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CONSERVATION LAWS IN POINCARÉ GAUGE THEORY*

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Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

Basic features of the conservation laws in the Hamiltonian approach to the Poincaré gauge theory are presented. It is shown that the Hamiltonian is given as a linear combination of ten first class constraints. The Poisson bracket algebra of these constraints is used to construct the gauge generators. By assuming that the asymptotic symmetry is the global Poincaré symmetry, we derived the improved form of the asymptotic generators, and discussed the related conservation laws of energy, momentum, *etc.*

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1. Introduction

Among various attempts to overcome the problem of quantization of the general relativity, gauge theories of gravity are especially attractive, as they are based on the concept of gauge symmetry which has been very successful in describing other fundamental interactions in nature. The importance of the Poincaré symmetry in particle physics leads one to consider the Poincaré gauge theory (PGT) as a natural framework for the description of the gravitational phenomena [1].

In this paper we shall

a) present the Hamiltonian structure of the general PGT [2, 3],

b) construct the gauge generators [4–6], and

c) clarify the relation between gauge symmetries and conservation laws, in case of asymptotically flat spacetimes [7].

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2. Hamiltonian dynamics

The Hamiltonian analysis of PGT leads to a simple form of the gravitational Hamiltonian, and yields a clear picture of the dynamical structure [3].

Basic gravitational variables in PGT are tetrad $b^i{}_\mu$ and Lorentz connection $A^{ij}{}_\mu$, and the corresponding field strengths are torsion and curvature: $T^i{}_{\mu\nu} = \partial_\mu b^i{}_\nu + A^i{}_{s\mu} b^s{}_\nu - (\mu \leftrightarrow \nu)$, $R^{ij}{}_{\mu\nu} = \partial_\mu A^{ij}{}_\nu + A^i{}_{s\mu} A^{sj}{}_\nu - (\mu \leftrightarrow \nu)$. The geometry of PGT is defined by the Riemann–Cartan spacetime U_4, with the general Lagrangian $\widetilde{\mathcal{L}} = b\mathcal{L}_G(R^{ij}{}_{kl}, T^i{}_{kl}) + b\mathcal{L}_M(\Psi, \nabla_k\Psi)$, where Ψ are matter fields and $\nabla_k\Psi$ is the covariant derivative. The gravitational Lagrangian which is at most quadratic in field strengths, *i.e.* of $R + T^2 + R^2$ type, depends on ten parameters (assuming parity invariance).

Constraints. The momentum variables $(\pi_k{}^\mu, \pi_{ij}{}^\mu, \pi)$, corresponding to $(b^k{}_\mu, A^{ij}{}_\mu, \Psi)$, are obtained from $\tilde{\mathcal{L}}$ in the usual way. Due to the fact that $T^i{}_{\mu\nu}$ and $R^{ij}{}_{\mu\nu}$ are defined through the antisymmetric derivatives of $b^k{}_\mu$ and $A^{ij}{}_\mu$, respectively, they do not involve velocities of $b^k{}_0$ and $A^{ij}{}_0$. As a consequence, one immediately obtains the following set of the so-called *sure* primary constraints:

$$\phi_k{}^0 \equiv \pi_k{}^0 \approx 0\,, \qquad \phi_{ij}{}^0 \equiv \pi_{ij}{}^0 \approx 0\,. \tag{1}$$

These constraints are always present, independently of the values of parameters in $\tilde{\mathcal{L}}$. Depending of a specific form of the Lagrangian, one may also have additional primary constraints in the theory.

The *canonical Hamiltonian* has the form $\mathcal{H}_c = \mathcal{H}_M + \mathcal{H}_G$, where $\mathcal{H}_M = \pi\Psi_{,0} - \widetilde{\mathcal{L}}_M$, $\mathcal{H}_G = \pi_k^\alpha b^k{}_{\alpha,0} + \frac{1}{2}\pi_{ij}{}^\alpha A^{ij}{}_{\alpha,0} - \widetilde{\mathcal{L}}_G$. The *total Hamiltonian* is

$$\mathcal{H}_T = \mathcal{H}_c + u^k{}_0\phi_k{}^0 + \tfrac{1}{2}u^{ij}{}_0\phi_{ij}{}^0 + (u\cdot\phi)\,, \tag{2}$$

where ϕ denotes all additional primary constraints, if they exist (if-constraints), and $H_T = \int d^3x\mathcal{H}_T$.

