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ON THE UNCERTAINTY RELATIONS OF κ-DEFORMED QUANTUM PHASE SPACE*

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The uncertainty relations associated to the covariant κ-deformation of $D = 4$ relativistic symmetries, with quantum "time" coordinate and modified Heisenberg algebra, are shown to be consistent with independent heuristic estimates of limitations on the measurability of space-time distances. Our analysis generalizes the one previously reported by one of us, which considered only the space-time coordinate sector.

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1. Introduction

Several authors (see, *e.g.*, [1-11]) have argued that the classical ideas about space-time structure might fail to describe physics below some minimal length. A frequently investigated minimal-length scenario is the one

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motivated by critical string theories; there the appearance of a minimal distance follows from the analysis of string collisions at Planckian energies, which are found to be characterized by the following modified uncertainty relation (see [2])

$$\Delta x \geq \frac{\hbar}{\Delta p} + \alpha G \Delta p\,, \tag{1}$$

where $G = c^2 l_p^2/\hbar$ is the gravitational coupling (Newton) constant, c and l_p are the speed-of-light and Planck-length constants respectively, and α is a constant related to the string tension (Regge slope). The relation (1) implies a lower bound on the measurability of distances

$$\min \Delta x \sim \sqrt{\hbar \alpha G}\,, \tag{2}$$

which fits well the expectation of certain heuristic studies [3,7] of measurability in quantum gravity[1]. It is also quite well understood that modifications of the ordinary Heisenberg uncertainty relations structure [4–7]; in particular, in Ref. [6] it was shown that the relations (1)-(2) can come from $\mathrm{SU}_q(n)$ covariance.

A more stringent alternative to the bound (2) was recently proposed [8] by one of the present authors[2], being motivated by an heuristic quantum-gravity analysis of the measurability of distances in which in particular it was found that gravitational effects prevent one from relying on *classical* agents for the measurement procedure. This originates [8] from the fact that in the framework of general relativity the classical (*i.e.* infinite-mass) limit leads to inconsistencies associated with the formation of horizons. By taking into account both the quantum nature of the agents involved in the measurement and the gravitational effects affecting the measurement, one finds [8] that the measurability of distances is bound by a quantity that grows with the time required by the measurement procedure, as needed for the decoherence mechanism discussed in Ref. [9]. Specifically, the following bound is found [8] for the measurability of a distance L:

$$\min\,[\Delta L] \sim l_p \sqrt{\frac{cT}{s}} \sim l_p \sqrt{\frac{L}{s}}\,, \tag{3}$$

where s is a length scale characterizing the spatial extension of the devices (*e.g.*, clocks) used in the measurement, T is the time needed to complete

[1] Here and in the following we qualify as "heuristic quantum-gravity studies" the ones that do not advocate a specific quantum-gravity model, but rather advocate a combination of ordinary quantum mechanics and general-relativity arguments.

[2] We also bring to the attention of the reader the Refs [9], in which related issues have been discussed, although the structures identified in those studies are *gravitational corrections* [8], rather than measurability bounds.

the procedure of measuring L, and on the right-hand-side we used the fact that, assuming the measurement procedure uses massless probes, one has typically $T \sim L$. Notice that for all acceptable values [8] of s, $l_p \geq s \geq L$, the bound (3) is more stringent than (2); this is a direct consequence of the fact that the analyses leading to (2) had implicitly relied on the availability of ideal classical agents in the measurement procedure.

While, as mentioned above, critical string theories provide a framework for the bound (2), it appears that noncritical string theories provide a framework for the bound (3). In particular, in the framework of "Liouville" noncritical string theories, with the target time identified with the Liouville mode [12], the nature of the dynamics of the light probes exchanged in a typical procedure of measurement of a distance was shown [10] to lead to a measurability bound of type q̊gboundgac.

