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CONNECTIONS ON SOLDERED PRINCIPAL BUNDLES*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

In Kaluza–Klein theory one usually computes the scalar curvature of the principal bundle manifold using the Levi–Civita connection. Here we consider a natural family of invariant connections on a *soldered* principal bundle which is then parallelizable and hence spinable. This 3-parameter family includes the Levi–Civita connection and the flat connection. By varying the connection instead of merely scaling the metric on the fibers, there is greater independence among the coupling constants in the scalar curvature. In particular, a large cosmological constant can be avoided in spite of tiny fibers.

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1. Preliminaries

For a Lie group G with Lie algebra $\mathfrak{g}$, let $\pi : P \to M$ be a principal G-bundle over an n-manifold M. For $A \in \mathfrak{g}$, there is a fundamental vertical vector field A^* on P given at $p \in P$ by

$$A^*_p = \frac{d}{dt}\left(p \exp\left(tA\right)\right)\Big|_{t=0} . \tag{1}$$

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A connection on P is a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P,\mathfrak{g})$, such that $\omega(A^*) = A$ for all $A \in \mathfrak{g}$ and $R_g^*\omega = Ad_{g^{-1}}\omega$ for all $g \in G$ where $R_g : P \to P$ is given by the right action $R_g(p) := pg$ and $Ad : G \to \mathrm{End}(\mathfrak{g})$ is the adjoint representation. Generally, we use the notation of [3] or [7]. For an insightful introduction to the use of bundles and modern geometry in physics which is particularly suited to this article, see [11].

We assume that there is a representation $\tau : G \to \mathrm{O}(n)$. More generally, we could consider the pseudo-orthogonal case $\tau : G \to \mathrm{O}(n-q,q)$, but for the sake of simplicity, we leave the straightforward modifications to the interested reader. We mention that for the most common physical applications, $G = G_0 \times \mathrm{O}(1,3)$ where G_0 is a compact internal symmetry group, such as $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$, $\mathrm{SU}(5)$, $\mathrm{Spin}(10)$, *etc.*, and $\tau : G_0\times\mathrm{O}(1,3) \to \mathrm{O}(1,3)$ is just the projection.

A $\mathbb{R}^n$-valued 1-form α on P is *equivariant with respect to* τ if $R_g^*\alpha = \tau^{-1}\alpha$ for all $g \in G$, and α is *horizontal* if $\alpha_p(A^*) = 0$ for all $A \in \mathfrak{g}$. We denote the vector space of all such smooth, horizontal, 1-forms equivariant with respect to τ by $\overline{\Omega}_\tau^1(P,\mathbb{R}^n)$. For the space of equivariant forms which are not necessarily horizontal we write $\Omega_\tau^1(P,\mathbb{R}^n)$, and we use similar notation for different representations and higher degree forms. A form $\varphi \in \overline{\Omega}_\tau^1(P,\mathbb{R}^n)$ is called a *soldering form* if $\varphi_p : T_pP \to \mathbb{R}^n$ is onto for all $p \in P$. There is a right action of G on $P\times\mathbb{R}^n$ given by $[p,v]\cdot g = \left[pg, \tau\left(g^{-1}\right)v\right]$. Let

$$V := P \times_\tau \mathbb{R}^n = \frac{P\times\mathbb{R}^n}{G} \tag{2}$$

be the quotient space. The obvious projection $V \to M$ defines a vector bundle over M, the vector bundle associated to $P \to M$ via τ. Soldering forms φ correspond to isomorphisms $TM \cong V$ via $\pi_*(X_p) \leftrightarrow [p,\varphi(X_p)]$ for $X_p \in T_pP$. Since V has a natural Riemannian structure, a soldering form induces a Riemannian metric on M. Many have profitably regarded soldering forms as more fundamental than metrics. Early on, Trautman [10] emphasized that the presence of a soldering form is what sets gravitational gauge theories apart from the rest.