The evaluation of the consistency conditions of the primary constraints, $\dot\pi_k{}^0 = \{\pi_k{}^0, H_T\} \approx 0$ and $\dot\pi_{ij}{}^0 = \{\pi_{ij}{}^0, H_T\} \approx 0$, is essentially simplified if we previously find out the dependence of $\mathcal{H}_T$ on the unphysical variables $b^k{}_0$ and $A^{ij}{}_0$. We shall show that $\mathcal{H}_c$ is linear in $b^k{}_0$ and $A^{ij}{}_0$,

$$\mathcal{H}_c = b^k{}_0\mathcal{H}_k - \tfrac{1}{2}A^{ij}{}_0\mathcal{H}_{ij} + \partial_\alpha D^\alpha\,, \tag{3}$$

where $\partial_\alpha D^\alpha$ is a three–divergence term, while possible extra primary constraints ϕ are independent of $b^k{}_0$ and $A^{ij}{}_0$. Consequently, the consistency conditions of the primary constraints will result in the *secondary* constraints:

$$\mathcal{H}_k \approx 0\,, \qquad \mathcal{H}_{ij} \approx 0\,. \tag{4}$$

The linearity of $\mathcal{H}_c$ in $b^k{}_0$ and $A^{ij}{}_0$ is closely related to the so–called (3+1) decomposition of spacetime. If $\boldsymbol{n}$ is the unit normal to the hypersurface $\Sigma_0 : x^0 = \text{const.}$, the four vectors $\{\boldsymbol{n}, \boldsymbol{e}_\alpha\}$ define the so-called ADM basis. Introducing the projectors on $\boldsymbol{n}$ and Σ_0, $(P_\perp)^l_k = n_k n^l$, $(P_\|)^l_k = \delta^l_k - n_k n^l$, we can express any vector in terms of its parallel and orthogonal components: $V_k = n_k V_\perp + V_{\bar{k}}$, where $V_{\bar{k}} \equiv (V_\|)_k = (P_\|)^l_k V_l$, $V_\perp = V^k n_k$. An analogous decomposition can be defined for any tensor.

The decomposition of $\boldsymbol{e}_0$ in the ADM basis yields $\boldsymbol{e}_0 = N\boldsymbol{n} + N^\alpha \boldsymbol{e}_\alpha$, where N and N^α are called lapse and shift functions, respectively. By using the fact that N and N^α are linear in $b^k{}_0$, $N = n_k b^k{}_0$, $N^\alpha = h_{\bar{k}}{}^\alpha b^k{}_0$, the canonical Hamiltonian (3) can be easily brought into an equivalent form:

$$\mathcal{H}_c = N\mathcal{H}_\perp + N^\alpha \mathcal{H}_\alpha - \tfrac{1}{2} A^{ij}{}_0 \mathcal{H}_{ij} + \partial_\alpha D^\alpha \,, \tag{5}$$

where $\mathcal{H}_\perp = n^k \mathcal{H}_k$, $\mathcal{H}_\alpha = b^k{}_\alpha \mathcal{H}_k$.

Matter Hamiltonian. Let us now turn to the proof of (5) for the matter Hamiltonian. First, we decompose $\nabla_k \Psi$ into the orthogonal and parallel components,

$$\nabla_k \Psi = n_k \nabla_\perp \Psi + \nabla_{\bar{k}} \Psi \equiv n_k h_\perp{}^\mu \nabla_\mu \Psi + h_{\bar{k}}{}^\alpha \nabla_\alpha \Psi \,.$$

Replacing this into $\mathcal{L}_M$ leads to $\mathcal{L}_M = \overline{\mathcal{L}}_M(\Psi, \nabla_{\bar{k}}\Psi; \nabla_\perp \Psi, n^k)$, where complete dependence on velocities and unphysical variables $(b^k{}_0, A^{ij}{}_0)$ is contained in $\nabla_\perp \Psi$. Second, since $b = \det(b^k{}_\mu) = NJ$, where J does not depend on $b^k{}_0$, the expression for π can be written as

$$\pi \equiv \frac{\partial(b\mathcal{L}_M)}{\partial \Psi_{,0}} = J \frac{\partial \overline{\mathcal{L}}_M}{\partial \nabla_\perp \Psi} \,.$$

