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CHERN–SIMONS SOLUTION OF THE ASHTEKAR CONSTRAINTS FOR THE TELEPARALLELISM EQUIVALENT OF GRAVITY*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

The teleparallelism equivalent $\mathrm{GR}_{\|}$ of gravity is considered. After its complexification via a canonical transformation, it becomes a true Yang–Mills theory of *translations*. It is shown that states constructed from the translational Chern–Simons term $\underline{\mathcal{C}}_{\mathrm{TT}}$ fully solve the corresponding Hamiltonian and diffeomorphism constraints.

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1. Introduction

The Ashtekar formulation of general relativity (GR) as well as that of its teleparallelism equivalent ($\mathrm{GR}_{\|}$) are both generated by the same *translational* boundary term $id\mathcal{C}_{\mathrm{TT}}$. Whereas the Hamiltonian constraints of complex GR with cosmological term can partially be mapped to SU(2) Chern–Simons theory, complexified $\mathrm{GR}_{\|}$ becomes a true Yang–Mills theory of *translations*. In this new approach, states constructed from the translational Chern–Simons term $\underline{\mathcal{C}}_{\mathrm{TT}}$ fully solve the constraints of teleparallelism without invoking a cosmological constant. Moreover, for $\mathrm{GR}_{\|}$ the (open) Wilson type loops get replaced by *Cartan circuits* carrying energy instead of spin as Noether charges. A further bonus is that, due to the teleparallelism constraint, the chiral anomaly for gravitationally coupled fermions seems to be absent.

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1.1. Complex variables for the 1D harmonic oscillator

In a 'nutshell', the Ashtekar approach to quantum gravity proceeds from a *canonical transformation.* In order to exhibit this on a most elementary level, we consider as a toy model the Lagrangian

$$\mathcal{L} = \frac{1}{2}\overset{\bullet}{q}{}^2 - U(q) = \frac{1}{2}(\overset{\bullet}{q}{}^2 - q^2) \tag{1}$$

of a one–dimensional harmonic oscillator of one Hertz, where $\bullet := \partial/\partial t$ denotes the time derivative. In terms of the canonical momentum $p := \partial\mathcal{L}/\partial\overset{\bullet}{q} = \overset{\bullet}{q}$ the Hamiltonian takes the form $\mathcal{H} = p\overset{\bullet}{q} - \mathcal{L} = \frac{1}{2}(p^2 + q^2)$.

Let us consider a canonical transformation $(q,p) \to (\widetilde{q}, \widetilde{p})$ induced by $\mathcal{C} = q^2/2$ as *generating function.* On the Lagrangian level this is equivalently achieved by adding the boundary term $i\overset{\bullet}{\mathcal{C}}$, resulting in the complex Lagrangian

$$\overset{\pm}{\mathcal{L}} = \mathcal{L} \pm i\overset{\bullet}{\mathcal{C}} = \mathcal{L} \pm i\overset{\bullet}{q}\, q\,. \tag{2}$$

The corresponding *complex momenta* are $\overset{\pm}{p} := \overset{\bullet}{q} \pm i\partial\overset{\bullet}{\mathcal{C}}/\partial\overset{\bullet}{q} = \overset{\bullet}{q} \pm i\partial\mathcal{C}/\partial q = p \pm iq \quad [= \pm i\sqrt{2}a^{(\dagger)}]$. As is shown below, in quantum mechanics (QM) this corresponds to the Bargmann representation of the harmonic oscillator (and is the main clever trick also in Ashtekar formulation, invented by a high **IQ**!). The Hamiltonian $\overset{\pm}{\mathcal{H}} := \overset{\pm}{p}\overset{\bullet}{q} - \overset{\pm}{\mathcal{L}} = \frac{1}{2}\overset{\pm}{p}\overset{\pm}{p} - i\overset{\pm}{p}\, q = \mathcal{H}$ turns out to be holomorphic in $\overset{\pm}{p}$.