An interesting problem is the one of finding a quantum-group (and quantum-Lie-algebra) framework for (3), just like Ref. [6] has provided a quantum-group framework for (2). The notion of quantum group as a Hopf algebra permits to consider deformed symmetries; in fact, the Hopf algebra axioms provide simultaneously an algebraic generalization of the definition of Lie group as well as of Lie algebra. As exemplified by the formulae in the following section, the phase space containing the coordinate and momentum sectors can be described in the quantum-deformed case as a semidirect product of two dual Hopf algebras describing the coordinates and momentum sectors. Such a definition of quantum phase space has been first proposed by Majid [4], and it is endoved with the property that in the undeformed case (coordinates and momenta described by Abelian Hopf algebra with primitive coproducts) one obtains the standard quantum mechanical Heisenberg commutation relations[3]. The so-called κ-deformations [14-18] provide an example of this type of quantum deformations of relativistic symmetries, and one of us recently argued [11] that κ-deformed symmetries might provide an algebraic abstraction of the measurability bound (3). The analysis reported in [11] was somewhat preliminary since only the coordinate sector was considered, but the bound (3) emerged rather compellingly, as a direct consequence of the noncommuting space-time coordinates of κ-deformed Minkowski space [16-18]. Encouraged by the findings of Ref. [11], in this Letter we explore further the relation between κ-Poincaré and (3); specifically, we extend the analysis of Ref. [11] from the confines of the space-time coordinate sector to the full structure of the κ-deformed phase space. We consider the κ-deformed Poincaré symmetries in the bicrossproduct basis [4, 17], which appears to be a very natural framework for the quantum

[3] In the literature sometimes the semidirect product construction for two dual Hopf algebras describing respectively quantum Lie group and quantum Lie algebra is called "Heisenberg double" (see, *e.g.*, Ref. [13]).

deformations of semidirect product algebras, and outside of the coordinate sector we identify two structures which could affect the analysis of Ref. [11]: the κ-deformed mass-shell condition, which is associated to the Casimir and modifies the propagation of the light probes exchanged during measurement, and the nontrivial commutation relation between three-momenta and quantum time coordinate, which we find to affect significantly the analysis of the propagation of heavy probes exchanged during measurement. As discussed below, our analysis uncovers new nonnegligible contributions to the bound on the measurability of distances. These contributions are however comparable to the one identified in Ref. [11], and therefore the order of magnitude of the effect discussed in Ref. [11] is confirmed by our analysis. These findings provide additional evidence of a relation between κ-Poincaré and the bound (3).

2. κ-deformed quantum relativistic phase space

The standard form of the covariant fourdimensional Heisenberg commutation relations, describing quantum-mechanical covariant phase space looks as follows:

$$[x_\mu, p_\nu] = i\hbar g_{\mu\nu}\,, \qquad g_{\mu\nu} = \mathrm{diag}(-1,1,1,1). \tag{4}$$

The space-time coordinates x_μ ($\mu = 0,1,2,3$) can be identified with the translation sector of the Poincaré group, and the fourmomenta p_μ ($\mu = 0,1,2,3$) are given by the translation generators of the Poincaré algebra. In considering quantum deformations of relativistic symmetries as describing the modification of space-time structure one is lead to the study of the possible quantum Poincaré groups[4]. The classification of quantum deformations of $D = 4$ Poincaré groups in the framework of Hopf algebras was given by Podleś and Woronowicz ([21]; see also [22]) and provides the most general class of noncommutative space-time coordinates $\hat{x}_\nu$ allowed by the quantum-group formalism. If we assume that the quantum deformation does not affect the nonrelativistic kinematics, i.e. we preserve the nonrelativistic O(3) rotations classical and O(3) covariance, the only consistent class of noncommuting space-time coordinates is described by the relations of the κ-deformed Minkowski space (see Refs. [16-18]) with commuting classical space coordinates In order to describe the κ-relativistic phase space we