Finally, using the relation $\nabla_0 \Psi \equiv N\nabla_\perp \Psi + N^\alpha \nabla_\alpha \Psi = \Psi_{,0} + \frac{1}{2} A^{ij}{}_0 \Sigma_{ij} \Psi$ to express the velocities $\Psi_{,0}$, the canonical Hamiltonian for matter fields takes the form (5), where

$$\begin{aligned} &\mathcal{H}^M_\alpha = \pi \nabla_\alpha \Psi \,, \qquad \mathcal{H}^M_{ij} = \pi \Sigma_{ij} \Psi \,, \\ &\mathcal{H}^M_\perp = \pi \nabla_\perp \Psi - J\overline{\mathcal{L}}_M \,, \qquad D^M_\alpha = 0 \,. \end{aligned} \tag{6}$$

Expressions for $\mathcal{H}^M_\alpha$ and $\mathcal{H}^M_{ij}$ are independent of unphysical variables. They do not depend on the specific form of $\mathcal{L}_M$, but only on the transformation properties of fields, and are called *kinematical* parts of the Hamiltonian. The term $\mathcal{H}^M_\perp$ is *dynamical*, as it depends on the choice of $\mathcal{L}_M$. After eliminating $\nabla_\perp \Psi$ with the help of the relation defining π, one finds that $\mathcal{H}^M_\perp$ does not depend on unphysical variables: $\mathcal{H}^M_\perp = \mathcal{H}^M_\perp(\Psi, \nabla_{\bar{k}}\Psi; \pi/J, n^k)$.

Additional primary constraints, if they exist, are also independent of unphysical variables.

Gravitational Hamiltonian. Construction of the gravitational Hamiltonian can be performed in a very similar way, the role of $\nabla_k \Psi$ being taken over by $T^i{}_{km}$ and $R^{ij}{}_{km}$. In the first step we decompose the torsion and the curvature, in last two indices, into the orthogonal and parallel components. The parallel components $T^i{}_{\bar{k}\bar{m}}$ and $R^{ij}{}_{\bar{k}\bar{m}}$ are independent of velocities and unphysical variables. The replacement in the gravitational Lagrangian yields $\mathcal{L}_G = \overline{\mathcal{L}}_G(T^i{}_{\bar{k}\bar{m}}, R^{ij}{}_{\bar{k}\bar{m}}; T^i{}_{\perp\bar{k}}, R^{ij}{}_{\perp\bar{k}}, n^k)$. The relations defining gravitational momenta take the form

$$\hat{\pi}_i{}^{\bar{k}} = J\frac{\partial \overline{\mathcal{L}}_G}{\partial T^i{}_{\perp\bar{k}}}\,, \qquad \hat{\pi}_{ij}{}^{\bar{k}} = J\frac{\partial \overline{\mathcal{L}}_G}{\partial R^{ij}{}_{\perp\bar{k}}}\,,$$

where $\hat{\pi}_i{}^{\bar{k}} \equiv \pi_i{}^{\alpha} b^k{}_{\alpha}$ and $\hat{\pi}_{ij}{}^{\bar{k}} \equiv \pi_{ij}{}^{\alpha} b^k{}_{\alpha}$ are "parallel" gravitational momenta. The velocities $b^i{}_{\alpha,0}$ and $A^{ij}{}_{\alpha,0}$ can be calculated from the definitions of $T^i{}_{0\alpha}$ and $R^{ij}{}_{0\alpha}$. After a simple algebra the canonical Hamiltonian takes the form (5), where

