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GAUGE THEORY OF GRAVITY: ELECTRICALLY CHARGED SOLUTIONS WITHIN THE METRIC–AFFINE FRAMEWORK*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

We find a class of electrically charged exact solutions for a toy model of metric-affine gravity. Their metric is of the Plebański-Demiański type and their nonmetricity and torsion are represented by a triplet of covectors with dilation, shear, and spin charges.

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1. Introduction

In analyzing the structure of Minkowski space, Kopczyński & Trautman [1] stressed its underlying *affine* structure which expresses the inertial properties of spacetime. Superimposed to it, there exists a flat *metric* which allows to calculate lengths, times, and angles. It is the combination of both structures, the affine and the metric one, which determines the unique features of the Minkowski space of special relativity.

In special relativity the effects of gravity are neglected. Then the group of motions of spacetime is the Poincaré group with its 4+6 parameters. This group is the semidirect product of the translation and the Lorentz group.

The source of gravity is the *mass–energy* of matter. Because of the Noether theorem, the *conservation* of mass–energy, in Minkowski space, is related to the translation group. Consequently the gauging of the translations should yield *gravity*, as foreseen by Feynman [2], amongst others. We

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understand here gauging as a heuristic concept, as developed by Yang-Mills, which allows to "deduce" an interaction (here gravity) from a conserved current (here mass-energy) and its associated invariance group (here the translation group), see O'Raifeartaigh [3].

Since the translations are an inseparable part of the Poincaré group, it is near at hand to gauge the Poincaré group altogether; even more general, one can set free the "frozen" affine degrees of freedom of the Minkowski space by gauging the affine group $A(4,R) = R^4 \rtimes GL(4,R)$, the semidirect product of the translations and the linear transformations. If we still keep a metric superimposed to that affine gauge ansatz, then we end up at the metric–affine gauge theory of gravity (MAG), see the reviews [4, 5] from which we also take our conventions.

Andrzej Trautman [6, 7] was one of the first to explore the relation between the metric and the affine properties of spacetime. He restricted himself to the use of the curvature scalar as a gravitational Lagrangian. Thus, he could not go beyond a Riemann–Cartan spacetime with its metric compatible connection, see theorem 3 on page 151 of Ref. [6]. If one allows pieces in the gravitational Lagrangian which are quadratic in the curvature, for instance, then such a restriction becomes unnatural and one arrives at the full potentialities of the metric–affine gauge theory of gravity.

2. A metric–affine model theory

In metric–affine gravity, the metric $g_{\alpha\beta}$, the coframe ϑ^α, and the connection $\Gamma_\alpha{}^\beta$ are considered to be independent gravitational field variables. In such a framework, one can recover general relativity by means of the gravitational Lagrangian

$$V_{\mathrm{GR}'} = -\frac{1}{2l^2}\left(R^{\alpha\beta}\wedge\eta_{\alpha\beta} + \beta Q\wedge{}^\star Q\right), \tag{1}$$

where we have $R_\alpha{}^\beta$ as curvature 2-form belonging to the connection $\Gamma_\alpha{}^\beta$, $\eta_{\alpha\beta} := {}^\star(\vartheta_\alpha\wedge\vartheta_\beta)$, the Weyl covector is $Q := Q_\gamma^{\ \gamma}/4$ with the nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, and the Hodge dual is denoted by a star *. Provided the matter Lagrangian L_{mat} doesn't couple to the connection, *i.e.*, if $\delta L_{\mathrm{mat}}/\delta\Gamma_\alpha{}^\beta = 0$, we fall back to general relativity. It is decisive for these considerations to have a non–vanishing β in (1), see [8,9] for a corresponding discussion.

In order to explore the potentialities of metric–affine gravity, we will choose the simple non–trivial dilation–shear Lagrangian,

$$V_{\mathrm{dil-sh}} = -\frac{1}{2l^2}\left(R^{\alpha\beta}\wedge\eta_{\alpha\beta} + \beta Q\wedge{}^\star Q + \gamma T\wedge{}^\star T\right) - \frac{1}{8}\alpha\, R_\alpha{}^\alpha\wedge{}^\star R_\beta{}^\beta\,, \tag{2}$$

with the dimensionless coupling constants α, β, and γ, and $T := e_\alpha \rfloor T^\alpha$, where T^α is the torsion of spacetime. In future we will choose units such that $l^2 = 1$. Observe that the last piece is of a pure *post*–Riemannian nature, in fact, it is proportional to the square of Weyl's segmental curvature which Weyl used in the context of his unsuccessful unified field theory of 1918. It can be alternatively written as $-(\alpha/2)\,\mathrm{d}Q \wedge {}^\star\mathrm{d}Q$, where Q is the Weyl covector.

