Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 4

WORLD SPINORS REVISITED*

DJ. ŠIJAČKI

Institute of Physics
P.O. Box 57, 11001 Belgrade, Yugoslavia

(Received February 11, 1998)

Dedicated to Andrzej Trautman in honour of his 64th birthday

World spinors are objects that transform w.r.t. double covering group $\overline{\mathrm{Diff}}(4,R)$ of the Group of General Coordinate Transformations. The basic mathematical results and the corresponding physical interpretation concerning these, infinite-dimensional, spinorial representations are reviewed. The role of groups Diff$(4,R)$, GA$(4,R)$, GL$(4,R)$, SL$(4,R)$, SO(3,1) and the corresponding covering groups is pointed out. New results on the infinite dimensionality of spinorial representations, explicit construction of the $\overline{\mathrm{SL}}(4,R)$ representations in the basis of finite-dimensional non-unitary SL$(2,C)$ representations, SL$(4,R)$ representation regrouping of tensorial and spinorial fields of an arbitrary spin lagrangian field theory, as well as its SL$(5,R)$ generalization in the case of infinite-component world spinor and tensor field theories are presented.

PACS numbers: 04.20. Gz, 03.65. Pm

1. Introduction

The basic wisdom of the standard approach to General Relativity is to start with the group of "General Coordinate Transformations" (GCT), *i.e.* the group of diffeomorphisms Diff$(4,R)$ of R^4. The theory is set upon the principle of general covariance. The GCT group has finite-dimensional tensorial representations only, and these representations characterize allowed world fields. A unified holonomic description of both tensors and spinors would require the existence of respectively tensorial and (double valued) spinorial representations of the GCT group. In other words one is interested in the corresponding single-valued representations of the double covering $\overline{\mathrm{GCT}}$ of the GCT group, since the topology of GCT is given by the

* Presented at the Workshop on Gauge Theories of Gravitation, Jadwisin, Poland, September 4–10, 1997.

topology of its linear compact subgroup. It is well known that the finite-dimensional representations of $\overline{\mathrm{GCT}}$ are characterized by the corresponding ones of the $\overline{\mathrm{SL}}(4,R) \subset \overline{\mathrm{GL}}(4,R)$ group, and $\overline{\mathrm{SL}}(4,R)$ does not have *finite* spinorial representations. However, *there are infinite-dimensional* $\overline{\mathrm{SL}}(4,R)$ *spinorial representations* that define the true "world" (holonomic) spinors [1].

There are two basic ways to introduce finite spinors in a generic curved space-time: *i*) One can make use of the nonlinear representations of the $\overline{\mathrm{GCT}}$ group, which are linear when restricted to the Poincaré subgroup [2] with metric as a nonlinear realizer field. *ii*) One can introduce a bundle of cotangent frames, *i.e.* a set of 1-forms e^a (tetrads; $a = 0,\dots,3$ the anholonomic indices) and define in this space an action of a physically distinct local Lorentz group. Owing to this Lorentz group one can introduce finite spinors, which behave as scalars w.r.t. $\overline{\mathrm{GCT}}$. The bundle of cotangent frames represents an additional geometrical construction corresponding to the physical constraints of a local gauge group of the Yang–Mills type, in which the gauge group is the isotropy group of the space-time base manifold.

In order to set up a framework for a unified description of both tensors and spinors one is now naturally led to enlarge the local Lorentz group to the whole linear group $\overline{\mathrm{GL}}(4,R)$, and together with translations one obtains the affine group $\overline{\mathrm{GA}}(4,R)$. The affine group translates and deforms the tetrads of the locally Minkowskian space-time [3], and provides one with either infinite-dimensional linear or finite-dimensional nonlinear spinorial representations [4].