$$\begin{aligned}
\mathcal{H}^G_{ij} &= 2\pi_{[i}{}^{\alpha} b_{j]\alpha} + 2\pi_{k[i}{}^{\alpha} A^k{}_{j]\alpha} + \partial_\alpha \pi_{ij}{}^{\alpha}\,,\\
\mathcal{H}^G_{\alpha} &= \pi_i{}^{\beta} T^i{}_{\alpha\beta} + \tfrac{1}{2}\pi_{ij}{}^{\beta} R^{ij}{}_{\alpha\beta} - b^k{}_{\alpha}\nabla_\beta \pi_k{}^{\beta}\,,\\
\mathcal{H}^G_{\perp} &= (\hat{\pi}_i{}^{\bar{m}} T^i{}_{\perp\bar{m}} + \tfrac{1}{2}\hat{\pi}_{ij}{}^{\bar{m}} R^{ij}{}_{\perp\bar{m}} - J\overline{\mathcal{L}}_G) - n^k \nabla_\beta \pi_k{}^{\beta}\,,\\
D^{\alpha}_G &= b^i{}_0 \pi_i{}^{\alpha} + \tfrac{1}{2}A^{ij}{}_0 \pi_{ij}{}^{\alpha}\,.
\end{aligned} \tag{7}$$

The expressions $T^i{}_{\perp\bar{m}}$ and $R^{ij}{}_{\perp\bar{m}}$ in $\mathcal{H}^G_\perp$ should be eliminated with the help of the equations defining momenta $\hat{\pi}_i{}^{\bar{m}}$ and $\hat{\pi}_{ij}{}^{\bar{m}}$.

Consistency of the theory. The fact that $\mathcal{H}_c$ is linear in unphysical variables implies the existence of the secondary constraints: $\mathcal{H}_\perp \approx 0$, $\mathcal{H}_\alpha \approx 0$ and $\mathcal{H}_{ij} \approx 0$. By working out the constraint algebra we shall see that these constraints are FC. As a consequence, the consistency conditions of the secondary constraints will be *automatically* satisfied.

3. Gauge symmetries

The correct definition of gauge generators enables one to clarify the relationship between gauge symmetries and conservation laws.

Constraint algebra. An explicit knowledge of the algebra of constraints is necessary for the investigation of the consistency of the theory, as well as for the construction of the gauge generators [4].

If extra constraints are not present in the theory, one can show that the Poisson bracket algebra of the secondary constraints takes the form

$$\{\mathcal{H}_{ij}, \mathcal{H}'_{kl}\} = \tfrac{1}{2} f_{ij}{}^{mn}{}_{kl}\mathcal{H}_{mn}\delta\,, \qquad \{\mathcal{H}_{ij}, \mathcal{H}'_{\alpha}\} = 0\,,$$

$$
\begin{aligned}
&\{\mathcal{H}_\alpha,\mathcal{H}'_\beta\} = (\mathcal{H}'_\alpha\partial_\beta + \mathcal{H}_\beta\partial_\alpha - \tfrac{1}{2}R^{ij}{}_{\alpha\beta}\mathcal{H}_{ij})\delta\,,\\
&\{\mathcal{H}_{ij},\mathcal{H}'_\perp\} = 0\,, \qquad \{\mathcal{H}_\alpha,\mathcal{H}'_\perp\} = (\mathcal{H}_\perp\partial_\alpha - \tfrac{1}{2}R^{ij}{}_{\alpha\perp}\mathcal{H}_{ij})\delta\,,\\
&\{\mathcal{H}_\perp,\mathcal{H}'_\perp\} = -({}^3g^{\alpha\beta}\mathcal{H}_\alpha + {}^3g'^{\alpha\beta}\mathcal{H}'_\alpha)\partial_\beta\delta\,. \qquad (8)
\end{aligned}
$$

In the presence of extra constraints the whole analysis becomes much more involved, but the results are essentially the same: *a*) the dynamical Hamiltonian $\mathcal{H}_\perp$ goes over into a redefined expression $\overline{\mathcal{H}}_\perp$, that includes the contributions of all primary second class constraints; *b*) the Poisson bracket algebra may contain primary FC terms (C_{PFC}). Therefore, consistency conditions of the secondary constraints are automatically satisfied.

