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A NOTE ON GAUGE COVARIANT TRANSLATIONS IN THE GAUGE APPROACH TO GRAVITY*

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Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

We point out that in the gauge approach to gravity it is not always possible to reduce translation invariance to diffeomorphism invariance. It is argued that a proper generator of translations on the spacetime manifold is given by a gauge covariant Lie derivative. A reduction to diffeomorphism invariance is obtained if the gauging of the translation group does not involve homogeneous frame transformations. Possible consequences are shortly discussed.

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1. Introduction

In this note we focus on the traditional gauge approach to gravity. This approach is the basis of any gauge approach to gravity. From our point of view, a "traditional" gauge approach to gravity includes the following features:

1. The four dimensional spacetime–manifold constitutes the base manifold of the gauge theory.
2. The translation group is gauged in some way in order to link the energy–momentum tensor of matter to the Riemannian geometry of spacetime.
3. The gauge symmetry might include homogeneous, linear frame transformations (Lorentz–transformations or, as the most general case, general linear transformations, *e.g.*).

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This setting developed from the pioneering works of Utiyama [21], Sciama [17, 18], and Kibble [10]; a concise historical summary of this development can be found in [15], for a modern review see [7]. The underlying mathematical structure, in particular in regard to the Einstein-Cartan theory, was analyzed by Trautman [19]. His analysis was based on the ideas of Cartan [1] but formulated in a more modern differential geometric framework [13].

Usually, traditions appear in various adaptations and modifications. The same is, unfortunately, also true for the gauging of the translation group. This is because the usual geometric framework of a gauge theory, as clearly described in the book of Trautman [20], cannot be straightforwardly applied to the translation group. This circumstance is sketched in Fig.1: A fiber bundle with spacetime as base manifold M carries the gauge freedom within its fibers. This picture is particularly clear if one concentrates on an associated vector bundle which contains the reference frames that are used to accomodate physical fields or to describe the geometry of the spacetime manifold itself. Then the vector bundle is associated to a principal bundle with the gauge group as structure group. It follows that the gauge group is represented by its action on the reference frames. A corresponding mapping from one set of reference frames to another is called a gauge transformation. As a consequence of this definition, gauge transformations act vertically, *i.e.*, they leave the base point of a particular reference frame invariant. It is thus not obvious how to interpret a translation as a vertical gauge transformation since, from its very definition, a translation does not leave the base points on the spacetime manifold invariant. This is the reason why it is difficult to gauge the translation group.

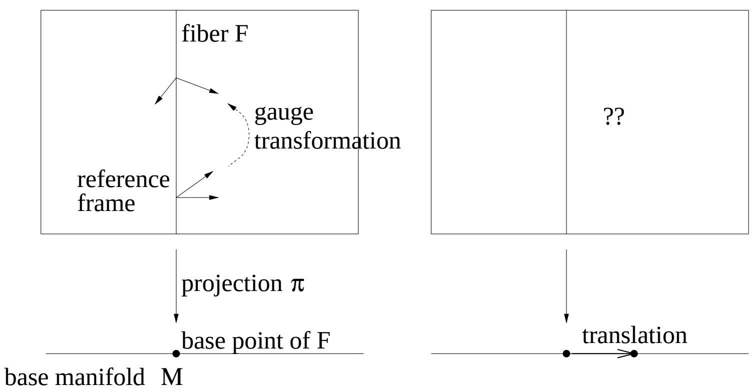

Fig. 1. The usual picture of a gauge theory is outlined on the left side of the figure. It is not obvious how to interpret a translation as a gauge transformation since, a priori, translations are no vertical transformations in some fiber but are defined on the base manifold.

2. Translations, diffeomorphisms, and frame transformations

There are two obvious ways to tackle the difficulty of gauging the translation group:

1. One can adopt a *passive* point of view and interpret the translations as coordinate transformations. This is feasible since it is always possible to compensate an active translation of base points by such a passive diffeomorphism. One might wonder why in this case a gauging makes sense since a Lagrangian, if written in exterior forms, is trivially invariant under passive diffeomorphisms. However, in exterior form language a corresponding Yang-Mills type gauge scheme of diffeomorphisms in their passive interpretation was proposed by Wallner [22].

2. The translations can be interpreted as *active* but *internal* translations within the fibers. A gauging of the translations is then immediate. *After* the gauging one then has to convert the active, internal translations to external translations, *i.e.* diffeomorphisms. The corresponding procedure is commonly called *soldering procedure*, (see below).