The existence and structure of spinors in a generic curved space have been the subject of more confusion than most issues in mathematical physics. The physics literature contains two common errors:

(i) For fifty years, it was wrongly believed that the double-covering of $\mathrm{GL}(n,R)$, $n \geq 3$, which we shall denote $\overline{\mathrm{GL}}(n,R)$ does not exist. Almost every textbook in general relativity theory, upon reaching the subject of spinors, contains a sentence such as "... there are no representations of $\mathrm{GL}(4,R)$, or even "representations up to a sign", which behave like spinors under the Lorentz subgroup". Ne'eman played a pioneer role in clarifying the issue of the double covering $\overline{\mathrm{SL}}(n,R) \subset \overline{\mathrm{GL}}(n,R)$ existence [5], and together with Hehl [6] envisaged a gauge theory of gravity with infinite-component spinorial matter fields. Though the correct answer has been known (and strengthened [7]) for the last twenty years, the same type of erroneous statement continues to appear in more recent texts. The complete list of the (infinite-dimensional) $\overline{\mathrm{SL}}(3,R)$ and $\overline{\mathrm{SL}}(4,R)$ unitary irreducible representations is known [8, 9], a formulation of (super-symmetric) spinning extended objects in a generic curved space is developed [10], as well as Gauge Affine and Metric Affine Gauge Theories of Gravity with tensor and spinor $\overline{\mathrm{GL}}(4,R)$ matter fields have been developed considerably [11, 12].

(*ii*) An additional reason for the overall confusion concerns the unitarity of the relevant spinor representations. In dealing with non-compact groups, it is customary to select infinite-dimensional unitary representations to describe the particle-states. However, in the standard (point-object) field theory of tensors or spinors the finite, non-unitary representations of $\mathrm{GL}(4,R)$ and $\mathrm{SL}(2,C)$ are used respectively. The correct answer for spinorial $\overline{\mathrm{GL}}(4,R)$ fields consists in using the infinite unitary representations in a physical base in which they become non-unitary for the $\mathrm{SL}(2,C)$ subgroup [13]. In this way one describes the experimental facts that elementary particles (say proton) when boosted do not turn into another particles (hadronic states) of the same infinite-component spinorial field. Field equations have been constructed for such infinite-component fields, "manifields", within Riemannian gravitational theory [1,14]. $\overline{\mathrm{SL}}(4,R)$ manifields have also been used in classifying the hadron spectrum [15].

2. World spinors existence

Let $g_0 = k_0 + a_0 + n_0$ be an Iwasawa decomposition of a semi-simple Lie algebra g_0 over R. Let G be any connected Lie group with Lie algebra g_0, and let K (compact), A (Abelian) and N (nilpotent) be the analytic subgroups of G with Lie algebras k_0,a_0 and n_0 respectively. The mapping $(k,a,n) \rightarrow kan \quad (k \in \mathrm{K}, a \in \mathrm{A}, n \in \mathrm{N})$ is an analytic diffeomorphism of the product manifold $\mathrm{K}\times \mathrm{A}\times \mathrm{N}$ onto G. The groups A and N are simply connected. Only K is not guaranteed to be simply-connected. There exists a universal covering group $\overline{\mathrm{K}}$ of K, and thus also a universal covering $\overline{\mathrm{G}} \simeq \overline{\mathrm{K}} \times \mathrm{A}\times \mathrm{N}$ of G. For the group of diffeomorphisms one has the following decomposition

$$\mathrm{Diff}(n,R) = \mathrm{GL}(n,R) \times \mathrm{H} \times R^n \,,$$

where the subgroup H is contractible to a point. As $\mathrm{O}(n)$ is the compact subgroup of $\mathrm{GL}(n,R)$, one finds that $\mathrm{O}(n)$ is a deformation retract of $\mathrm{Diff}(n,R)$. Thus, there exists a universal covering of the Diffeomorphism group

$$\overline{\mathrm{Diff}}(n,R) \simeq \overline{\mathrm{GL}}(n,R) \times \mathrm{H} \times R^n.$$

Summing up, we note that for $n \geq 3$ both $\mathrm{SL}(n,R)$ and on the other hand $\mathrm{GL}(n,R)$ and $\mathrm{Diff}(n,R)$ will all have double coverings, defined by $\overline{\mathrm{SO}}(n) \simeq \mathrm{Spin}(n)$ and $\overline{\mathrm{O}}(n) \simeq Pin(n)$ respectively, the double-coverings of the $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ maximal compact subgroups.