[4] We take into consideration here only the genuine 10-generator quantum deformations of $D = 4$ Poincaré symmetries. In particular, the "standard" q-deformations are not allowed. In such a case the scheme requires adding an eleventh (dilatation) generator, *i.e.* one deals with the dilatation extended Poincaré algebra [19]. In such a case the corresponding quantum phase space is much more complicated (see, *e.g.*, [20]), and the deformation parameter is dimensionless, rendering difficult the physical separation between the ordinary regime of commutative space-time coordinates and the short-distance regime in which non-commutativity sets in.

start with the κ-deformed Hopf subalgebra of fourmomenta $\hat{p}_\mu$ written in bicrossproduct basis [17, 18]

$$
\begin{aligned}
[\hat{p}_0, \hat{p}_k] &= 0 && (5)\\
\Delta(\hat{p}_0) &= \hat{p}_0 \otimes 1 + 1 \otimes \hat{p}_0 \\
\Delta(\hat{p}_k) &= \hat{p}_k \otimes 1 + e^{\frac{\hat{p}_0}{\kappa c}} \otimes \hat{p}_k && (6)
\end{aligned}
$$

with antipode and counit given by

$$S(\hat{p}_k) = -e^{-\hat{p}_0/\kappa c}\,\hat{p}_k \qquad S(\hat{p}_\mu) = -\hat{p}_\mu \qquad \varepsilon(\hat{p}_\mu) = 0\,. \tag{7}$$

Note that both the fundamental constant c (speed of light) and the (mass-like) deformation parameter κ are present in the coproduct $\mathring{2}$.4b.

Using the duality relations involving the second fundamental constant $\hbar$ (Planck's constant)

$$\langle \hat{x}_\mu, \hat{p}_\nu \rangle = -i\hbar g_{\mu\nu} \qquad g_{\mu\nu} = (-1, 1, 1, 1) \tag{8}$$

we obtain the noncommutative κ-deformed configuration space $\mathcal{X}_\kappa$ as a Hopf algebra with the following algebra and coalgebra structure

$$[\hat{x}_0, \hat{x}_k] = \frac{i\hbar}{\kappa c}\hat{x}_k\,, \qquad [\hat{x}_k, \hat{x}_l] = 0 \tag{9}$$

$$
\begin{aligned}
\Delta(\hat{x}_\mu) &= \hat{x}_\mu \otimes 1 + 1 \otimes \hat{x}_\mu\,, && (2.10a)\\
S(\hat{x}_\mu) &= -\hat{x}_\mu \qquad \varepsilon(\hat{x}_\mu) = 0 && (2.10b)
\end{aligned}
$$

The κ-deformed phase space can be considered as the vector space $\mathcal{X}_\kappa \otimes \mathcal{P}_\kappa$ with the product (see [4])

$$(x \otimes p)(\tilde{x} \otimes \tilde{p}) = x(p_{(1)} \triangleright \tilde{x}) \otimes p_{(2)}\tilde{p} \tag{2.11}$$

where left action is given by

$$p \triangleright x = \left\langle p, x_{(2)} \right\rangle x_{(1)} \tag{2.12}$$

The product $\mathring{2}$.7 can be rewritten as the commutators between coordinates and momenta by using the obvious isomorphism $x \sim x \otimes 1$, $p \sim 1 \otimes p$. Application to the case of κ-Poincaré algebra provides the following relations (see also [23, 24])

$$
\begin{aligned}
[\hat{x}_k, \hat{p}_l] &= i\hbar\delta_{kl}\,, & [\hat{x}_k, \hat{p}_0] &= 0\,, \\
[\hat{x}_0, \hat{p}_k] &= -\tfrac{i\hbar}{\kappa c}\hat{p}_k\,, & [\hat{x}_0, \hat{p}_0] &= -i\hbar\,.
\end{aligned} \tag{2.13}
$$

The set of relations $\mathring{2}.4$a, $\mathring{2}.6$a and (2.9) describes the κ-deformed relativistic quantum phase space, which is κ-Poincaré covariant[5].