Gauge generators. In PGT, the gauge generator has the form $G = \dot\varepsilon(t)G_1 + \varepsilon(t)G_0$, where G_0, G_1 are phase space functions satisfying the conditions [5]

$$
\begin{aligned}
G_1 &= C_{PFC}\,,\\
G_0 + \{G_1, H_T\} &= C_{PFC}\,,\\
\{G_0, H_T\} &= C_{PFC}\,.
\end{aligned}
$$

It is clear that the construction of the gauge generator demands the knowledge of the algebra of constraints. Since the Poincaré gauge symmetry is always present, independently of a specific form of the action, one naturally expects that all essential features of the gauge generator can be obtained by considering the simple case of the theory without extra constraints. In that case the primary constraints $\pi_k{}^0$ and $\pi_{ij}{}^0$ are FC, and the Poincaré gauge generator takes the form [6]

$$
\tilde G = \int d^3x[\dot\xi^\mu(b^k{}_\mu\pi_k{}^0 + \tfrac{1}{2}A^{ij}{}_\mu\pi_{ij}{}^0) + \xi^\mu\mathcal{P}_\mu + \tfrac{1}{2}\dot\omega^{ij}\pi_{ij}{}^0 + \tfrac{1}{2}\omega^{ij}S_{ij}]\,, \qquad (9)
$$

where

$$
\begin{aligned}
&\mathcal{P}_\mu = b^k{}_\mu\mathcal{H}_k - \tfrac{1}{2}A^{ij}{}_\mu\mathcal{H}_{ij} + b^k{}_{0,\mu}\pi_k{}^0 + \tfrac{1}{2}A^{ij}{}_{0,\mu}\pi_{ij}{}^0\,,\\
&S_{ij} = -\mathcal{H}_{ij} + 2b_{[i0}\pi_{j]}{}^0 + 2A^s{}_{[i0}\pi_{sj]}{}^0\,.
\end{aligned}
$$

Note that $\mathcal{P}_0 = \mathcal{H}_T - \partial_\alpha D^\alpha$, since $\dot b^k{}_0 = u^k{}_0\,,\ \dot A^{ij}{}_0 = u^{ij}{}_0$, on shell.

The action of the gauge generator on the fields $(\Psi, b^k{}_\mu, A^{ij}{}_\mu)$ produces the correct Poincaré gauge transformations. These transformations are symmetry transformations of the action not only when extra constraints are absent, but also in the general case. This fact leads to the conclusion that the expression (9) is the correct generator of the Poincaré gauge symmetry for any choice of the parameters of the theory.

4. Conservation laws

Now, we are going to consider one of the most important problems of the classical theory of gravity — the definition of the gravitational energy, and other conserved quantities [7].

The asymptotic symmetry. We assume that the symmetry of the U_4 theory in the asymptotic region is the global Poincaré symmetry. The global Poincaré transformations can be obtained from the gauge transformations by the following replacement of parameters:

$$\omega^{ij}(x) \to -\omega^{ij}\,, \qquad \xi^\mu(x) \to -\omega^\mu{}_\nu x^\nu - \varepsilon^\nu \equiv -\xi^\mu\,,$$

where ω^{ij} and ε^ν are constants, $\omega^\mu{}_\nu = \delta^\mu_i \omega^{ij}\eta_{j\nu}$. The related generator can be obtained from the gauge generator (9) in the same manner, leading to

$$G = \tfrac{1}{2}\omega^{ij}M_{ij} - \varepsilon^\nu P_\nu\,, \tag{10}$$

where

$$\begin{aligned}
P_\mu &= \int d^3x \mathcal{P}_\mu,\\
M_{\alpha\beta} &= \int d^3x (x_\alpha \mathcal{P}_\beta - x_\beta \mathcal{P}_\alpha - S_{\alpha\beta})\,,\\
M_{0\beta} &= \int d^3x (x_0 \mathcal{P}_\beta - x_\beta \mathcal{P}_0 - S_{0\beta} + b^k{}_\beta \pi_k{}^0 + \tfrac{1}{2}A^{ij}{}_\beta \pi_{ij}{}^0)\,.
\end{aligned}$$

Since the generators act on basic dynamical variables via Poisson brackets, they are required to have *well defined functional derivatives*. As this is not always the case with the generator (10), we shall try to improve its form so as to obtain the expression with well defined functional derivatives. The first step in that direction is to define precisely the phase space in which the generator (10) acts.

The phase space. The choice of asymptotics will become more clear if we first express the asymptotic structure of spacetime in certain geometric terms. Here we shall be concerned with *isolated* physical systems, characterized by matter fields that decrease sufficiently fast at large distances, so that their contribution to surface integrals vanishes. The spacetime outside an isolated system is said to be *asymptotically flat* if the following two conditions are satisfied:

(a) $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}_1$, where $\eta_{\mu\nu}$ is the Minkowskian metric, $\mathcal{O}_n$ decreases like r^{-n} or faster for large r, and $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

(b) $R^{ij}{}_{\mu\nu} = \mathcal{O}_{2+\alpha}$ $(\alpha > 0)$ (the absolute parallelism for large r).