Suppose that $k_\mathfrak{g}$ is an Ad_G-invariant inner product on $\mathfrak{g}$ and $k_{\mathbb{R}^n}$ is the usual inner product on $\mathbb{R}^n$. If $\varphi \in \overline{\Omega}_\tau^1(P,\mathbb{R}^n)$ is a soldering form, one can define the metric

$$g_P := k_\mathfrak{g}(\omega,\omega) + k_{\mathbb{R}^n}(\varphi,\varphi) \tag{3}$$

on P. This is analogous to the sort of bundle metric used in Kaluza–Klein theory, except that our P is soldered to M via φ and our group G is not usually interpreted as a purely internal symmetry group G_0. In Kaluza–Klein theory one computes the scalar curvature R_{P_0} of the Levi–Civita connection for a metric on P_0, where $\pi_0 : P_0 \to M$ is a principal bundle with compact internal symmetry group G_0. The metric on P_0 is $g_{P_0} := k_{\mathfrak{g}_0}(\omega_0, \omega_0) + \pi_0^* g_M$, where $k_{\mathfrak{g}_0}$ is an Ad_{G_0}-invariant scalar product on $\mathfrak{g}_0$, ω_0 is a connection 1-form on P_0, and g_M is a metric on M. One finds that the scalar curvature for g_{P_0} is

$$R_{P_0} = R_{G_0} + \pi_0^* (R_M) - \tfrac{1}{4} (k_{\mathfrak{g}_0} \otimes g_M) (\Omega^{\omega_0}, \Omega^{\omega_0}), \tag{4}$$

where R_{G_0} is the scalar curvature of the Levi–Civita connection for the bi-invariant metric on G_0 induced by $k_{\mathfrak{g}_0}$, R_M is the scalar curvature of M with respect to the Levi–Civita connection for the metric g_M on M, and $\Omega^{\omega_0} := d\omega_0 + \frac{1}{2}[\omega_0, \omega_0] \in \overline{\Omega}^2 (P_0, G_0)$ is the field strength of the gauge potential ω_0. As the scalar curvature R_{P_0} is G_0-invariant, it projects to a well-defined function on M which serves as an action density. Setting the first variation of the total action with respect to g_M equal to zero yields Einstein's equations

$$\begin{aligned} &(R_M)_{ij} - \tfrac{1}{2} (R_M + R_{G_0}) (g_M)_{ij} \\ = \; &\tfrac{1}{2} (g_M)^{hk} k_{\mathfrak{g}_0} \left(\Omega^{\omega_0}_{hi}, \Omega^{\omega_0}_{kj} \right) - \tfrac{1}{8} (k_{\mathfrak{g}_0} \otimes g_M) (\Omega^{\omega_0}, \Omega^{\omega_0}) (g_M)_{ij} \end{aligned} \tag{5}$$

with Yang–Mills source originating from ω_0 and a cosmological constant due to R_{G_0} (which many are content to remove by hand). The first variation with respect to ω_0 yields the Yang–Mills equation $\delta^{\omega_0} \Omega^{\omega_0} = 0$, where δ^{ω_0} is the covariant codifferential operator. For more details on this, see [3] or [4].

One can also compute the scalar curvature of the Levi–Civita connection of g_P in (3), and we will present the result. However, there are many other natural G-invariant linear metric connections on P for which one can compute the scalar curvature. Indeed, there is a natural 3-parameter family of G-invariant linear connections for P which we will examine. We have computed the scalar curvature of these connections as a quadratic function of these 3 parameters. Essentially the result generalizes the standard Kaluza–Klein result. Conceivably this could be useful to modelers who want to get an early start building the next universe before time runs out.

2. Invariant connections

A vector $v \in \mathbb{R}^n$ gives rise to a *standard horizontal vector field* v^* which is determined by

$$\omega(v^*) = 0 \text{ and } \varphi(v^*) = v. \tag{6}$$

Given an orthonormal basis $e_1, ..., e_n$ for $\mathbb{R}^n$ with the usual metric and an orthonormal basis $u_{1'}, ..., u_{N'}$ for $\mathfrak{g}$ relative to the Ad_G-invariant inner product