Therefore, with both approaches we have reduced translation invariance to diffeomorphism invariance - and this is why the group of diffeomorphisms is often referred to as the gauge group of gravity.

We now ask what happens if the gauge symmetry additionally includes homogeneous, linear frame transformations, generated by a group H, like Lorentz–transformations ($H = \mathrm{SO}(1,3)$) or general linear transformations ($H = \mathrm{GL}$). In both approaches it is possible to gauge these *active* frame transformations independently of the translations T^4 in complete analogy to a conventional Yang–Mills theory. This leaves the gauge procedure 1. or 2. for the translations untouched.

From a physical point of view it is more plausible to include the frame transformations by means of a *semidirect* product $T^4 \otimes H$, *i.e.*, to directly gauge, for example, the Poincaré group ($H = \mathrm{SO}(1,3)$) or the more general affine group ($H = \mathrm{GL}(4,R)$). In this case it is not clear how to generalize the gauge procedure 1. since the frame transformations are not regarded as passive ones. However, it is immediate to generalize the gauge procedure 2. in this respect, and this is what we will focus on in the following.

3. Lie derivatives

Since we want to talk about diffeomorphism invariance on the spacetime manifold we have to know how to actually measure the effect of a diffeomorphism on physical fields. A passive diffeomorphism of a geometric object $\mathcal{O}$, with the diffeomorphism defined by a vector flow, "pointing" from a point p

with coordinates x to a point $p + dp$ with coordinates $\tilde{x}$, means taking the value of $\mathcal{O}$ at p in the dragged coordinate system $\tilde{x}$ of $p + dp$. This is opposed to an active diffeomorphism, where the value of the actively dragged $\mathcal{O}$ is taken at $p + dp$ in the coordinate system $\tilde{x}$. Both (passive) dragging of the coordinate system $\tilde{x}$ to $\mathcal{O}$ or (active) dragging of $\mathcal{O}$ to the coordinate system $\tilde{x}$, with subsequent comparison to the original value of $\mathcal{O}$ at p or $p + dp$, respectively, can be uniquely described by a limiting process which leads to the definition of a Lie derivative [2,16]. The Lie derivative describes the change of a geometric object along the flow of a vector field.

Since physical fields can be described as exterior forms with values in some vector space it is convenient to have a corresponding operator for the Lie derivative at hand. We first introduce the anticommutator between the interior product $\rfloor$ and the exterior derivative,

$$\ell_v ... \ := \ v \rfloor d ... + d(v \rfloor ...) \,, \tag{1}$$

with some vector field v. For scalar valued p-forms Ψ this operator coincides with the Lie derivative, *i.e.*,

$$\mathcal{L}_v \Psi \ := \ v \rfloor d\Psi + d(v \rfloor \Psi) \tag{2}$$

yields the value of the Lie derivative of Ψ with respect to v.

Next we consider tensor-valued p-forms $\Phi_{\alpha ...}{}^{\beta ...}$ where the indices α, β refer to some vector basis e_α (or the corresponding cobasis ϑ^α). In this case the operator for the Lie derivative turns out to be

$$\mathcal{L}_v \Phi_{\alpha ...}{}^{\beta ...} \ = \ \ell_v \Phi_{\alpha ...}{}^{\beta ...} + \Phi_{\mu ...}{}^{\beta ...}(e_\alpha \rfloor \ell_v \vartheta^\mu) + ... - \Phi_{\alpha ...}{}^{\mu ...}(e_\mu \rfloor \ell_v \vartheta^\beta) - ... \,. \tag{3}$$

In the case of orthonormal bases an analogous formula is valid for spinor-valued forms [8]. The Lie derivative of other geometric objects, like bases or connections, can be calculated from the defining limiting process. A collection of useful formulas involving different operators of the Lie derivative can be found in [6].

The expression (3) is not covariant under a change of frames. That is, if gauge transformations involve homogeneous frame transformations the operator for the Lie derivative is in general not gauge covariant. Therefore, from a physical point of view, it is useful to add to the concept of a Lie derivative the notion of a gauge covariant Lie derivative. This is similar to the introduction of a gauge covariant derivative in addition to the usual derivative. To this end we first introduce the anticommutator between the interior product and the gauge covariant exterior derivative D^Γ,

$$\text{Ł}_v ... \ := \ v \rfloor D^\Gamma ... + D^\Gamma (v \rfloor ...) \,. \tag{4}$$

Then the gauge covariant Lie derivative of a tensor-valued p-form $\Phi_{\alpha\ldots}{}^{\beta\cdots}$ is defined as

$$Ł_v \Phi_{\alpha\ldots}{}^{\beta\cdots} := v\rfloor D^\Gamma \Phi_{\alpha\ldots}{}^{\beta\cdots} + D^\Gamma (v\rfloor \Phi_{\alpha\ldots}{}^{\beta\cdots}) \,, \qquad (5)$$

with an analogous formula for spinor-valued p-forms.