The second condition can be easily satisfied by demanding $A^{ij}{}_{\mu} = \mathcal{O}_{1+\alpha}$. In the Einstein–Cartan (EC) theory the connection behaves as $\partial g_{\mu\nu}$, so that $A = \mathcal{O}_2$. The same law holds in the general U_4 theory when the field A is massive, while massless A can have a slower decrease. We shall study here, for simplicity, the EC theory, *i.e.* we shall assume that

$$b^k{}_{\mu} = \delta^k_{\mu} + \mathcal{O}_1 \,, \qquad A^{ij}{}_{\mu} = \mathcal{O}_2 \,. \tag{11}$$

To ensure the global Poincaré invariance of these conditions we demand $b^k{}_{\mu,\nu} = \mathcal{O}_2$, $A^{ij}{}_{\mu,\nu} = \mathcal{O}_3$, *etc.*

The asymptotic behaviour of *momenta* is determined by requiring $p - \partial\mathcal{L}/\partial\dot{q} = \widehat{\mathcal{O}}$, where $\widehat{\mathcal{O}}$ denotes a term that decreases sufficiently fast. From the definitions of the gravitational momenta in EC theory one obtains

$$\pi_k{}^0, \pi_{ij}{}^0 = \widehat{\mathcal{O}} \,, \qquad \pi_k{}^{\alpha} = \widehat{\mathcal{O}} \,, \quad \pi_{ij}{}^{\alpha} = -4aJn_{[i}h_{j]}{}^{\alpha} + \widehat{\mathcal{O}} \,. \tag{12}$$

Similar arguments lead to the consistent determination of the asymptotic behaviour of the Hamiltonian multipliers.

Improving the Poincaré generators. The generators act on dynamical variables via Poisson brackets, defined in terms of functional derivatives. A functional $F[\varphi,\pi] = \int d^3x f(\varphi, \partial_\mu\varphi, \pi, \partial_\nu\pi)$ has well defined functional derivatives if its variation can be written as $\delta F = \int d^3x[A\delta\varphi + B\delta\pi]$, where terms $\delta\varphi_{,\mu}$ and $\delta\pi_{,\mu}$ are *absent*.

The variation of the spatial translation generator has the form

$$\begin{aligned} &\delta P_\alpha = -\delta E_\alpha + R \,, \\ &E_\alpha \equiv \omega int ds_\gamma (\pi_{ij}{}^{\beta} A^{ij}{}_{[\alpha}\delta_{\beta]}{}^{\gamma}) \,, \end{aligned} \tag{13}$$

where the integration domain is the boundary of the three-dimensional space, and R denotes regular terms, not containing $\delta\varphi_{,\mu}$, $\delta\pi_{,\nu}$. Therefore, we can redefine the generator P_α,

$$P_\alpha \to \widetilde{P}_\alpha \equiv P_\alpha + E_\alpha \,, \tag{14}$$

so that $\tilde{P}_\alpha$ has well defined functional derivative. The assumed asymptotic behaviour of phase–space variables ensures *finitness* of E_α.

In a similar way we find $\widetilde{P}_0 \equiv P_0 + E_0$, where

$$E_0 \equiv \omega int ds_\gamma (-2aJh_a{}^{\alpha} h_b{}^{\gamma} A^{ab}{}_{\alpha}) \,. \tag{15}$$

The surface term E_0 is finite under the adopted asymptotic conditions, and represents the value of the energy of the system.

The spatial rotation generator reads $\widetilde{M}_{\alpha\beta} = M_{\alpha\beta} + E_{\alpha\beta}$, where

$$E_{\alpha\beta} \equiv \omega intds_\gamma[-\pi_{\alpha\beta}{}^\gamma + x_{[\alpha}(\pi_{ij}{}^\gamma A^{ij}{}_{\beta]})]\,. \tag{16}$$

A detailed analysis shows that the adopted asymptotic conditions do not guarantee the finitness of $E_{\alpha\beta}$, as the integrand contains $\mathcal{O}_1$ terms. These troublesome terms are seen to vanish if we impose the *asymptotic* gauge condition $a_{[ij]} = \mathcal{O}_2$ on the gauge potentials $a^k{}_\mu = b^k{}_\mu - \delta^k_\mu$, and certain *parity conditions*. These conditions are invariant under the global Poincaré transformations, and they restrict the remaining gauge symmetry. After that $E_{\alpha\beta}$ is seen to be finite and, consequently, $\widetilde{M}_{\alpha\beta}$ is well defined.