$k_{\mathfrak{g}}$, we have a globally defined framing $u_{1'}^*, ..., u_{N'}^*, e_1^*, ..., e_n^*$ of fundamental vertical vector fields and standard horizontal vector fields. Note that we have primed $1', ..., N'$ so that $\{1', ..., N'\} \cap \{1, ..., n\} = \phi$. In sums, we let the Greek indices $\alpha, \beta, \gamma, ...$ run from $1'$ to N' and lower-case Latin indices $h, i, j, ...$ run from 1 to n. Writing

$$\omega = \sum_{\alpha=1'}^{N'} \omega^\alpha u_\alpha = \sum_\alpha \omega^\alpha u_\alpha \text{ and } \varphi = \sum_{i=1}^{n} \varphi^i e_i = \sum_i \varphi^i e_i, \tag{7}$$

we see that $\omega^{1'}, ..., \omega^{N'}, \varphi^1, ..., \varphi^n$ form the coframe field dual to frame field $u_{1'}^*, ..., u_{N'}^*, e_1^*, ..., e_n^*$. These are *global* orthonormal framings with respect to g_P. For non-soldered Kaluza–Klein theories, one does not necessarily have a global framings, let alone natural ones. It is also convenient to define

$$(v_1, ..., v_N, v_{N+1}, ..., v_{N+n}) := (u_{1'}, ..., u_{N'}, e_1, ..., e_n), \tag{8}$$

and introduce the $(\mathfrak{g} \oplus \mathbb{R}^n)$-valued form $\varpi := (\omega, \varphi)$. This form is an example of a Cartan connection (see [8]). Via the choice of bases, we can identify ϖ with the $\mathbb{R}^{N+n}$-valued form (global coframe)

$$\left(\varpi^1, ..., \varpi^N, \varpi^{N+1}, ..., \varpi^{N+n}\right) := \left(\omega^{1'}, ..., \omega^{N'}, \varphi^1, ..., \varphi^n\right) \tag{9}$$

which is dual to the framing of vertical and horizontal vector fields $\left(v_1^*, ..., v_{N+n}^*\right)$. We let the upper-case Latin indices H, I, J, K,... run from 1 to $N + n$.

Let ∇ denote the covariant derivative for a linear connection on P. Associated with ∇ is a globally-defined $(N + n) \times (N + n)$ matrix-valued 1-form θ on P, defined by

$$\nabla_{v_K^*} v_J^* = \sum_K \theta_{IJ} \left(v_K^*\right) v_I^*. \tag{10}$$

Alternatively (in basis-free notation), for A, $B \in \mathfrak{g} \oplus \mathbb{R}^n$ and with $A^* := \varpi^{-1}(A)$, we have

$$\nabla_{A^*} B^* = \left(\theta\left(A^*\right) B\right)^* \tag{11}$$

for $\theta \in \Omega^1 \left(P, \mathrm{End}\left(\mathfrak{g} \oplus \mathbb{R}^n\right)\right)$. The *flat* connection for g_P, is the one for which $\theta = 0$. By definition, a *metric connection* (relative to g_P) is one for which ∇ satisfies

$$A^* \left[g_P \left(B^*, C^*\right)\right] = g_P \left(\nabla_{A^*} B^*, C^*\right) + g_P \left(B^*, \nabla_{A^*} C^*\right) \tag{12}$$

or

$$0 = g_P\left(\left(\theta\left(A^*\right)B\right)^*, C^*\right) + g_P\left(B^*, \left(\theta\left(A^*\right)C\right)^*\right), \tag{13}$$

i.e., the associated matrix θ of connection 1-forms is anti-symmetric. The *torsion* of ∇ is the $(\mathfrak{g} \oplus \mathbb{R}^n)$-valued 2-form T^θ on P given by

$$T^\theta := d\varpi + \theta \wedge \varpi = d\omega \oplus d\varphi + \theta \wedge (\omega \oplus \varphi). \tag{14}$$

Here $\theta \wedge \varpi$ is the matrix product of θ with ϖ, where the entries are multiplied via wedge product, or equivalently

$$(\theta \wedge \varpi)(X, Y) = \theta(X)\,\varpi(Y) - \theta(Y)\,\varpi(X). \tag{15}$$

For the Levi–Civita connection $T^\theta = 0$, and for the flat connection clearly $T^\theta = d\varpi$. For a given 2-form $T \in \Omega^2(P, \mathfrak{g} \oplus \mathbb{R}^n)$, the equation

$$d\varpi + \theta \wedge \varpi = T \tag{16}$$

determines θ uniquely. To find θ, we proceed as follows. We define the *curvature* of ω by

$$\Omega^\omega := d\omega + \frac{1}{2}[\omega, \omega], \tag{17}$$

and the *torsion of* ω *relative to* φ by

$$\Phi := D^\omega\varphi := d\varphi + \tau_*(\omega) \wedge \varphi. \tag{18}$$