Below we will look for exact solutions of the field equations belonging to the Lagrangian

$$L = V_{\mathrm{dil-sh}} + V_{\mathrm{Max}}\,, \qquad \text{with} \qquad V_{\mathrm{Max}} = -(1/2)F \wedge {}^\star F \tag{3}$$

as the Lagrangian of the Maxwell field $F = dA$. We will only be able to find non–trivial solutions, if the coupling constants fulfill the constraint

$$\gamma = -\frac{8}{3}\frac{\beta}{\beta + 6}\,. \tag{4}$$

This is not completely satisfactory but, up to now, we cannot do better. The search for exact solution in metric affine gravity has been pioneered by Tresguerres [10, 11] and by Tucker and Wang [12].

3. Starting with the Plebański–Demiański metric of general relativity

Using the Eq. (3.30) of Ref. [13], see also [14,15], the orthonormal coframe can be expressed in terms of the coordinates (τ, q, p, σ) as follows,

$$\begin{aligned}
\vartheta^{\hat{0}} &= \frac{1}{H}\sqrt{\frac{\mathcal{Q}}{\Delta}}\left(d\tau - p^2 d\sigma\right)\,,\\
\vartheta^{\hat{1}} &= \frac{1}{H}\sqrt{\frac{\Delta}{\mathcal{Q}}}\,dq,\\
\vartheta^{\hat{2}} &= \frac{1}{H}\sqrt{\frac{\Delta}{\mathcal{P}}}\,dp,\\
\vartheta^{\hat{3}} &= \frac{1}{H}\sqrt{\frac{\mathcal{P}}{\Delta}}\left(d\tau + q^2 d\sigma\right)\,,
\end{aligned} \tag{5}$$

with the metric

$$g = -\vartheta^{\hat{0}} \otimes \vartheta^{\hat{0}} + \vartheta^{\hat{1}} \otimes \vartheta^{\hat{1}} + \vartheta^{\hat{2}} \otimes \vartheta^{\hat{2}} + \vartheta^{\hat{3}} \otimes \vartheta^{\hat{3}}\,, \tag{6}$$

or

$$g = \frac{1}{H^2}\left\{ -\frac{\mathcal{Q}}{\Delta}\,(d\tau - p^2 d\sigma)^2 + \frac{\Delta}{\mathcal{Q}}\,dq^2 + \frac{\Delta}{\mathcal{P}}\,dp^2 + \frac{\mathcal{P}}{\Delta}\,(d\tau + q^2 d\sigma)^2 \right\}. \quad (7)$$

The unknown functions are polynomials and read:

$$\begin{aligned} \mathcal{P} &:= \left(\gamma - g_o^2 - \frac{\lambda}{6}\right) + 2n\,p - \varepsilon\,p^2 + 2m\,p^3 - \left(\gamma + e_o^2 + \frac{\lambda}{6}\right) p^4 , \\ \mathcal{Q} &:= \left(\gamma + e_o^2 - \frac{\lambda}{6}\right) - 2m\,q + \varepsilon\,q^2 - 2n\,q^3 - \left(\gamma - g_o^2 + \frac{\lambda}{6}\right) q^4 , \\ \Delta &:= p^2 + q^2 , \\ H &:= 1 - p\,q . \end{aligned} \quad (8)$$

The electromagnetic potential A appropriate for this solution can be expressed as follows:

$$\begin{aligned} A &= \frac{1}{\Delta}\left[(e_o\,q + g_o\,p)d\tau + (g_o\,q - e_o\,p)\,p\,q d\sigma\right] \\ &= \frac{H}{\sqrt{\Delta}}\left(\frac{e_o\,q}{\sqrt{\mathcal{Q}}}\,\vartheta^{\hat{0}} + \frac{g_o\,p}{\sqrt{\mathcal{P}}}\,\vartheta^{\hat{3}}\right). \end{aligned} \quad (9)$$