We have proven previously [7] that $\overline{\mathrm{SL}}(4,R)$ cannot be embedded into either $\mathrm{SL}(4,C)$ or any other classical semi-simple Lie group. Here we demonstrate on the simplest $\overline{\mathrm{SL}}(3,R)$ example how infinite matrices appear. Let J_i ($i = 1,2,3$; angular momentum) and T_k ($k = 1,\ldots,5$; shear) be the

$\overline{\mathrm{SL}}(3,R)$ generators. For the simplest (multiplicity free) representations, one obtains in the spherical basis the following reduced matrix elements of the non-compact (shear) generators [8]

$$\langle j-2||T||j\rangle = -i(-)^{2j}(\sigma_1 + i\sigma_2 + 2j - 1)\sqrt{\frac{j(j-1)}{2j-1}}\,,$$

$$\langle j||T||j\rangle = +i(-)^{2j}(\sigma_1 + i\sigma_2)\sqrt{\frac{2j(j+1)(2j+1)}{3(2j+3)(2j-1)}}\,,$$

$$\langle j+2||T||j\rangle = -i(\sigma_1 + i\sigma_2 - 2j - 3)\sqrt{\frac{(j+1)(j+2)}{2j+3}}\,,$$

where $\sigma_1, \sigma_2 \in R$. One can have angular momentum $j = 1/2$ provided $\sigma_1 = \sigma_2 = 0$ ($\langle j_{\min} - 2||T||j_{\min}\rangle = 0$, $j_{\min} = 1/2$), however in this case one obtains all $j = 5/2, 9/2, \ldots$ as well, and an infinite non-compact matrix for the shear generators.

3. General affine particles and world spinor fields

The finite-dimensional world tensor field components are characterized by the non-unitary representations of the homogeneous group $\mathrm{GL}(4,R) \subset \mathrm{Diff}(4,R)$. In the flat-space limit they split up into non-unitary $\mathrm{SL}(2,C)$ irreducible pieces. The particle states are defined in the tangent flat-space only. They are characterized by the unitary irreducible representations of the (inhomogeneous) Poincaré group $\mathrm{P}(4) = \mathrm{T}_4 \wedge \mathrm{SL}(2,C)$, and they are enumerated by the "little" group unitary representations (*e.g.* $\mathrm{T}_3\otimes$ $\mathrm{SU}(2)$ for $m \neq 0$). In the generalization to world spinors, the $\mathrm{SL}(2,C)$ group is enlarged to the $\overline{\mathrm{SL}}(4,R) \subset \overline{\mathrm{GL}}(4,R)$ group, while $\mathrm{GA}(4,R) = \mathrm{T}_4 \wedge \overline{\mathrm{GL}}(4,R)$ is to replace the Poincaré group. Affine "particles" are characterized by the unitary irreducible representations of the $\overline{\mathrm{GA}}(4,R)$ group, whose unitarity is provided by the unitarity of the relevant "little" group (*e.g.* $T_3 \otimes \overline{\mathrm{SL}}(3,R) \supset \mathrm{T}_3 \otimes \mathrm{SU}(2)$). A mutual particle–field matching is achieved by requiring the subgroup of the homogeneous group, that is isomorphic to the homogeneous part of the "little" group (say, $\mathrm{SU}(2)$ of $\mathrm{SL}(2,C)$), to be represented unitarily. Furthermore, one has to project away all representations of this group except a single one that is realized for the particle states (say $D^{(j)}$ of $\mathrm{SU}(2) \subset T_3\otimes$ $\mathrm{SU}(2)$).