Here $\frac{1}{2}[\omega, \omega] \in \Omega^2(P, \mathfrak{g})$ is given by

$$\frac{1}{2}[\omega, \omega](X, Y) := \frac{1}{2}\left(\left[\omega(X), \omega(Y)\right] - \left[\omega(Y), \omega(X)\right]\right) = \left[\omega(X), \omega(Y)\right], \tag{19}$$

and for $\mathfrak{g}$ a Lie algebra of matrices, $\frac{1}{2}[\omega, \omega] = \omega \wedge \omega$. Also, $\tau_* : \mathfrak{g} \to \mathfrak{so}(n)$ is the Lie algebra map induced by $\tau : G \to \mathrm{O}(n)$. We say that ω is *torsion-free relative to* φ, if $\Phi = 0$. This is the usual notion of "torsion-free" when φ is the canonical 1-form on the bundle of orthonormal frames (relative to some metric g_M on M) with connection 1-form ω (i.e., in this case, if ω is torsion-free relative to φ, then ω is the Levi–Civita connection for g_M). We remain in the general setting, not assuming that $\Phi = 0$. We can write (16) as

$$\left(\Omega^\omega - \frac{1}{2}[\omega, \omega]\right) \oplus \left(D^\omega\varphi - \tau_*(\omega) \wedge \varphi\right) + \theta \wedge (\omega \oplus \varphi) = T. \tag{20}$$

We also write

$$\Omega^{\omega} = \frac{1}{2}\sum_{i,j}\Omega_{ij}\varphi^i \wedge \varphi^j = \frac{1}{2}\sum_{i,j,\alpha}\Omega_{\alpha ij}u_\alpha \varphi^i \wedge \varphi^j \tag{21}$$

and

$$D^{\omega}\varphi = \Phi = \frac{1}{2}\sum_{i,j}\Phi_{ij}\varphi^i \wedge \varphi^j = \frac{1}{2}\sum_{i,j,k}\Phi_{kij}e_k\varphi^i \wedge \varphi^j. \tag{22}$$

The structure constants $c_{\alpha\beta\gamma}$ are defined by

$$[u_\beta, u_\gamma] = \sum_{\alpha} c_{\alpha\beta\gamma}u_\alpha. \tag{23}$$

For any Ad_G-invariant metric $k_{\mathfrak{g}}$ on $\mathfrak{g}$, we have $c_{\alpha\beta\gamma}$ totally antisymmetric in α, β, γ. Now

$$\begin{aligned} d\varpi &= d\omega \oplus d\varphi = \left(\Omega^{\omega} - \frac{1}{2}[\omega,\omega]\right) \oplus \left(D^{\omega}\varphi - \tau_*(\omega)\wedge\varphi\right) \\ &= \sum_{\alpha}\left(\frac{1}{2}\sum_{i,j}\Omega_{\alpha ij}\varphi^i\wedge\varphi^j - \frac{1}{2}\sum_{\beta,\gamma}c_{\alpha\beta\gamma}\omega^\beta\wedge\omega^\gamma\right)u_\alpha \\ &\quad \oplus \sum_i\left(\frac{1}{2}\sum_{j,k}\Phi_{ijk}\varphi^j\wedge\varphi^k - \sum_{\alpha,j}\tau_*(u_\alpha)_{ij}\,\omega^\alpha\wedge\varphi^j\right)e_i. \end{aligned} \tag{24}$$

In general, suppose that

$$d\varpi^I = \frac{1}{2}\sum_{J,K} b_{IJK}\varpi^J \wedge \varpi^K, \tag{25}$$

where $b_{IJK} = -b_{IKJ}$. Then the unique constants θ_{IJK}, such that

$$\theta_{IJ} = \sum_K \theta_{IJK}\varpi^K \tag{26}$$

satisfies

$$d\varpi + \theta\wedge\varpi = T = \frac{1}{2}\sum_{I,J,K}T_{IJK}v^I\,\varpi^J\wedge\varpi^K \tag{27}$$

for given T_{IJK} (with $T_{IKJ} = -T_{IJK}$), are given by

$$\theta_{IKJ} = -\frac{1}{2}\left(b_{IJK} + b_{KIJ} - b_{JKI}\right) + \frac{1}{2}\left(T_{IJK} + T_{KIJ} - T_{JKI}\right). \tag{28}$$