By varying the expression for the boost generator one finds

$$\begin{aligned} E_{0\beta} \;\equiv\; & \omega intds_\gamma[-\pi_{0\beta}{}^\gamma + x_0(\pi_{ij}{}^\alpha A^{ij}{}_{[\beta}\delta_{\alpha]}{}^\gamma) \\ & -x_\beta(2aJh_a{}^\alpha h_b{}^\gamma A^{ab}{}_\alpha)]\,. \end{aligned} \tag{17}$$

Additional gauge and parity conditions guarantee the finitness of $E_{0\beta}$.

All these results are referred to the EC theory. Analogous considerations in the general $R + T^2 + R^2$ theory show that the boost generator cannot be redefined by adding a surface term. Therefore, it is not a well defined generator under the adopted boundary conditions.

Conservation laws. The improved asymptotic Poincaré generators satisfy the standard Poincaré algebra, up to squares (or higher powers) of constraints and surface terms. This results proves the asymptotic Poincaré symmetry of the theory. We now wish to see whether this symmetry implies, as usually, the existence of certain conserved quantities.

One can prove that a phase-space functional $G[\varphi,\pi,t]$ is a generator of global symmetries if and only if

$$\{G,\widetilde{H}_T\} + \frac{\partial G}{\partial t} = C_{PFC}\,, \qquad \{G,\varphi_s\} \approx 0\,,$$

where $\widetilde{H}_T$ is the improved Hamiltonian, φ_s are all constraints, and, as before, the equality means an equality up to the zero generators. The first equation represents the Hamiltonian form of the conservation law. Indeed, it implies $dG/dt \equiv \{G,H_T\} + \partial G/\partial t \approx S$, so that G is conserved *if the surface term S is absent*.

One finds in this way that the generators $\widetilde{P}_0$, $\widetilde{P}_\alpha$ and $\widetilde{M}_{\alpha\beta}$ are conserved, and that the surface terms E_0, E_α and $E_{\alpha\beta}$ represent the values of energy, linear momentum and angular momentum as conserved quantities. On the other hand, the boost generator is not a conserved quantity. This result is a consequence of an explicit, linear time dependence of $\widetilde{M}_{0\beta}$, and the existence of a non–vanishing surface term in $\widetilde{P}_\beta$.

Comparison with the Lagrangian formalism. In order to compare the form of the surface terms with those obtained by the Lagrangian treatment, one should express all momentum variables in terms of fields and their derivatives, with the help of the constraints and the equations of motion. One finds that

i) the energy–momentum in EC theory is given by the same expressions as in GR,

ii) the angular momentum also coincides with the GR expression.

In the general $R+T^2+R^2$ theory, the result for the energy–momentum is of the same form, while the angular momentum remains the same only when all tordions are massive. When massless tordions exist, then *a*) the spatial angular momentum $E_{\alpha\beta}$ becomes different from the GR expression, and *b*) the boost $E_{0\beta}$ is not even defined in this case.

5. Concluding remarks

1) We constructed the Hamiltonian for the general PGT. The Hamiltonian constraints $\mathcal{H}_\perp, \mathcal{H}_\alpha, \mathcal{H}_{ij}$ are found to be first class.

2) The Poisson bracket algebra of constraints is calculated and used to construct the Poincaré gauge generators.

3) In case of the Minkowskian asymptotics, we obtained the conservation of energy–momentum and angular momentum. Other interesting asymptotic conditions (*e.g.* de Sitter spacetime) have not yet been studied.

4) Depending on the structure of $\tilde{\mathcal{L}}$, one may have extra FC constraints in the theory. The related extra gauge symmetries have been studied only in the linear approximation [8].

5) The Hamiltonian approach may be very useful in clarifying the dynamical structure of the teleparallelism theory.

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