A physically correct picture, in the affine case, is obtained by making use of the $\overline{\mathrm{SA}}(4,R) \subset \overline{\mathrm{GA}}(4,R)$ group unitary irreducible representations for "affine" particles, with particular states characterized by the $T_3 \otimes \overline{\mathrm{SL}}(3,R)$ "little" group representations. The corresponding affine fields are described by the non-unitary infinite-dimensional $\overline{\mathrm{SL}}(4,R) \subset \overline{\mathrm{GL}}(4,R)$ representations, that are unitary when restricted to the homogeneous "little" subgroup $\overline{\mathrm{SL}}(3,R)$. Therefore, the first step towards world spinor fields is a construction of infinite-dimensional non-unitary $\overline{\mathrm{SL}}(4,R)$ representations, that are unitary when restricted to $\overline{\mathrm{SL}}(3,R)$. These fields reduce to an infinite sum of (non-unitary) finite-dimensional $\mathrm{SL}(2,C)$ fields.

The world spinor fields transform w.r.t. $\overline{\mathrm{Diff}}(4,R)$ as follows

$$(D(a,\bar{f})\Phi_A)(x) = (D_{\{\overline{\mathrm{Diff}_0}\}})_A^B(\bar{f})\Phi_B(f^{-1}(x-a)), \quad (a,\bar{f}) \in T_4 \wedge \overline{\mathrm{Diff}_0},$$

where $\overline{\mathrm{Diff}_0}$ is the homogeneous part of $\overline{\mathrm{Diff}}$, and $D_{\{\overline{\mathrm{Diff}_0}\}} = \sum^{\oplus} D_{\{\overline{\mathrm{SL}}\}}$. The affine "particle" states transform according to the following representation

$$D((a,\bar{s})) \to e^{i(sp)\cdot a} D_{\{\overline{\mathrm{SL}}\}}(L^{-1}(sp)\bar{s}L(p)), \quad (a,\bar{s}) \in T_4 \wedge \overline{\mathrm{SL}}(4,R),$$

and $L \in \overline{\mathrm{SL}}(4,R)/\overline{\mathrm{SL}}(3,R)$ The unitarity properties of various representations in these expressions is as described above.

4. Spinorial $\mathbf{SL(2,\boldsymbol{C}) \subset \overline{SL}(4,\boldsymbol{R})}$ representations

In order to analyze the representations, as well as to make use of them in a gauge theory, it is convenient to have the matrix elements of the group generators. Also, in that case the task of determining the scalar products of the irreducible representations is considerably simplified. Let $M_{\mu\nu}$ and $T_{\mu\nu}$ be the $\overline{\mathrm{SL}}(4,R)$ generators, with $M_{\mu\nu}$ generating the Lorentz subgroup $\mathrm{SL}(2,C) \simeq \overline{\mathrm{SO}}(3,1)$. In the $3+1$ notation one has $M_{\mu\nu} \to J_i = \epsilon_{ijk}M_{jk}$ (angular momentum), $K_i = M_{0i}$ (boost), and $T_{\mu\nu} \to T_{ij}$ (3-shear), $N_i = T_{0i} = T_{i0}$, and T_{00}. At this point it is convenient (as in the Lorentz covariant field theory) to introduce $J_i^{(1)}$, and $J_i^{(2)}$ that generate an $\mathrm{SU}(2)\otimes\ \mathrm{SU}(2)$ group with the corresponding representation labels (j_1,j_2), $j_1,\ j_2 = 0,\ 1/2,\ 1\ ....$