Writing

$$\begin{aligned}
\theta_{\alpha\beta} &= \sum_\gamma \theta_{\alpha\beta\gamma}\omega^\gamma + \sum_k \theta_{\alpha\beta k}\varphi^k\,, \\
\theta_{\alpha j} &= -\theta_{j\alpha} = \sum_\gamma \theta_{\alpha j\gamma}\omega^\gamma + \sum_k \theta_{\alpha jk}\varphi^k\,, \\
\theta_{ij} &= \sum_\gamma \theta_{ij\gamma}\omega^\gamma + \sum_k \theta_{ijk}\varphi^k,
\end{aligned} \tag{29}$$

(28) yields

$$\begin{aligned}
\theta_{\alpha\beta\gamma} &= \frac{1}{2}c_{\alpha\gamma\beta} + \tfrac{1}{2}\left(T_{\alpha\gamma\beta} + T_{\beta\alpha\gamma} - T_{\gamma\beta\alpha}\right)\,, \\
\theta_{\alpha\beta i} &= \tfrac{1}{2}\left(T_{\alpha i\beta} + T_{\beta\alpha i} - T_{i\beta\alpha}\right)\,, \\
\theta_{\alpha ji} &= -\tfrac{1}{2}\Omega_{\alpha ij} + \tfrac{1}{2}\left(T_{\alpha ij} + T_{j\alpha i} - T_{ij\alpha}\right)\,, \\
\theta_{\alpha j\gamma} &= \tfrac{1}{2}\left(T_{\alpha\gamma j} + T_{j\alpha\gamma} - T_{\gamma j\alpha}\right)\,, \\
\theta_{j\alpha i} &= \tfrac{1}{2}\Omega_{\alpha ij} + \tfrac{1}{2}\left(T_{ji\alpha} + T_{\alpha ji} - T_{i\alpha j}\right)\,, \\
\theta_{j\alpha\beta} &= \tfrac{1}{2}\left(T_{j\beta\alpha} + T_{\alpha j\beta} - T_{\beta\alpha j}\right)\,, \\
\theta_{ikj} &= -\tfrac{1}{2}\left(\Phi_{ijk} + \Phi_{kij} - \Phi_{jki}\right) + \tfrac{1}{2}\left(T_{ijk} + T_{kij} - T_{jki}\right)\,, \\
\theta_{ik\alpha} &= \tau_*\left(u_\alpha\right)_{ik} - \tfrac{1}{2}\Omega_{\alpha ik} + \tfrac{1}{2}\left(T_{i\alpha k} + T_{ki\alpha} - T_{\alpha ki}\right)\,.
\end{aligned} \tag{30}$$

We obtain the Levi–Civita connection, say θ^L, by setting all components of T equal to 0. Thus,

$$\begin{aligned}
\theta^L_{\alpha\beta} &= \frac{1}{2}\sum_\gamma c_{\alpha\gamma\beta}\omega^\gamma\,, \\
\theta^L_{\alpha j} &= -\theta^L_{j\alpha} = -\frac{1}{2}\sum_k \Omega_{\alpha ij}\varphi^i\,, \\
\theta^L_{ij} &= \sum_\gamma \left(\tau_*\left(u_\gamma\right)_{ij} - \tfrac{1}{2}\Omega_{\gamma ij}\right)\omega^\gamma - \frac{1}{2}\sum_k \left(\Phi_{ikj} + \Phi_{jik} - \Phi_{kji}\right)\varphi^k\,.
\end{aligned} \tag{31}$$