In QM, where q and p become operators obeying $[q,p] = i\hbar$, the canonical transformation is achieved by the *similarity transformation*

$$\begin{cases} \widetilde{q} = NqN^{-1} = q \\ \widetilde{p} = NpN^{-1} = p \pm i\partial\overset{\bullet}{\mathcal{C}}/\partial\overset{\bullet}{q} = \overset{\pm}{p} \end{cases} \Rightarrow N = \underbrace{\mathrm{e}^{\pm\mathcal{C}/\hbar}}_{\text{non-unitary}}\,. \tag{3}$$

The construction of the non-unitary operator N proceeds via $\mathrm{e}^{\mathcal{C}}p\mathrm{e}^{-\mathcal{C}} = p + [\mathcal{C},p] + \frac{1}{2!}[\mathcal{C},[\mathcal{C},p]] + \frac{1}{3!}[\mathcal{C},[\mathcal{C},[\mathcal{C},p]]] + \cdots = p \pm iq$. Since $[q^2,p] = q[q,p] + [q,p]q = 2i\hbar q$, we necessarily recover $\mathcal{C} = q^2/2\hbar$. Observe that in the Schrödinger representation this corresponds to the *renormalization* of the wave function

$$\psi = N\widetilde{\psi} = \exp\left(\pm q^2/2\hbar\right)\widetilde{\psi}\,, \quad \widetilde{\psi}_n = A_n\, H_n\left(q/\sqrt{\hbar}\right)\,, \tag{4}$$

where H_n are the Hermite polynomials; for reality constraints and Wick rotation to the creation and annihilation operators a and $a^{\dagger}$, see Ref. [31]. In the Ashtekar formulation of gravity with complex variables, the *"triad densities"* and the complex momenta will become the generalized coordinates and momenta according to $q \to {}^*\underline{\vartheta}^{\alpha}$ and $\overset{\pm}{p} \to {}^*\overset{(\pm)}{\underline{\Pi}}{}_{\alpha} = \overset{(\pm)}{\underline{\mathcal{A}}}{}_{\alpha}$.

1.2. Ashtekar variables from the translational Chern–Simons term

Originally Ashtekar [2, 3] found his complex variables in the Hamiltonian formulation. In the equivalent Lagrangian approach, the change of variables is likewise induced by a *generating function* [18, 19] which involves the boundary term $d\mathcal{C}_{\mathrm{TT}}$ multiplied by the imaginary unit.

Similarly as in Maxwell's theory, where the Chern–Simons (CS) term reads $\mathcal{C}_{\mathrm{em}} := A \wedge F$, the *translational Chern–Simons term* in gravity is constructed from the coframe ϑ^α as a soldered translational gauge potential [20] and torsion $T^\alpha := D\vartheta^\alpha$ as its corresponding field strength:

$$\mathcal{C}_{\mathrm{TT}} := \frac{1}{2\ell^2}\left(\vartheta^\alpha \wedge T_\alpha\right). \tag{5}$$

A fundamental length ℓ necessarily occurs for dimensional reasons. The corresponding boundary or Nieh–Yan term [26]

$$d\mathcal{C}_{\mathrm{TT}} \equiv \frac{1}{2\ell^2}\left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) \tag{6}$$

suggest two options for a viable gravitational Lagrangian: After distributing a Hodge star between the two 2–forms, Hilbert's original choice

$$V_{\mathrm{HE}} = -\frac{1}{2\ell^2} R^{\{\}}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) \tag{7}$$

emerges, where $R^{\{\}}_{\alpha\beta}$ denotes the Riemannian curvature and $T^\alpha \wedge T_\alpha = 0$ as in general relativity (GR). Secondly, a torsion square Lagrangian [15]

$$V_{\|} := \frac{1}{2\ell^2} T^\alpha \wedge {}^*\left(-{}^{(1)}T_\alpha + 2\,{}^{(2)}T_\alpha + \frac{1}{2}\,{}^{(3)}T_\alpha\right) \tag{8}$$

with the *teleparallelism* ($\mathrm{GR}_{\|}$) constraint $R_{\alpha\beta} = 0$ of vanishing Riemann–Cartan curvature could also be employed, as has already been suggested by Einstein. Due to the geometric identity

$$V_{\|} \equiv V_{\mathrm{HE}} + \frac{1}{2\ell^2} R_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{1}{2\ell^2} d\left(\vartheta^\alpha \wedge {}^*T_\alpha\right), \tag{9}$$

$\mathrm{GR}_{\|}$ is equivalent to GR up to a boundary term. Note that both Lagrangians are subcases of the Poincaré gauge (PG) theory [4, 12].