The modified covariant Heisenberg uncertainty relations follow from the relations $\mathring{2}.7$ and $\mathring{2}.8$. Introducing the dispersion of the observable a in quantum mechanical sense by

$$\Delta(a) \;=\; \sqrt{\langle a^2\rangle - \langle a\rangle^2} \tag{2.14}$$

we have

$$\Delta(a)\Delta(b) \geq \frac{1}{2}|\,\langle c\rangle\,|\,, \qquad \text{where} \qquad c=[a,b] \tag{2.15}$$

We obtain κ-deformed uncertainty relations

$$\Delta\hat{t}\Delta\hat{x}_k \;\geq\; \frac{\hbar}{2\kappa c^2}|\,\langle\hat{x}_k\rangle\,| = \frac{1}{2}\frac{l_\kappa}{c}\,|\,\langle\hat{x}_k\rangle\,|\,, \tag{2.16a}$$

$$\Delta\hat{p}_k\Delta\hat{x}_l \;\geq\; \frac{1}{2}\hbar\delta_{kl}\,, \tag{2.16b}$$

$$\Delta\hat{E}\Delta\hat{t} \;\geq\; \frac{1}{2}\hbar\,, \tag{2.16c}$$

$$\Delta\hat{p}_k\Delta\hat{t} \;\geq\; \frac{\hbar}{2\kappa c^2}|\,\langle\hat{p}_k\rangle\,| = \frac{1}{2}\frac{l_\kappa}{c}\;|\langle\hat{p}_k\rangle|\;. \tag{2.16d}$$

where $l_\kappa = \frac{\hbar}{\kappa c}$ describes the fundamental length at which the time variable should already be considered noncommutative. In the recent estimates $\kappa > 10^{12}GeV$ (see, *e.g.*, Ref. [26]) i.e. $l_\kappa < 10^{-26}$cm; in particular one can put κ equal to the Planck mass which implies that $l_\kappa = l_p \simeq 10^{-33}$cm.

In comparison with the discussion in Ref. [11], which only considered the coordinate sector, the significant new element emerged in our present analysis is the relation $\mathring{2}.11$d. Interestingly, multiplying the three relations $\mathring{2}.11$a, $\mathring{2}.11$b and $\mathring{2}.11$d one obtains

$$(\Delta\hat{t})^2(\Delta\hat{x}_l\Delta\hat{p}_l)^2 \;\geq\; \frac{\hbar}{8}\frac{l_\kappa^2}{c^2}\;|\langle\hat{x}_l\rangle\,\langle\hat{p}_l\rangle|\;, \tag{2.17}$$

(where no sum over the index l is to be understood). This indicates that a wave packet with minimal standard ($\Delta x\Delta p$) uncertainty has the largest uncertainty in the localization of time. (In ordinary quantum mechanics $l_\kappa = 0$ and there is no such correlation.)

It is also interesting to consider the relation $\mathring{2}.11$d under the assumption that the three-momenta $\hat{p}_k$ can be expressed by a general formula $\hat{p}_i =$

[5] The κ-covariance of the relations $\mathring{2}.6$a has been shown firstly in Ref. [17]. The κ-covariance of the whole quantum κ-deformed Heisenberg algebra follows from the general properties of the semidirect product, defined by the relations $\mathring{2}.6$a and $\mathring{2}.9$. (see, *e.g.*, [25])

$\mathcal{M}(v^2)v_i$, in which case $\Delta\hat{p}_i = \mathcal{M}_{ij}\Delta v_k$ with $\mathcal{M}_{ij} = \mathcal{M}[\delta_{ij} + 2v_iv_j(\ln\mathcal{M})']$. Then 2̊.11d implies

$$\Delta\hat{t}\Delta v_i \;\geq\; \frac{l_\kappa}{c}\,\mathcal{M}(v)\,\mathcal{M}^{-1}_{ij}(v)\,v_j\,|\langle\hat{x}_l\rangle\,\langle\hat{p}_l\rangle|\;. \tag{2.18}$$