If $\hat{u}_\alpha := k_{\mathfrak{g}}(u_\alpha, \cdot)$ and $\hat{e}_i := k_{\mathbb{R}^n}(e_i, \cdot)$, then

$$\begin{aligned} \theta^L &= \sum_{\alpha,\beta} \theta^L_{\alpha\beta} u_\alpha \otimes \hat{u}_\beta + \sum_{\alpha,j} \theta^L_{\alpha j}\left(u_\alpha \otimes \hat{e}_j - e_j \otimes \hat{u}_\alpha\right) + \sum_{i,j} \theta^L_{ij} e_i \otimes \hat{e}_j \\ &= \frac{1}{2}\sum_{\alpha,\beta,\gamma} c_{\alpha\gamma\beta}\omega^\gamma \otimes u_\alpha \otimes \hat{u}_\beta - \frac{1}{2}\sum_{\alpha,j,i} \Omega_{\alpha ij}\varphi^i \otimes \left(u_\alpha \otimes \hat{e}_j - e_j \otimes \hat{u}_\alpha\right) \\ &\quad + \sum_{i,j,\gamma}\left(\tau_*\left(u_\gamma\right)_{ij} - \frac{1}{2}\Omega_{\gamma ij}\right)\omega^\gamma \otimes e_i \otimes \hat{e}_j \\ &\quad - \frac{1}{2}\sum_{i,j,k}\left(\Phi_{ikj} + \Phi_{jik} - \Phi_{kji}\right)\varphi^k \otimes e_i \otimes \hat{e}_j. \end{aligned} \tag{32}$$

Besides the Levi–Civita connection and the flat connection (where $\theta = 0$), there are other natural choices. For example, even if $\Phi \neq 0$, we may choose $T_{ijk} = \Phi_{ijk}$ to produce $\theta_{ikj} = 0$. We can also choose $T_{\alpha\gamma\beta} = -c_{\alpha\gamma\beta}$ to get $\theta_{\alpha\beta\gamma} = \frac{1}{2}c_{\alpha\gamma\beta} - \frac{1}{2}\left(c_{\alpha\gamma\beta} + c_{\beta\alpha\gamma} - c_{\gamma\beta\alpha}\right) = 0$. We study more general possibilities below. For the Levi–Civita connection, we have

$$\begin{aligned} \nabla_{u^*_\alpha} u^*_\beta &= \frac{1}{2}\sum_\gamma c_{\gamma\alpha\beta} u^*_\gamma = \frac{1}{2}\left[u_\alpha, u_\beta\right]^*, \\ \nabla_{e^*_j} u^*_\beta &= \frac{1}{2}\sum_i \Omega_{\beta ji} e^*_i, \\ \nabla_{u^*_\alpha} e^*_j &= \sum_i \left(\tau_*\left(u_\alpha\right)_{ij} - \frac{1}{2}\Omega_{\alpha ij}\right) e^*_i, \\ \nabla_{e^*_j} e^*_k &= \frac{1}{2}\sum_\gamma \Omega_{\gamma kj} u^*_\gamma - \frac{1}{2}\sum_i \left(\Phi_{ijk} + \Phi_{kij} - \Phi_{jki}\right) e^*_i. \end{aligned} \tag{33}$$

Since we want the scalar curvature for ∇ to be G-invariant, it is desirable to require that the covariant derivative operator ∇ determined by θ be G-invariant on P, in the sense that

$$R_{g*}\left(\nabla_{A^*} B^*\right) = \nabla_{R_{g*}(A^*)}\left(R_{g*}\left(B^*\right)\right). \tag{34}$$

One can show that this is equivalent to the $Ad_{(Ad\times\tau)}$-equivariance of θ, where $Ad_{(Ad\times\tau)} : G \to \mathrm{O}(\mathfrak{so}\left(\mathfrak{g} \oplus \mathbb{R}^n\right))$ is given (for $E \in \mathrm{End}(\mathfrak{g} \oplus \mathbb{R}^n)$) by

$$Ad_{(Ad\times\tau)}(g)(E) = \left(Ad_g \times \tau(g)\right) \circ E \circ \left(Ad_{g^{-1}} \times \tau\left(g^{-1}\right)\right). \tag{35}$$

To say that $\theta \in \Omega^1_{Ad_{(Ad\times\tau)}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n))$ means that for all $A \in \mathfrak{g}\oplus\mathbb{R}^n$ and $g \in G$, we have

$$(Ad_g \times \tau(g))^{-1}\theta(A^*)(Ad_g \times \tau(g)) = \theta\left(\left((Ad_g \times \tau(g))^{-1}(A)\right)^*\right). \quad (36)$$