In the self-dual or *chiral* formulation [21, 22], the Lagrangian of the Einstein–Cartan (EC) theory of gravity [32] plus cosmological term then takes the form

$$\overset{(\pm)}{V}_{\mathrm{EC}} := V_{\mathrm{EC}} \pm i d\mathcal{C}_{\mathrm{TT}} = -\frac{1}{2\ell^2} \overset{(\pm)}{R}_{\alpha\beta} \wedge {}^*(\vartheta^\alpha \wedge \vartheta^\beta) + \frac{\Lambda}{\ell^2}\eta\,. \tag{10}$$

Without fermions as matter source, it suffices to restrict to a Riemannian connection $\Gamma_{\alpha\beta} \equiv \Gamma^{\{\}}_{\alpha\beta}$ as part of the linear connection [12]. In the case of gravitationally coupled Dirac spinors in the EC theory as well as for coupled Rarita–Schwinger fields in simple supergravity, the translational boundary term induces "on shell", *i.e.* via Cartan's algebraic torsion relation, also a *chiral* decomposition in the fermionic part of the Lagrangian, *cf.* [21, 22] for details. There exists also solutions of the corresponding Hamiltonian constraints by loop states [1, 24].

2. Hamiltonian formulation

The general form of the PG Hamiltonian reads [19]

$$\mathcal{H} \cong n^\alpha \, \underline{\frac{\delta L}{\delta \vartheta^\alpha}} + \Gamma_\perp{}^{\alpha\beta} \, \underline{\frac{\delta L}{\delta \Gamma^{\alpha\beta}}} =: n^\alpha \, \mathcal{G}_\alpha + \Gamma_\perp{}^{\alpha\beta} \, \mathcal{G}_{\alpha\beta} \,, \tag{11}$$

where $n^\alpha := n\rfloor\vartheta^\alpha$ comprises the *lapse and shift* vector and $\Gamma_\perp{}^{\alpha\beta}$ are local Lorentz boosts. Together these are $4+6$ parameters, which surface as *Lagrange multipliers* for the generators $\mathcal{G}_\alpha$ and $\mathcal{G}_{\alpha\beta}$ in phase space.

Quite generally, the *Poisson brackets* of the generators at "equal times" read [4, 27, 29]

$$\{\mathcal{G}_\alpha(t,\vec{x}),\ \mathcal{G}_\beta(t,\vec{y})\} \ = \ (-T_{\alpha\beta}{}^\gamma \mathcal{G}_\gamma + R_{\alpha\beta}{}^{\gamma\delta}\mathcal{G}_{\gamma\delta})\cdot\delta(\vec{x}-\vec{y}) \,, \tag{12}$$

$$\{\mathcal{G}_\alpha(t,\vec{x}),\ \mathcal{G}_{\beta\gamma}(t,\vec{y})\} \ = \ \eta_{\alpha[\beta}\mathcal{G}_{\gamma]}\delta(\vec{x}-\vec{y}) \,, \tag{13}$$

$$\{\mathcal{G}_{\alpha\beta}(t,\vec{x}),\ \mathcal{G}_{\gamma\delta}(t,\vec{y})\} \ = \ (\eta_{\gamma[\alpha}\mathcal{G}_{\beta]\delta} - \eta_{\delta[\alpha}\mathcal{G}_{\beta]\gamma})\cdot\delta(\vec{x}-\vec{y}) \,. \tag{14}$$