4. Soldering procedure

The gauging of internal translations in a Yang-Mills like mode introduces a translational gauge potential $\Gamma^{(T)}$, but this is not sufficient to reproduce gravity. Additionally, a soldering procedure is required which induces an identification of internal translations and diffeomorphisms on the base manifold. Only after the soldering the translational gauge potential is coupled to the energy-momentum tensor. For the fairly general framework of metric-affine gravity [7], which is based on a gauging of the n-dimensional affine group $A(n,R) = T^n \otimes \mathrm{GL}(n,R)$, a down-to-earth introduction into the gauge and soldering process was given in [4]. We shortly repeat the essential steps to see how the notion of gauge covariant translations emerges.

The gauging of the affine group presupposes that the corresponding fibers, compare Fig.1, are represented by affine tangent spaces. Within any fiber a gauge transformation is an affine transformation which relates two affine frames. An affine frame is a pair (e_i, p) consisting of a linear frame e_i and a point p. The gauging itself is accomplished by the introduction of a $\mathrm{GL}(n,R)$-valued linear connection $\Gamma^{(L)}$ and a R^n-valued translational connection $\Gamma^{(T)}$. Both $\Gamma^{(L)}$ and $\Gamma^{(T)}$ form a so called *generalized affine connection* which establishes parallelism of affine frames of different fibers. This is all what the gauging of the affine group is about.

The subsequent soldering process is more complicated. The essential idea is to identify in each fiber one point p with the corresponding base point $x \in M$. This is also called the zeroth order approximation of the base manifold by affine tangent spaces. Due to the presence of a generalized affine connection it is then possible to identify, to first order points, of affine tangent spaces with points of the base manifold. This identification is made precise by *Cartan's development* [13], *i.e.* the development of a curve on the manifold to a curve in an affine tangent space. It follows that a vector flow induces both an internal translation within the affine tangent spaces (the fibers) and an external translation (diffeomorphism) on the manifold. This makes it possible to identify an internal translation with an external diffeomorphism. It is important to note that this identification presupposes a proper gauging of the affine group.

There is a nice illustration of the interplay between internal and external translations: Suppose that we already performed the gauging and the soldering. Now we conduct an infinitesimal internal translation ε and focus on a particular fiber, *i.e.*, a particular affine tangent space A_xM. Within this affine tangent space the origin o_x gets shifted by an amount ε and is no longer the origin of the affine tangent space A_xM. However, due to the presence of the generalized affine connection, this former origin can be identified with a point of another affine tangent space $A_{x+\varepsilon}M$ which became an origin $o_{x+\varepsilon}$ after the infinitesimal translation. Now we turn to the external diffeomorphism which corresponds to the internal translation ε: If additionally performed it corrects the coordinates on the manifold in a way such that the point with coordinates $x+\varepsilon$ becomes the point with coordinates x. It follows that the former origin o_x (before the combined internal and external translation) again becomes an origin o_x (after the combined internal and external translation). This really is as much translational invariance as we can expect!

In view of a physical theory it is clearly not enough to discuss the invariance of points. The building of a translational invariant theory requires the introduction of a coframe ϑ^α which automatically embodies translational invariance. If expressed with respect to this tetrad, physical fields are automatically translation invariant. In metric-affine gravity the construction of the tetrad goes along the following line [4]: One first considers an infinitesimal affine transformation, acting on affine frames (e_i, p) within the fibers,

$$\delta e_i \;=\; \varepsilon_i{}^j e_j\,, \quad \delta p \;=\; \varepsilon^i e_i\,. \tag{6}$$

Under such a transformation the generalized affine connection transforms according to

$$\delta \Gamma^{(T)i} \;=\; -\varepsilon_j{}^i \Gamma^{(T)j} - D^\Gamma \varepsilon^i\,, \tag{7}$$

$$\delta \Gamma_i^{(L)j} \;=\; -D^\Gamma \varepsilon_i{}^j\,. \tag{8}$$

The desired GL(n,R)-covariant transformation behavior of ϑ^α reads

$$\delta\vartheta^\alpha \;=\; -\varepsilon_\beta{}^\alpha \vartheta^\beta\,. \tag{9}$$