Because in part of our measurement analysis we shall consider light probes, we now discuss the modification of the kinematics of κ-deformed photons. We shall assume that the generators of the κ-deformed Poincaré algebra in bicrossproduct basis describes the "physical" generators of space-time symmetries. In the bicrossproduct basis the κ-deformed mass Casimir takes the form

$$C_2^\kappa = \frac{1}{c^2}\vec{P}^2 e^{-\frac{P_0}{\kappa c}} - (2\kappa\sinh\frac{P_0}{2\kappa c})^2 = -M^2\,, \tag{2.19}$$

where P_μ are the generators of space-time translations and M denotes the κ-invariant mass parameter. For $M = 0$ (κ-deformed photons) from 2̊.14 one obtains that (we identify $P_\mu \equiv \hat{p}_\mu$)

$$\hat{p}_0 = \kappa c\ln(1 + \frac{|\vec{\hat{p}}|}{\kappa c}) = |\vec{\hat{p}}| - \frac{|\vec{\hat{p}}|^2}{2\kappa c} + O(\frac{1}{\kappa^2}) \tag{2.20}$$

and in particular the velocity formula for massless κ-deformed quanta looks as follows[6] ($E = c\hat{p}_0$)

$$v_i = \frac{\partial E}{\partial\hat{p}_i} = \frac{c}{1 + \frac{|\vec{\hat{p}}|}{\kappa c}}\frac{\hat{p}_i}{|\vec{\hat{p}}|} \tag{2.21a}$$

or

$$v = |\vec{v}| = \frac{c}{1 + \frac{|\vec{\hat{p}}|}{\kappa c}} = c - \frac{|\vec{\hat{p}}|}{\kappa} + O(\frac{1}{\kappa^2}) \tag{2.21b}$$

The inverse formula, which can be inserted in 2̊.new2 looks as follows

$$\hat{p}_i = \kappa\frac{c}{v}(\frac{c}{v} - 1)v_i \tag{2.22}$$

and it is linear in the deformation parameter κ.

This three-momentum-dependent (i.e. energy-dependent) "speed of light" is a completely novel phenomenon that arises in the formalism here considered. Interestingly, it has the same functional form (upon appropriate identification between κ and the string scale) as the energy-dependent speed

[6] The relation (2.21a) is valid as a consequence of the Hamiltonian equation of motion $\dot{x}_i = \partial H/\partial p_i - (x_i/\kappa)\partial H/\partial x_0$. [See Ref. [18], Eq. (4.22).] For the κ-photon here considered, since $H = H(p_i)$, the velocities are classical ($[v_i, v_j] = 0$).

of light recently discussed [9] in the non-critical ("Liouville") string literature. Both in the κ-Poincaré and in the string theory contexts the deviation from ordinary physics, while very significant at the conceptual level, is rather marginal from the phenomenological viewpoint. For example, for photons of energies of order 1 GeV the Eq. 2̊.16b entails a minuscule $10^{-19}c$ correction with respect to the ordinary scenario with constant speed of light. As discussed in greater detail in [9], at least when κ is identified with the Planck scale, the Eq.2̊.16b is completely consistent with available experimental data. As manifest in the relations 2̊.11a-2̊.11d, the κ-modifications of the covariant Heisenberg commutations relations are of quantum mechanical nature, *i.e.* proportional to the Planck constant $\hbar$. This suggests that the κ-deformation (together with its exotic energy-dependent speed of light) can be related with the quantum corrections to the classical dynamics of space-time.