They are the counterparts of the *local* Poincaré algebra

$$\begin{aligned} [D_\alpha, D_\beta] &= -T_{\alpha\beta}{}^\gamma(x)\, D_\gamma + R_{\alpha\beta}{}^{\gamma\delta}(x)\, L_{\gamma\delta} \,, \\ [D_\alpha, L_{\beta\gamma}] &= \eta_{\alpha[\beta}\, D_{\gamma]} \,, \\ [L_{\alpha\beta}, L_{\gamma\delta}] &= \eta_{\gamma[\alpha}\, L_{\beta]\delta} - \eta_{\delta[\alpha}\, L_{\beta]\gamma} \,, \end{aligned} \tag{15}$$

where $D_\alpha := e_\alpha\rfloor D$ is the gauge covariant derivative. Instead of "structure constants", in the translational part of both algebras there arises torsion and curvature. Thus only a "soft gauge algebra" [28] emerges, a fact which is gradually also realized by the Ashtekar community [25].

2.1. Complex GR constraints

Following Schwinger [30], we can apply a 3+1 decomposition [19] and use tetrads in the time or *temporal gauge*, for which the coframe and curvature with $A, B, C, \cdots = 1, 2, 3$ *tangential* to the hypersurface satisfy

$$\underline{\vartheta}^{\hat{0}} = 0 \,, \qquad \overset{(\pm)}{\underline{R}}{}_A := \frac{1}{2}\eta_{ABC}\,\overset{(\pm)}{\underline{R}}{}^{BC} \,. \tag{16}$$

Then in the Ashtekar formulation [2], the Gauss, diffeomorphism and Hamiltonian constraints read

$$\overset{\pm}{\underline{D}}\,{}^{*}\underline{\vartheta}^{A} \;\cong\; 0\,, \tag{17}$$

$$^{*}\overset{(\pm)}{\underline{R}}{}_{AB} \wedge {}^{*}\underline{\vartheta}^{B} \;\cong\; 0\,, \tag{18}$$

$$\mathcal{H}_{\Lambda} := \eta_{ABC}\,{}^{*}\left[\overset{(\pm)}{\underline{R}}{}^{A} - \frac{\Lambda}{6}\,{}^{*}\underline{\vartheta}^{A}\right] \wedge \underline{\vartheta}^{B} \wedge \underline{\vartheta}^{C} \;\cong\; 0\,. \tag{19}$$

In the transition to quantum gravity, one works in the connection representation, for which the tangential Ashtekar connection $\overset{(\pm)}{\underline{A}}{}_{B}$ ("momentum") is represented by itself, whereas the "densitized" triads ${}^{*}\underline{\vartheta}^{B}$ become differential operators:

$$\begin{aligned} \overset{\pm}{p}: \quad \overset{(\pm)}{\underline{A}}{}_{B}\,\Psi(A) &= \overset{(\pm)}{\underline{A}}{}_{B}\Psi(A)\,, \\ q: \quad {}^{*}\underline{\vartheta}^{B}\,\Psi(A) &= \frac{\delta}{\delta \overset{(\pm)}{\underline{A}}{}_{B}}\Psi(A)\,. \end{aligned} \tag{20}$$

Quantum-theoretical factor ordering problems are ignored here.

3. Chern–Simons solution of quantum gravity?

Since in the Hamiltonian formulation, chiral GR becomes essentially 3D on the hypersurface, on can formally transfer results from SU(2) Chern–Simons field theory [9, 11, 35] to gravity. In fact, the state vector

$$\begin{aligned} \Psi_{\Lambda}(A) &= \exp\left(\frac{3}{\Lambda}\int_{M_3} \overset{(\pm)}{\underline{C}}{}_{\mathrm{RR}}{}^{\{\}}\right) \\ &= \exp\left(\frac{3}{\Lambda}\int_{M_3}\left[\overset{(\pm)}{\underline{A}}{}^{B} \wedge \overset{(\pm)}{\underline{R}}{}_{B}{}^{\{\}} - \frac{1}{3!}\eta_{BCD}\,\overset{(\pm)}{\underline{A}}{}^{B} \wedge \overset{(\pm)}{\underline{A}}{}^{C} \wedge \overset{(\pm)}{\underline{A}}{}^{D}\right]\right) \end{aligned} \tag{21}$$