The angular momentum and boost generators are given by $J_i = J_i^{(1)} + J_i^{(2)}$, and $K_i = i(J_i^{(2)} - J_i^{(1)})$. The remaining $\overline{\mathrm{SL}}(4,R)$ generators transform as a $(1,1)$ $\mathrm{SU}(2)\otimes\ \mathrm{SU}(2)$ irreducible tensor operator $Z_{pq},\ p,q = 0,\pm 1$. The most general $\overline{\mathrm{SL}}(4,R)$ representations are obtained in the $\left|\begin{array}{cc} j_1 & j_2 \\ k_1\ m_1 & k_2\ m_2 \end{array}\right\rangle$ basis of the $\mathrm{SU}(2)\otimes\ \mathrm{SU}(2)$ representations. In the reduction to the $\mathrm{SL}(2,C)$ subgroup they contain an infinite direct sum of corresponding irreducible representations $D^{(j_1,j_2)}_{\mathrm{SL}(2,C)}$.

The matrix elements of the Lorentz group generators are well known, and we list only the matrix elements of the Z_{pq} generators obtained by the appropriate generator redefinition as compared to $\overline{\mathrm{SL}}(4,R)/\overline{\mathrm{SO}}(4)$ representations [9].

$$\left\langle \begin{array}{cc} j_1' & j_2' \\ k_1'\ m_1' & k_2'\ m_2' \end{array} \right| Z_{pq} \left| \begin{array}{cc} j_1 & j_2 \\ k_1\ m_1 & k_2\ m_2 \end{array} \right\rangle = (-)^{j_1'-m_1'}(-)^{j_2'-m_2'}$$

$$\times \begin{pmatrix} j_1' & 1 & j_1 \\ -m_1' & p & m_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -m_2' & q & m_2 \end{pmatrix} \left\langle \begin{array}{cc} j_1' & j_2' \\ k_1' & k_2' \end{array} \right\| Z \left\| \begin{array}{cc} j_1 & j_2 \\ k_1 & k_2 \end{array} \right\rangle ,$$

where

$$\begin{aligned}
&\left\langle \begin{array}{cc} j_1' & j_2' \\ k_1' & k_2' \end{array} \right\| Z \left\| \begin{array}{cc} j_1 & j_2 \\ k_1 & k_2 \end{array} \right\rangle \\
&= (-)^{j_1'-k_1'}(-)^{j_2'-k_2'}\frac{i}{2}\sqrt{(2j_1'+1)(2j_2'+1)(2j_1+1)(2j_2+1)} \\
&\times \Big\{ [e+4-j_1'(j_1'+1)+j_1(j_1+1)-j_2'(j_2'+1)+j_2(j_2+1)] \\
&\times \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 0 & k_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 0 & k_2 \end{pmatrix} \\
&-(c+k_1-k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 1 & k_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & -1 & k_2 \end{pmatrix} \\
&-(c-k_1+k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & -1 & k_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 1 & k_2 \end{pmatrix} \\
&+(d+k_1+k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & 1 & k_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & 1 & k_2 \end{pmatrix} \\
&+(d-k_1-k_2) \begin{pmatrix} j_1' & 1 & j_1 \\ -k_1' & -1 & k_1 \end{pmatrix} \begin{pmatrix} j_2' & 1 & j_2 \\ -k_2' & -1 & k_2 \end{pmatrix} \Big\} .
\end{aligned}$$

The representation labels c, d, e are arbitrary complex numbers.

A class of infinite-dimensional spinorial/tensorial representations of the $\mathrm{GL}(4,R)$ group in the basis of its Lorentz subgroup were recently constructed [16] by extending the $\mathrm{SL}(2,C)$ Naimark representations. These representations are made up of infinite-dimensional $\mathrm{SL}(2,C)$ representations and fail to meet the necessary physical requirements.

5. Minimal field configurations for arbitrary spin lagrangian theory

The task of constructing a lagrangian formulation for relativistic fields of unique mass and arbitrary spin turns out to be rather non-trivial. There is no unique Lorentz covariant field to be associated to a given $[m, J]$ particle Poincaré representation. Moreover, as a rule, lagrangian formulation requires quite a number of additional auxiliary fields.