3. Measurement of distance and covariant κ-deformed phase space

In this section we analyze the measurement of the distance L between two bodies as it results from a plausible physical interpretation of the uncertainty relations 2̊.11a-2̊.11d. Like the related studies [8–10] we consider the procedure of measurement of distances set out by Wigner [27], which relies on the exchange of a probe/signal between the bodies. The distance is therefore measured as $L = v\,T/2$, where v is the velocity of the probe and T is the time (being measured by a clock) spent by the probe to go from one body to the other and return. In general the quantum mechanical nature of the agents intervening in the experiment introduces uncertainties in the measurement of L, and in particular one finds that [7]

$$\Delta L \geq [\Delta L]_{\text{clock}} + [\Delta L]_{\text{probe}} \ , \tag{3.23}$$

i.e. the uncertainty in the measurement of L receives of course contributions that originate from the quantum mechanical nature of the clock (*i.e.* the timing/triggering device employed in the measurement) and from the quantum mechanical nature of the probe exchanged between the bodies.

A significant contribution to ΔL_{clock} was uncovered in Ref. [8]; this results in the relation

$$[\Delta L]_{\text{clock}} \geq l_p \sqrt{\frac{cT}{s}} \ , \tag{3.24}$$

[7] Of course there are other contributions to ΔL (*e.g.*, coming from the quantum mechanical nature of the other devices used in the experiment [8]); however, since they obviously contribute additively to the total uncertainty in the measurement of L, these uncertainties could only make stronger the bound derived in the following.

where s is a length scale characterizing the spatial extension of the clock (*e.g.*, the radius of a spherically-symmetric clock) and T is the time needed to complete the procedure of measuring L (*i.e.* T is time that the clock measures).

Within ordinary quantum mechanics the quantum mechanical nature of the probe (while contributing in general to the uncertainty) does not contribute to the bound on the measurability of L (*i.e.* a suitable measurement set up can be found so that the quantum mechanical nature of the probe does not lead to a contribution to ΔL). It was shown in Ref. [11] that instead the kinematics of quantum κ-Minkowski space-time does lead to a nontrivial $[\Delta L]_{probe}$, and interestingly this turns out to be of the same type of the $[\Delta L]_{\rm clock}$ in (3.24). As announced in the Introduction we are interested in extending the analysis of Ref. [11] to include structure from the full κ-deformed phase space. We are also more general than Ref. [11] and other related work (see, *e.g.*, Ref. [8–10]) in that we not only consider massless particles as the probes exchanged in the Wigner measurement, but we also consider the opposite limit in which the probes are ultra-heavy.

3.1. Using a heavy probe

In general combining the contribution (3.24) originating from the quantum mechanical nature of the clock with uncertainties due to the quantum mechanical nature of the probe one finds that

$$\Delta L \geq l_p \sqrt{\frac{cT}{s}} + \Delta x + v\,\Delta t + T\Delta v \tag{3.25}$$

where Δx and Δt are the uncertainties on the space-time position [8] of the probe at the "final time" T, while Δv is the uncertainty on the velocity of the probe.

The first contribution on the right-hand-side of (3.25) originates from the quantum mechanical nature of the clock, and it is interesting to notice that in the case of a heavy probe the proportionality to $\sqrt{T}$ of that term, which always signals decoherence effects (*e.g.*, the more time goes by, the more the quantum clock decoheres according to the ideas in Refs. [8, 9]), can be turned into a proportionality to $\sqrt{L/v}$, *i.e.* the uncertainty actually diverges in the limit of vanishing velocity as expected in a context involving decoherence since there small velocities imply large times.

Concerning the contributions on the right-hand-side of (3.25) that originate from the quantum mechanical nature of the probe, it is interesting to

[8] As implicit in the terminology here adopted, the Wigner measurement procedure is essentially one-dimensional, and the only relevant spatial coordinate is the one along the axis passing through the bodies whose distance is being measured.

