{1}{6at}(t\psi_{\mu\nu} - 4\psi_{\mu\alpha}\psi_\nu{}^\alpha) \text{ and } P^{\mu\nu} = \frac{1}{6a}\left[\left(\frac{B}{t} - \frac{3t}{4}\right)\gamma^{\mu\nu} + \frac{4}{t}\psi^{\mu\nu} - g^{\mu\nu}\right]. \tag{24}$$

Finally these relations allow to simplify the Lagrange equations of motion. After a lengthy computation one arrives at

$$\begin{aligned}
&E^{\mu\nu} = L^{\mu\nu} - P^{\mu\nu} = \gamma^{\alpha(\mu}\nabla^{\nu)}\nabla_\alpha \ln t \\
&+\gamma^{\alpha\beta}[-\tfrac{1}{2}g^{\mu\nu}\nabla_\alpha\nabla_\beta \ln t + \nabla_\alpha J^{\mu\nu}{}_\beta - \gamma^{\lambda\sigma}J^{\mu\nu}{}_\lambda\psi_{\alpha\sigma;\beta} + (\tfrac{1}{2}g^{\mu\nu}\gamma^{\lambda\sigma}\psi_{\alpha\lambda;\sigma} \\
&-\gamma^{\lambda(\mu}\psi_{\alpha\lambda}{}^{;\nu)})\nabla_\beta \ln t] + \gamma^{\alpha(\mu}\gamma^{\nu)\beta}[-\tfrac{1}{2}\Box\psi_{\alpha\beta} + \gamma^{\lambda\rho}(\tfrac{1}{2}\psi_{\alpha\lambda}{}^{;\sigma}\psi_{\beta\rho;\sigma} \\
&+2\psi^\sigma{}_{[\alpha;\lambda]}\psi_{\sigma[\beta;\rho]})] - P^{\mu\nu} = 0\,,
\end{aligned} \tag{25}$$

where $J^{\mu\nu}{}_\alpha \equiv 2\gamma^{\lambda(\mu}\psi^{\nu)}{}_{[\lambda;\alpha]}$. There are only 9 algebraically independent equations since $L^{\mu\nu}\psi_{\mu\nu} = 0 = P^{\mu\nu}\psi_{\mu\nu}$ identically. Complexity of the equations, though considerable, is not hopeless. One can replace $\psi_{\mu\nu}$ by $\phi_{\mu\nu} \equiv \psi_{\mu\nu} - \frac{1}{2}tg_{\mu\nu}$. It follows from the constraints that $\phi_{\mu\nu}$ is divergenceless, $\phi_{\mu\nu}{}^{;\nu} = 0$, it is *not*, however, traceless since $g^{\mu\nu}\psi_{\mu\nu} = -t$. For this field the usual condition of vanishing trace is replaced by the constraint $4A = t$ and the field has 5 degrees of freedom.

4. Open problems

1. Explicit proof of consistency. The fourth-order Lagrange equations for any NLG theory are consistent in the sense that their divergence vanishes identically (as being a Bianchi identity following from the coordinate invariance of the Lagrangian L). Consequently Lagrange equations (25) arising from the Legendre transformation are consistent too. Can their consistency be directly proven without invoking the original theory?

2. Stability of the ground state solution. The candidate ground states for the theory are Minkowski, de Sitter and anti-de Sitter spaces [13]. Rather little is known which of them are true vacuum states, *i.e.* which solutions are stable.

3. To find simple "realistic" solutions for $\psi_{\mu\nu}$ and $g_{\mu\nu}$ and study their physical content. In particular to study their energy density.

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