involving the tangential complexified Chern–Simons term $\underline{C}^{\{\}}_{\mathrm{RR}}$ is known [14] to solve the Hamiltonian constraint $\mathcal{H}_{\Lambda}\,\Psi_{\Lambda}(A) = 0$ of gravity in the non-perturbative loop approach [6, 10]. This is due to the curvature identity

$$\frac{\delta}{\delta \overset{(\pm)}{\underline{A}}{}^{B}}\Psi_{\Lambda}(A) = \frac{6}{\Lambda}\overset{(\pm)}{\underline{R}}{}_{B}\,\Psi_{\Lambda}(A)\,. \tag{22}$$

Note that a *non-zero* cosmological constant Λ is crucial for this construction. Therefore, one would surmise that the state vector is dominated by configurations winding around compact 3-manifolds as hypersurfaces.

Because of the Bianchi identity $\overset{(\pm)}{\underline{D}}\,\overset{(\pm)}{\underline{R}}_B \equiv 0$ for the 3D curvature, also the Gauss constraint (17) holds. However, for the diffeomorphism constraint there arises a non-trivial Lanczos type quadratic curvature relation which has to be satisfied.

3.1. Full solution of the Ashtekar constraints of the teleparallelism equivalent in terms of Cartan circuits

On the other hand, "...gravity is that field which corresponds to a gauge invariance with respect to displacement transformations", according Feynman. The teleparallel version (8) of Einstein's GR has recently been cast [19, 21, 22] into a Yang–Mills type *gauge theory of translations* after a change of variables induced again via $\overset{(\pm)}{V}_{\parallel} := V_{\parallel} \pm i dC_{\rm TT}$. The teleparallelism constraint can be enforced in the proper Lagrangian $\widetilde{V}_{\parallel} := V_{\parallel} - R^{\alpha\beta}\wedge\lambda_{\alpha\beta}$ via the Lagrange multipliers $\lambda_{\alpha\beta}$. In the Hamiltonian formulation, there would then arise the extra term $\underline{D\lambda_{\alpha\beta}}$ in the *Lorentz constraint*, see (11.2) of Ref. [19]. However, as suggested by the geometric identity (9), this constraint is identically satisfied by $\underline{\lambda}_{\alpha\beta} = (1/2\ell^2)\,\underline{{}^*(\vartheta_\alpha\wedge\vartheta_\beta)} = -(1/N\ell^2) n_{[\alpha}\,\underline{{}^*\vartheta}_{\beta]}$. This is only implicitly related to the teleparallel connection $\Gamma_{\parallel}^{\alpha\beta}$ which is of pure gauge type and therefore can be gauged to zero.

After this trivialization of the Lorentz constraint, the following Hamiltonian surfaces [19]

$$\overset{(\pm)}{\mathcal{H}} = -n^\alpha\wedge\underline{\overset{(\pm)}{D}\,\overset{(\pm)}{\varPi}}_\alpha + \underline{d}(n^\alpha\,\overset{(\pm)}{\underline{\varPi}}_\alpha)\,. \tag{23}$$

If we denote by $\mathcal{K} := N\rfloor V_{\parallel}$ the normal part of the GR$_{\parallel}$–Lagrangian, the *"triad densities"* $\underline{{}^*\vartheta}^\alpha$ and the Ashtekar connection $\overset{(\pm)}{\underline{\mathcal{A}}}_\alpha := \underline{{}^*\overset{(\pm)}{\varPi}}_\alpha$, being 3–dual to the translational field momentum, form a *canonical pair*: $\partial\overset{(\pm)}{\mathcal{K}}/\partial(\ell_n\,\underline{{}^*\vartheta}^\alpha) = -\overset{(\pm)}{\underline{\mathcal{A}}}_\alpha$.