A minimal Lorentz covariant Fierz–Pauli lagrangian formulation of an massive arbitrary-spin s boson field is achieved in terms of traceless, symmetric tensor fields [17, 18]. Let $\phi^{(s)}$ be a Lorentz covariant field that transforms w.r.t. the $D^{(\frac{s}{2},\frac{s}{2})}$ SL$(2,C)$ irreducible representation (a symmetric traceless tensor of rank s) satisfying the Klein–Gordon equation with mass m, *i.e.* $\left(\Box + m^2\right)\phi^{(s)}_{\nu_1\nu_2\cdots\mu_s}(x) = 0$. Representation $D^{(\frac{s}{2},\frac{s}{2})}$ is reducible under the SO(3) subgroup of spatial rotations, $D^{(\frac{s}{2},\frac{s}{2})} = \sum_{l=0}^{s} D^{(l)}$, and thus one imposes the "Lorentz condition" $\partial^{\mu_1}\phi^{(s)}_{\mu_1\mu_2\cdots\nu_s} = 0$ in order to eliminate all lower spin values. In order to have enough field components to vary, it is necessary to introduce certain auxiliary fields. The simplest viable choice is to introduce, besides the starting field $\phi^{(s)}$, the following set of auxiliary fields: $\phi^{(s-2)}, \phi^{(s-3)}, \cdots, \phi^{(0)}$. The total field is $\Phi^{(s)} = \{\phi^{(s)}, \phi^{(s-2)}, \phi^{(s-3)}, \cdots \phi^{(0)}\}$, and consists of $(s+1)^2 + \frac{1}{6}s(s+1)(2s-1)$ field components.

The fields of a generic curved-space theory formulation transform w.r.t. linear SL$(4,R) \subset$ GL$(4,R)$ representations, that provide a space for nonlinear realization of the complete Diff$(4,R)$ transformation group. Therefore, the first step in formulating a generic curved-space lagrangian field theory for arbitrary spin is to embed the space of all fields of the above minimal Lorentz formulation into an appropriate SL$(4,R)$ representation space. An analysis of SL$(2,C)$ and SL$(4,R)$ representations shows that the basic field $\phi^{(s)}$ as well as all accompanying auxiliary fields can be reorganized to fit into two SL$(4,R)$ irreducible representations when $s \geq 3$, while for $s = 0, 1, 2$ a single SL$(4,R)$ representation suffice. In the Young tableau notation of SL$(4,R)$ irreducible representations, we find

$$\Phi^{(s)} \quad \sim \quad \underbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}}_{s} \quad \bigoplus \quad \underbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}}_{s-3}$$

when $s \geq 3$, while $\Phi^{(0)} \sim \bullet$ (scalar representation; also the second representation when $s = 3$), $\Phi^{(1)} \sim \Box$, and $\Phi^{(2)} \sim \Box\Box$.

A minimal Lorentz covariant Fierz–Pauli lagrangian formulation of an massive arbitrary-spin $j = \frac{1}{2} + s$ fermion field is achieved in terms of Rarita–Schwinger spinor-tensor fields [17, 18]. Let $\psi^{(s)}$ be a symmetric traceless spinor-tensor field that transforms w.r.t. $D^{(\frac{1}{2}(s+1),\frac{1}{2}s)} \oplus D^{(\frac{1}{2}s,\frac{1}{2}(s+1))}$ representation of the $\mathrm{SL}(2,C)$ group and satisfies $(i\gamma\cdot\partial - m)\psi^{(s)}(x) = 0$, and the spinor trace condition $\gamma^{\mu_1}\psi^{(s)}_{\mu_1\mu_2\cdots\mu_s}(x) = 0$. The lagrangian formulation is achieved for a field $\Psi^{(s)} = \{\psi^{(s)},\ \psi^{(s-1)},\ 2\times\psi^{(s-2)},\ 2\times\psi^{(s-3)},\cdots 2\times\psi^{(0)}\}$, transforming w.r.t. $D^{(\frac{1}{2}(s+1),\frac{1}{2}s)} \oplus D^{(\frac{1}{2}s,\frac{1}{2}(s+1))} \oplus D^{(\frac{1}{2}s,\frac{1}{2}(s-1))} \oplus D^{(\frac{1}{2}(s-1),\frac{1}{2}s)}$ $\oplus\ 2\ \sum_{l=0}^{s-2}\,[D^{(\frac{1}{2}(l+1),\frac{1}{2}l)} \oplus D^{(\frac{1}{2}l,\frac{1}{2}(l+1))}]$. representation of the Lorentz group, and consists of the starting field $\psi^{(s)}$ and the necessary auxiliary fields.