It can be achieved by the coupling (we use the Kronecker symbol δ_i^α to carefully shift from holonomic to anholonomic coordinates)

$$\vartheta^\alpha := \delta_i^\alpha (dx^i + \Gamma^{(T)i}) \tag{10}$$

if the holonomic frame dx^i transforms as

$$\delta dx^i \;=\; D^\Gamma{}_j \varepsilon^i dx^j - \varepsilon_j{}^i dx^i \tag{11}$$

$$\;=\; \text{Ł}_\varepsilon dx^i - \varepsilon_j{}^i dx^j\,. \tag{12}$$

Therefore the construction of ϑ^α is connected with a gauge covariant translation of dx^i. This is due to the presence of the gauge covariant exterior derivative in (7). More general: A transition from the holonomic coframe dx^i to the locally translation invariant and anholonomic coframe ϑ^α requires a transition from internal translations in the fibers to gauge covariant translations on M.

5. Consequences

Gauge covariant translations do not show up in General Relativity (GR). This is a trivial fact since in this case there is no linear connection present which is independent of the Riemannian geometry of spacetime: GR can be deduced from a mere gauging of the translation group and thus only requires a translational potential which determines an orthonormal coframe. This is also the reason why in GR we cannot accomodate vector or spinor fields but only spinless matter.

Classically, in a general metric-affine gravity model (with independent linear connection) the difference between gauge covariant translations and ordinary diffeomorphisms does neither affect the specific form of a specific Lagrangian nor influence the field equations. However, a difference shows up in the corresponding Noether identities: Invariance under gauge covariant translations yields a gauge covariant Noether identity [11, 12], in contrast to the ordinary case.

In view of quantum aspects one is mainly interested in the gauge algebra of a theory [3, 9]. The gauge algebra is given by the commutators of the generators of the Noether identity. Thus, in metric affine gravity, one is interested in the commutator of two translations generated by $Ł_{\varepsilon_1}, Ł_{\varepsilon_2}$, the commutator of a translation $Ł_{\varepsilon_1}$ and a general linear transformation $\delta_{\varepsilon_{2\alpha}{}^\beta}$, and the commutator of two general linear transformations $\delta_{\varepsilon_{1\alpha}{}^\beta}, \delta_{\varepsilon_{2\gamma}{}^\delta}$. This yields the commutation relations

$$\begin{aligned}
[Ł_{\varepsilon_1}, Ł_{\varepsilon_2}] &= Ł_{[\varepsilon_1,\varepsilon_2]} + \delta_{(\varepsilon_2\rfloor(\varepsilon_1\rfloor d\Gamma_\alpha{}^\beta)+(\varepsilon_1\rfloor\Gamma_\gamma{}^\beta)(\varepsilon_2\rfloor\Gamma_\alpha{}^\gamma)-(\varepsilon_2\rfloor\Gamma_\gamma{}^\beta)(\varepsilon_1\rfloor\Gamma_\alpha{}^\gamma))} \\
&= Ł_{[\varepsilon_1,\varepsilon_2]} + \delta_{(\varepsilon_2\rfloor(\varepsilon_1\rfloor R_\alpha{}^\beta))}\,, \qquad (13) \\
[Ł_{\varepsilon_1}, \delta_{\varepsilon_{2\alpha}{}^\beta}] &= \delta_{\varepsilon_1\rfloor(D^\Gamma\varepsilon_{2\alpha}{}^\beta)}\,, \qquad (14) \\
[\delta_{\varepsilon_{1\alpha}{}^\beta}, \delta_{\varepsilon_{2\gamma}{}^\delta}] &= \delta_{\varepsilon_{1\delta}{}^\rho\varepsilon_{2\lambda}{}^\delta-\varepsilon_{2\gamma}{}^\rho\varepsilon_{1\lambda}{}^\gamma}\,, \qquad (15)
\end{aligned}$$

This gauge algebra is still irreducible and closed, but exhibits field-dependent structure functions. The commutator (13), which generalizes the infinite Ogievetsky algebra, is analogous to that of two "anholonomized coordinate transformations" [14, 15] which also play an important part in supergravity.

A general covariant quantization scheme is the BRST-antifield formalism [3]. It can be straightforwardly applied to metric-affine gravity with gauge algebra (13)–(15) [5]. The result is that due to the use of the gauge covariant Lie derivative complicated couplings to the linear connection occur which make any attempt to quantization very difficult. This might be the reason why the use of gauge covariant translations, though desirable from a geometric point of view, is not really common.

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