Then the Hamiltonian ($\alpha = 0$) and diffeomorphism constraints ($\alpha = A$) of complexified GR$_{\parallel}$ are, in terms of this Sen type connection, already of Yang–Mills form:

$$\overset{(\pm)}{\underline{D}}\,\underline{{}^*}\overset{(\pm)}{\underline{\mathcal{A}}}_\alpha = 0\,. \tag{24}$$

In the transition to quantum gravity, we use, in contrast to the Hilbert–Einstein case, the Schrödinger representation, for which the Ashtekar connection $\overset{(\pm)}{\underline{\mathcal{A}}}{}_B$ ("momentum") becomes a differential operator

$$
\begin{aligned}
\overset{\pm}{p}: \quad \overset{(\pm)}{\underline{\mathcal{A}}}{}_\alpha\, \Psi_{\|}(\vartheta) &= \frac{\delta}{\delta\, {}^{*}\!\underline{\vartheta}_\alpha} \Psi_{\|}(\vartheta)\,, \\
q: \quad \underline{\vartheta}^B\, \Psi_{\|}(\vartheta) &= \underline{\vartheta}^B\, \Psi_{\|}(\vartheta)\,,
\end{aligned} \tag{25}
$$

whereas the triads remain generalized coordinates.

The Hamiltonian constraint $\mathcal{H}_{\|} = \overset{(\pm)}{\underline{D}}\, {}^{*}\overset{(\pm)}{\underline{\mathcal{A}}}{}_{\hat{0}} = 0$, *i.e.* (24) for $\alpha = \hat{0}$, vanishes identically, if we adopt again tetrads in the time gauge (16).

Let us try for the remaining Gauss constraint (24) the state vector

$$
\begin{aligned}
\Psi_{\|}(\vartheta) &= \exp\left(\int_{M_3} \overset{(\pm)}{\underline{C}}{}_{\mathrm{TT}}\right) \\
&= \exp\left(\frac{1}{2\ell^2}\int_{M_3}\left[\underline{\vartheta}^B \wedge \overset{(\pm)}{\underline{T}}{}_B\right]\right) = \exp\left(\frac{1}{2\ell^2}\int_{M_3}\left[{}^{*}\!\underline{\vartheta}_B \wedge {}^{*}\overset{(\pm)}{\underline{T}}{}^B\right]\right)
\end{aligned} \tag{26}
$$

which involves the tangential complexified *translational* Chern–Simons term $\overset{(\pm)}{\underline{C}}{}_{\mathrm{TT}}$ in terms of the self- or anti-selfdual torsion $\overset{(\pm)}{T}{}^\alpha := \frac{1}{2}\,(T^\alpha \pm i\,{}^{*}T^\alpha)$. Due to the torsion identity, we have

$$
\frac{\delta}{\delta\, {}^{*}\!\underline{\vartheta}_B} \Psi_{\|}(\vartheta) = {}^{*}\overset{(\pm)}{\underline{T}}{}^B\, \Psi_{\|}(\vartheta)\,. \tag{27}
$$

For the Sen covariant tangential diffeomorphism constraint we consequently find

$$
\overset{(\pm)}{\underline{D}}\, {}^{*}\left(\frac{\delta}{\delta\, {}^{*}\!\underline{\vartheta}_A}\right) = \overset{(\pm)}{\underline{D}}\,\overset{(\pm)}{\underline{T}}{}^A \equiv \overset{(\pm)}{\underline{R}}{}_B{}^A \wedge \underline{\vartheta}^B = 0\,, \tag{28}
$$

which vanishes due to the first Bianchi identity and the teleparallelism constraint of zero Riemann–Cartan curvature. (The latter needs also to be fulfilled on the quantum level.)

Note that our new approach has the advantage that the state vector does not depend on any cosmological constant which in the Kodama approach [6, 10, 14] becomes *singular* for $\Lambda \to 0$, but merely smoothly on the Planck length ℓ.