In this case, we find that the tensor part of the spinor-tensor field can, again, be described by two $\mathrm{SL}(4,R)$ irreducible representations. In the Young tableau notation we write

$$\Psi^{(s)} \ \sim\ \left[\ D^{(\frac{1}{2},0)} \bigoplus D^{(0,\frac{1}{2})}\ \right] \bigotimes \left[\ \underbrace{\square\square\cdots\square}_{s} \bigoplus \underbrace{\square\square\cdots\square}_{s-2}\ \right]$$

when $s \geq 2$, while $\Psi^{(0)} \sim [D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}] \otimes \bullet$, and $\Psi^{(1)} \sim [D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}]$ $\otimes\square$. Here, the spinor and tensor parts transforms w.r.t. $\mathrm{SL}(2,C)$ and $\mathrm{SL}(4,R)$ representations respectively.

6. World spinor field choice and $\overline{\mathbf{SL}}(5,R)$

Let us consider world tensor and spinor fields that transform according to infinite-dimensional $\overline{\mathrm{SL}}(4,R)$ representations that consists of finite-dimensional, non-unitary $\mathrm{SL}(2,C)$ subgroup representations. Owing to the fact that each infinite-dimensional $\overline{\mathrm{SL}}(4,R)$ representation contains an infinite set of Lorentz representations, *i.e.* an infinite set of tensors or spinors, one has a structure that should contain $\sum_{s=0}^{\infty}\Phi^{(s)}$ or $\sum_{s=0}^{\infty}\Psi^{(s)}$ fields at least.

The simplest tensorial case is based on the multiplicity-free (ladder) $\mathrm{SL}(4,R)$ representations [13] $D^{\mathrm{ladd}}_{\mathrm{SL}(4,R)}(0,0)$ and $D^{\mathrm{ladd}}_{\mathrm{SL}(4,R)}(\frac{1}{2},\frac{1}{2})$ that contain each $D^{(\frac{s}{2},\frac{s}{2})}$, $s = 0,1,\ldots$, $\mathrm{SL}(2,C)$ representation once. However, each Lorentz covariant field $\phi^{(s)}$ is accompanied by the appropriate auxiliary fields: $\phi^{(s)} \to \Phi^{(s)}$, resulting in an infinite repetition of the starting $\mathrm{SL}(4,R)$ representations. One obtains a structure resembling that of the leading and daughter Regge trajectories of hadrons. We find that *all these field components* (physical and auxiliary) *can be obtained from a single infinite-dimensional SL(5, R) multiplicity-free representation.*

$$\tilde{\Phi}\ \sim\ \ D^{(\mathrm{ladd})}_{\mathrm{SL(5,R)}}\ \supset\ \sum\nolimits^{\oplus}\left[D^{(\mathrm{ladd})}_{\mathrm{SL(4,R)}}(0,0) \bigoplus D^{(\mathrm{ladd})}_{\mathrm{SL(4,R)}}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\right].$$