observe that in ordinary quantum mechanics Δx, Δt and Δv are not correlated and therefore they do not lead to a contribution to the bound on the measurability of L. However, the κ deformation induces correlations between Δx, Δt and Δv. In particular, we observe that $\mathring{2}$.11a-$\mathring{2}$.11d imply (for an ideal heavy/nonrelativistic probe with $p = Mv$ and interpreting the x on the right-hand-side of $\mathring{2}$.11a as the distance traveled by the probe)

$$\Delta v \geq \frac{l_\kappa \, v \, \Delta t}{2c} \tag{3.26}$$

and

$$\Delta x \geq \frac{l_\kappa \, L}{2\kappa c \, \Delta t} \, . \tag{3.27}$$

This relations together with the fact that $v \sim L/T$ allow to rewrite (3.25) as

$$\Delta L \geq l_p \sqrt{\frac{cT}{s}} + \frac{l_\kappa \, L}{2\kappa c \, \Delta t} + \frac{L}{T} \Delta t + \frac{l_\kappa \, L}{2\kappa c \, \Delta t} \ . \tag{3.28}$$

This uncertainty can be minimized by preparing the probe in a state with $v \sim c l_p/\sqrt{s l_\kappa}$, *i.e.* $T \sim L\sqrt{s l_\kappa}/(c l_p)$, and $\Delta t \sim \sqrt{l_\kappa T/c}$, and this results in the measurability bound

$$\min[\Delta L] \sim \sqrt{L l_p \sqrt{\frac{l_\kappa}{s}}} \ . \tag{3.29}$$

The fact that this bound emerging from our analysis of Wigner measurement using a heavy probe manifests the same $\sqrt{L}$ behavior encountered in the heuristic quantum-gravity analysis of the clock involved in the measurement is a rather nontrivial aspect of the covariantly κ-deformed phase space. In fact, the κ-deformed kinematics of the heavy probe leads to an uncertainty with this $\sqrt{L}$ behavior just as a direct result of Eq.(3.26), which reflects the specific structure of the κ-deformed commutation relation between three-momenta and quantum time coordinate. This effect could not be uncovered in Ref. [11], since it requires the introduction of the κ-deformed four-momentum sector.

3.2. Using a massless probe

Of course, also in the case of a Wigner measurement involving a massless probe one finds that

$$\Delta L \geq l_p \sqrt{\frac{cT}{s}} + \Delta x + c\,\Delta t + T \Delta v \ , \tag{3.30}$$

and again the κ deformation induces correlations between Δx, Δt and Δv. In particular, concerning the correlation between Δx and Δt using again $\mathring{2}$.11a one finds

$$\Delta t \geq \frac{\hbar L}{2\kappa c^2\,\Delta x}\,. \tag{3.31}$$

Moreover, if the probe is massless with modified velocity[9] $\mathring{2}$.16b one finds that

$$\Delta v \sim \frac{\Delta P}{\kappa} \sim \frac{\hbar}{2\kappa\,\Delta x}\,, \tag{3.32}$$

where on the right-hand-side we used $\mathring{2}$.11b.

Using $\mathring{3}$starb and $\mathring{\text{d}}$eltavofE one can rewrite $\mathring{3}$.1light as

$$\Delta L \geq l_p\sqrt{\frac{cT}{s}} + \Delta x + \frac{\hbar L}{2\kappa c\,\Delta x} + \frac{\hbar T}{2\kappa\,\Delta x}\,, \tag{3.33}$$

and therefore, also taking into account that $L \sim cT/2$ and $l_\kappa \equiv \hbar/(\kappa c)$, one finds that the minimal value of ΔL is obtained if $(\Delta x)^2 \sim L l_\kappa$ and this implies that the minimal uncertainty in the measurement of the distance L is

$$\min[\Delta L] \sim \sqrt{\frac{L l_p^2}{s}} + \sqrt{L l_\kappa} \tag{3.34}$$

Again we find the $\sqrt{L}$ behavior, and again the full structure of the covariantly κ-deformed phase space advocated here plays a rather central role in obtaining this result; in fact, the relation (2.16) ensures that the fourth term on the right-hand side of Eq. $\mathring{3}$.1light (which was not considered in Ref. [11]) is of the same order as the second term on the right-hand side of Eq. $\mathring{3}$.1light, which is the one considered in Ref. [11].