4. Generating solutions in metric-affine gravity

It has been pointed out by Dereli, Tucker, *et al.*, see [16, 17], and by Obukhov *et al.*, see [18], that exact solutions of metric-affine gravity can be generated from electrovacuum solutions of general relativity if one assumes, besides the coframe (5), (6), a fairly simple form for nonmetricity and torsion, namely the so-called triplet ansatz (see [19]) patterned after the electromagnetic potential in (9):

$$\begin{aligned} Q &= k_0 \frac{H}{\sqrt{\Delta}}\left(\frac{N_{\rm e}\,q}{\sqrt{\mathcal{Q}}}\,\vartheta^{\hat{0}} + \frac{N_{\rm g}\,p}{\sqrt{\mathcal{P}}}\,\vartheta^{\hat{3}}\right), \\ T &= k_1 \frac{H}{\sqrt{\Delta}}\left(\frac{N_{\rm e}\,q}{\sqrt{\mathcal{Q}}}\,\vartheta^{\hat{0}} + \frac{N_{\rm g}\,p}{\sqrt{\mathcal{P}}}\,\vartheta^{\hat{3}}\right), \\ \Lambda &= k_2 \frac{H}{\sqrt{\Delta}}\left(\frac{N_{\rm e}\,q}{\sqrt{\mathcal{Q}}}\,\vartheta^{\hat{0}} + \frac{N_{\rm g}\,p}{\sqrt{\mathcal{P}}}\,\vartheta^{\hat{3}}\right). \end{aligned} \quad (10)$$

The Weyl covector Q and the torsion trace T were defined above, the Λ covector represents one irreducible piece of the shear part of the nonmetricity $\Lambda := \vartheta^\alpha e^\beta \rfloor (Q_{\alpha\beta} - Q g_{\alpha\beta})$.

The result of Ref. [18] is even stronger: Let the starting point be a general metric-affine Lagrangian which is quadratic in curvature, torsion, and nonmetricity. Provided this Lagrangian possesses, like (2), only one curvature square piece built up from Weyl's *segmental* curvature $R_\alpha{}^\alpha$, then the triplet (10) represents the general solution for the post-Riemannian pieces of the connection. In other words, if one wants to find exact solutions encompassing also the other $2+2$ irreducible pieces of nonmetricity and torsion, respectively, then one has to enrich the Lagrangian by other curvature square pieces.

For our Lagrangian (2), the constants in the triplet (10) read

$$k_0 = -\frac{24}{\beta+6}\,, \quad k_1 = -\frac{36}{\beta+6}\,, \qquad k_2 = 6\,, \tag{11}$$

and $N_{\rm e}$ and $N_{\rm g}$ are the quasi-electric and the quasi-magnetic dilation–shear–spin charges, respectively. Now, in MAG, the polynomials depend also on these quasi-charges in a fairly trivial way,

$$\begin{aligned}
\mathcal{P} &:= \left(b - g_o^2 - \mathcal{G}_o^2\right) + 2np - \varepsilon p^2 + 2m\mu p^3 - \left[\mu^2\left(b + e_o^2 + \mathcal{E}_o^2\right) + \frac{\lambda}{3}\right] p^4,\\
\mathcal{Q} &:= \left(b + e_o^2 + \mathcal{E}_o^2\right) - 2mq + \varepsilon q^2 - 2n\mu q^3 - \left[\mu^2\left(b - g_o^2 - \mathcal{G}_o^2\right) + \frac{\lambda}{3}\right] q^4,\\
\mathit{\Delta} &:= p^2 + q^2,\\
H &:= 1 - \mu\, p\, q\,,
\end{aligned} \tag{12}$$

where we introduced $b := \gamma - \lambda/6$. We slightly generalized the solution by means of the parameter μ, which can take the values $-1, 0, +1$. The post-Riemannian charges $N_{\rm e}$ and $N_{\rm g}$ are related to the post-Riemannian pieces $\mathcal{E}_o$ and $\mathcal{G}_o$, entering the polynomials, according to

$$\mathcal{E}_o = k_0\sqrt{\frac{\alpha}{2}}\,N_{\rm e}\,, \qquad \mathcal{G}_o = k_0\sqrt{\frac{\alpha}{2}}\,N_{\rm g}\,, \qquad \text{with} \qquad \alpha > 0\,. \tag{13}$$

The solution (5), (6), (10) to (13) has been thoroughly checked with the help of our computer algebra programs written in Reduce for the exterior calculus package Excalc, see [20]. This solution seems to exhaust all the possibilities one has with the Plebański-Demiański metric and the triplet ansatz.