We find, in the spinor field case, an analogous result. For a spinor-tensor field $\tilde{\Psi}$ that transforms as a Dirac field w.r.t. the Lorentz group, and as a tensor w.r.t. $\mathrm{SL}(4,R)$, we have $\tilde{\Psi} \sim \left[D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}\right] \otimes D^{(\mathrm{ladd})}_{\mathrm{SL(5,R)}} \supset$

$\sum^{\oplus}\left[D^{(\frac{1}{2},0)}\oplus D^{(0,\frac{1}{2})}\right]\otimes\left[D^{\rm (ladd)}_{\rm SL(4,R)}(0,0)\oplus D^{\rm (ladd)}_{\rm SL(4,R)}(\frac{1}{2},\frac{1}{2})\right]$. Finally, in order to obtain a genuine world spinor field transforming according to the $\overline{\rm SL}(4,R)\subset\overline{\rm Diff}(4,R)$ group spinorial representation, we make the appropriate changes and find

$$\tilde{\Psi}\;\sim\;D^{\rm (ladd)}_{\overline{\rm SL}(5,R)}\;\supset\;\sum\nolimits^{\oplus}\left[D^{\rm (ladd)}_{\overline{\rm SL}(4,R)}(\tfrac{1}{2},0)\bigoplus D^{\rm (ladd)}_{\overline{\rm SL}(4,R)}(0,\tfrac{1}{2})\right]\;.$$

REFERENCES

[1] Y. Ne'eman, Dj. Šijački, *Phys. Lett.* **B157**, 275 (1985).

[2] V.I. Ogievetsky, I.V. Polubarinov, *Sov. Phys. JETP* **21**, 1093 (1965).

[3] F.W. Hehl, G.D. Kerlick, P. von der Heyde, *Phys. Lett.* **B63**, 446 (1976).

[4] Y. Ne'eman, Dj. Šijački, *Ann. Phys. (N.Y.)* **120**, 292 (1979).

[5] Y. Ne'eman, *Proc. Natl. Acad. Sci. USA* **74**, 4157 (1977); *Ann. Inst. Henri Poincaré Phys. Theor.* **A28**, 369 (1978).

[6] F.W. Hehl, E.A. Lord, Y. Ne'eman, *Phys. Rev.* **D17**, 428 (1978).

[7] Y. Ne'eman, Dj. Šijački, *Int. J. Mod. Phys.* **A2**, 1655 (1987).

[8] Dj. Šijački, *J. Math. Phys.* **16**, 298 (1975).

[9] Dj. Šijački, $\overline{\rm SL}(n,R)$ *Spinors for Particles, Gravity and Superstrings*, in *Spinors in Physics and Geometry*, eds. A. Trautman and G. Furlan, World Scientific Pub., 1988.

[10] Y. Ne'eman, Dj. Šijački, *Phys. Lett.* **B174**, 165 (1986); *Phys. Lett.* **B174**, 171 (1986); *Phys. Lett.* **B206**, 458 (1988).

[11] Dj. Šijački, *Phys. Lett.* **B109**, 435 (1982); Y. Ne'eman, Dj. Šijački, *Phys. Lett.* **B200**, 489 (1988).

[12] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne'eman, *Phys. Rep.* **258**, 1 (1995).

[13] Dj. Šijački, Y. Ne'eman, *J. Math. Phys.* **26**, 2475 (1985).

[14] A. Cant, Y. Ne'eman, *J. Math. Phys.* **26**, 3180 (1985).

[15] Y. Ne'eman, Dj. Šijački, *Phys. Rev.* **D37**, 3267 (1988); *Phys. Rev.* **D47**, 4133 (1993).

[16] V.I. Marchenko, *JETP Lett.* **64**, 334 (1996).

[17] L.P.S. Singh, C.R. Hagen, *Phys. Rev.* **D9**, 898 (1974); *Phys. Rev.* **D9**, 910 (1974).

[18] C. Fronsdal, *Phys. Rev.* **D18**, 3624 (1978); J. Fang, C. Fronsdal, *Phys. Rev.* **D18**, 3630 (1978).