While the $\sqrt{L}$ behavior is of course the most robust outcome of these analyses, it is interesting to notice the interplay between the scale l_κ, which characterizes the κ deformation, and the scales s and l_p, which characterize heuristic quantum-gravity arguments. The magnitude of these scales is actually quite important for the issue of the phenomenological implications [10] of this type of measurability bounds, but unfortunately very little is known about them. However, it is quite natural to guess that if κ deformations

[9] It is interesting to notice that κ-deformed mass-shell condition and κ-commutation relation between three-momenta and quantum time coordinate are somewhat related. In fact, for a minimum-uncertainty state in the framework of κ-deformed kinematics one has $\Delta E\,\Delta t \sim \hbar/2$ and $\Delta p\,\Delta t \sim l_\kappa p/(2c)$, and this is consistent with a given dispersion relation $E(p)$ only if $E(p) \sim (c\hbar/l_\kappa)\ln(p/p^*)$ (with p^* a constant to be otherwise determined) which coincides with the asymptotic behavior of the κ-deformed dispersion relation (see $\mathring{2}$.15).

were to have physical applications it might be that $l_\kappa \sim l_p$. Moreover, from the role of s in the measurement procedure it is clear [8, 11] that $s \geq l_p$, and since the measurability bound should be a general property of the theory it is quite conceivable that also $s \sim l_p$. This for example appears to fit rather well the schemes, such as the one discussed in Ref. [28], in which "fundamental clocks" are intrinsic to the formulation of the quantum-gravity approach. For $l_\kappa \sim l_p \sim s$ the heavy probe and the massless probe considered in this and in the previous subsection lead to exactly (up to an overall numerical factor of order 1) the same bound in the context of the Wigner measurement, and even the heuristic quantum-gravity measurement analysis of Ref. [8] reproduces this bound exactly (again up to an overall numerical factor of order 1). Nevertheless, especially in light of the fact that very little will be known about s until a fully consistent (and genuinely quantum) theory of gravity is available, it is interesting to observe that if $s \neq l_p$ (*i.e.* $s > l_p$) the Wigner measurement using a heavy probe is actually a "better measurement" (weaker bound) than its counterpart using a massless probe. Since most of the previous studies of quantum-gravity measurability bounds have relied on massless probes, our results suggest that a reanalysis of those studies might be necessary.

4. Closing remarks

The covariant κ-deformation of relativistic symmetries here considered, and the associated covariant κ-deformation of the Heisenberg algebra $\mathring{2}.9$, has several appealing properties as a candidate for the high-energy modification of classical relativistic symmetries. As a dimensionful deformation it is relevant only to the description of processes characterized by energies of order κ or higher. In addition, in an appropriate sense, it provides a rather moderate (at least in comparison with some of its alternatives) deformation of classical relativistic symmetries, which in particular reflects the reasonable expectation that, if any of the space-time coordinates is to be special, the special coordinate should be time. (Interestingly this intuition appears to be also realized in certain approaches to string theory, see *e.g.* Ref. [12].)

In extending the analysis of Ref. [11] from the space-time coordinate sector to the full structure of the κ-deformed phase space, our analysis has provided additional evidence that the bounds on the measurability of distances associated with the uncertainty relations characterizing the κ-deformed covariant Heisenberg algebra $\mathring{2}.4$a, $\mathring{2}.6$a and (2.9) are consistent with independent heuristic quantum-gravity analyses of such measurability bounds. The consistency between heuristic quantum-gravity measurability analysis and κ-Poincaré measurability analysis might signal that below some length scale characterized by the deformation parameter κ (and possibly related or even identified with the Planck length) the κ-deformations of Poincaré symmetries might play a role in the description of gravity. Also important is the nature of the κ-dependent kinematics of massless particles that we employed here. We expect that the experimental consequences of this modified photon kinematics should be a main ground for the physical testing of the idea of κ-deformation.

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