Finally, we would like to link up our new solution with the more special cases known from the literature [19, 21–23].

5. Reduction to a solution with mass, angular momentum, electric charge and quasi-electric post-Riemannian triplet

In order to recover known solutions, we change to more familiar coordinates, namely to the Boyer-Lindquist coordinates of the Kerr solution:

$$(\tau, q, p, \sigma) \longrightarrow (t, y, x, \phi) \longrightarrow (t, r, \theta, \phi)\,. \tag{14}$$

More exactly, we have

$$\begin{aligned}
\tau &= j_o(t - j_o\phi)\,,\\
\sigma &= -j_o^2\,\phi\,,\\
q &= \frac{y}{j_o} = \frac{r}{j_o}\,,\\
p &= x = -\cos\theta\,.
\end{aligned} \tag{15}$$

By means of these transformations, the coframe (5) has the same form as that of the Vtoh-solution of Ref. [21], Eq. (3.1). In fact, we can identify the coframes, provided we have

$$\frac{1}{H}\sqrt{\frac{\mathcal{Q}j_o^2}{\Delta}} \equiv \sqrt{\frac{\Delta_{\rm Vtoh}}{\Sigma_{\rm Vtoh}}}\,, \qquad \frac{1}{H}\sqrt{\frac{\mathcal{P}}{\Delta}} \equiv \sqrt{\frac{f_{\rm Vtoh}\sin^2\theta}{\Sigma_{\rm Vtoh}}}\,. \tag{16}$$

These identities can be fulfilled by the ansatz

$$H = 1\,, \quad \Delta j_o^2 = \Sigma_{\rm Vtoh}\,, \quad \mathcal{Q}j_o^4 = \Delta_{\rm Vtoh}\,, \quad \mathcal{P}j_o^2 = f_{\rm Vtoh}\sin^2\theta\,. \tag{17}$$

Therefore we have first to kill the μ–parameter: $\mu = 0$. Furthermore, the magnetic and the quasi–magnetic charges and the NUT–parameter have to vanish:

$$g_o = 0\,, \quad \mathcal{G}_o = 0\,, \quad N_{\rm g} = 0\,, \quad n = 0\,. \tag{18}$$

A modification of $\varepsilon = 1 - \lambda j_o^2/3$ is also necessary.

Then, by a suitable redefinition of the constants, the electromagnetic potential reduces to (*cf.* [24])

$$A = \frac{e_o\, r}{r^2 + j_o^2\cos^2\theta}\left(dt - j_o\sin^2\theta d\phi\right)\,, \tag{19}$$

and the Vtoh-functions (for $a_0 = 1$ and $z_4 = \alpha$) turn out to be:

$$\begin{aligned}
\Delta_{\rm Vtoh} &= r^2 + j_o^2 - 2Mr - \frac{\lambda}{3}r^2\left(r^2 + j_o^2\right) + \alpha\frac{(k_0 N_{\rm e})^2}{2} + e_o^2\,,\\
\Sigma_{\rm Vtoh} &= r^2 + j_o^2\cos^2\theta\,,\\
f_{\rm Vtoh} &= 1 + \frac{\lambda}{3}j_o^2\cos^2\theta\,.
\end{aligned} \tag{20}$$

Note that we kept the electric charge e_o. Correspondingly, we found the charged version of the Vtoh-solution. Putting $e_o = 0$, we finally recover the Vtoh-solution of [21]: It carries only mass, angular momentum, and the quasi–electric dilation–shear–spin charge $N_{\rm e}$. Our solution of Sec.4 has, additionally, the NUT–parameter, electric and magnetic charges, the acceleration parameter, and the quasi–magnetic dilation–shear–spin charge.

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