3.2. Cartan circuits versus Wilson loops

Instead of loops, we encounter for $\mathrm{GR}_{\|}$ *Cartan circuits* [7, 16] with dislocations at the Planck scale. Since these Cartan loops carry triads along, they are inherently *framed.*

Moreover, it is a general feature of *open* Wilson loops that there ends carry the Noether charge corresponding to the connection transported along the loop. In the Ashtekar approach this is the spin as an su(2)-valued "charge", leading eventually to a 'spin network'. This intuitively *non-gravitational* feature originates from the intertwined coupling of the energy–momentum and spin currents to torsion and curvature in the EC theory. In contradistinction, $\mathrm{GR}_{\|}$ has the bonus that the corresponding loops transport along the translational connection, *i.e.* the soldered coframe ϑ^α. Consequently, their ends carry energy(–momentum) $\Sigma_\alpha := \partial L/\partial\vartheta^\alpha$ as Noether charge as it should be in a proper gauge theory of gravity.

If one converts the tetrads into the *true* translational connection $\Gamma^{(T)\alpha} = \vartheta^\alpha - D\xi^\alpha$ of Cartan [7,20], there arises also the prospect of a more coherent formulation of the *gravitational* Aharanov–Bohm effect [16].

Moreover, the usual *axial anomaly* [23]

$$\langle dj_5\rangle = 2im\langle P\rangle - \frac{1}{48\pi^2}\left[R^{\{\}\alpha\beta}\wedge R^{\{\}}_{\alpha\beta} + \frac{1}{2}dA\wedge dA\right] \tag{29}$$

in the coupling to fermions is absent due to the teleparallelism constraint $R_{\alpha\beta}=0$ if we disregard topological defects for the axial torsion $A := {}^*(\vartheta^\alpha\wedge T_\alpha)$ and the contortion $K_{\alpha\beta} := \Gamma^{\{\}}_{\alpha\beta} - \Gamma_{\alpha\beta}$. This can be infered from the deformation $R^{\{\}\alpha\beta}\wedge R^{\{\}}_{\alpha\beta} = R^{\alpha\beta}\wedge R_{\alpha\beta} + d[K_{\alpha\beta}\wedge(R^{\{\}\alpha\beta} + D^{\{\}}K^{\alpha\beta} - \frac{2}{3}K_\gamma{}^\alpha\wedge K_\beta{}^\gamma)]$, *cf.* [34]. (Observe that no Nieh–Yan term (6) contributes to (29), despite recent claims in Ref. [8]).

On the other hand, on the 3D hypersurface of the Hamiltonian formulation there exist the intriguing mapping $\underline{\vartheta}^A \to \underline{\Gamma}^{\star A} := \frac{1}{2}\eta^{ABC}\,\underline{\Gamma}_{BC}$ of the triads to the corresponding 3-dual of the su(2)-valued connection [5]. This allows to establish the corresponding mapping $\underline{\mathcal{C}}_{\mathrm{TT}} \to \underline{\mathcal{C}}_{\mathrm{RR}} + \Lambda'\underline{\eta}$ for the associate Chern–Simons term plus induced cosmological term. Thus we expect that the complexified translational CS term $\overset{(\pm)}{\underline{C}}_{\mathrm{TT}}$ will induce the same knot invariants as for the Ashtekar connection.

4. Outlook: knitwear spacetime?

The full correlation function for Wilson loops is related to *knot invariants*, such as the Jones polynomial or the Kauffman bracket [6, 10]. One should note however, that the construction of a knot polynomial which is different for all knots is an open problem. The Jones polynomial, for instance, coincides for some of the different prime knots up to 13 crossings. It is a conjecture that the Vassiliev invariant [13, 33] classifies the knots *uniquely.*

Commonly, Wilson type loops live on simply–connected spacetime. However, it would be interesting to see applications to spacetimes with *knot wormholes* [17], for which generalizations of torus knots and links should occur which cannot shrink to a point.

In conclusion, our macroscopic view of a smooth spacetime may be much too simple on the Planck scale $\ell \simeq 10^{-33}$ cm of quantum gravity. Do we live rather in a *"knitwear spacetime"* with dislocations and other topological defects? A silk tie with one macroscopic trefoil knot looks also very smooth and elegant, however, even under a modest microscope the "knitting and knotting" of the texture will reveal itself!

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