A straightforward computation shows that

$$T^\theta \in \Omega^2_{Ad\times\tau}(P, \mathfrak{g}\oplus\mathbb{R}^n) \Leftrightarrow \theta \in \Omega^1_{Ad_{Ad\times\tau}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n)). \quad (37)$$

In particular, the Levi–Civita connection form θ^L is in $\Omega^1_{Ad_{Ad\times\tau}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n))$. The last two sums in (32) each split into two separate sums in $\Omega^1_{Ad_{Ad\times\tau}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n))$, and hence θ^L is a sum of six terms, each one of which is in $\Omega^1_{Ad_{Ad\times\tau}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n))$. There is actually another element of $\Omega^1_{Ad_{Ad\times\tau}}(P, \mathfrak{so}(\mathfrak{g}\oplus\mathbb{R}^n))$ not occurring in θ^L, but worthy of consideration, namely

$$\sum_{\alpha,j}\tau_*(u_\alpha)_{ij}\,\varphi^i\otimes(u_\alpha\otimes\hat{e}_j - e_j\otimes\hat{u}_\alpha). \quad (38)$$

By giving each of the six sums in θ^L its own real coefficient and including the term (38) as well, we have a connection form $\theta(a, b_\Omega, b_\tau, c_\Omega, c_\tau, c_1, c_2)$ defined by

$$\begin{aligned}
&\theta(a, b_\Omega, b_\tau, c_\Omega, c_\tau, c_1, c_2) := \frac{1}{2}a\sum_{\alpha,\beta,\gamma}c_{\alpha\gamma\beta}\omega^\gamma\otimes u_\alpha\otimes\hat{u}_\beta\\
&-\frac{1}{2}b_\Omega\sum_{\alpha,j}\Omega_{\alpha ij}\varphi^i\otimes(u_\alpha\otimes\hat{e}_j - e_j\otimes\hat{u}_\alpha)\\
&+\frac{1}{2}b_\tau\sum_{\alpha,j}\tau_*(u_\alpha)_{ij}\,\varphi^i\otimes(u_\alpha\otimes\hat{e}_j - e_j\otimes\hat{u}_\alpha)\\
&-\frac{1}{2}c_\Omega\sum_{i,j,\gamma}\Omega_{\gamma ij}\omega^\gamma\otimes e_i\otimes\hat{e}_j + c_\tau\sum_{i,j,\gamma}\tau_*(u_\gamma)_{ij}\,\omega^\gamma\otimes e_i\otimes\hat{e}_j\\
&+\frac{1}{2}c_1\sum_{i,j,k}(\Phi_{ijk}-\Phi_{jik})\,\varphi^k\otimes e_i\otimes\hat{e}_j - \frac{1}{2}c_2\sum_{i,j,k}\Phi_{kij}\varphi^k\otimes e_i\otimes\hat{e}_j. \quad (39)
\end{aligned}$$

It is possible (although not necessarily desirable) to construct more terms involving contracted polynomial expressions of higher degree in $\Omega_{\alpha ij}$, $\tau_*(u_\gamma)_{ij}$, Φ_{kij}, and $c_{\alpha\gamma\beta}$. For simplicity (and the fact that time did not permit us to sufficiently check our computations in the more general case), we will consider the restricted family

$$\begin{aligned}
\theta^{(a,b,c)} &:= a\sum_{\alpha,\beta}\theta^{L}_{\alpha\beta}u_\alpha\otimes\hat{u}_\beta + b\sum_{\alpha,j}\theta^{L}_{\alpha j}\left(u_\alpha\otimes\hat{e}_j - e_j\otimes\hat{u}_\alpha\right) + c\sum_{i,j}\theta^{L}_{ij}e_i\otimes\hat{e}_j \\
&= \frac{1}{2}a\sum_{\alpha,\beta,\gamma}c_{\alpha\gamma\beta}\omega^\gamma\otimes u_\alpha\otimes\hat{u}_\beta - \frac{1}{2}b\sum_{\alpha,i,j}\Omega_{\alpha ij}\varphi^i\otimes\left(u_\alpha\otimes\hat{e}_j - e_j\otimes\hat{u}_\alpha\right) \\
&\quad + c\sum_{i,j}\begin{pmatrix}\sum_\gamma\left(\tau_*\left(u_\gamma\right)_{ij} - \frac{1}{2}\Omega_{\gamma ij}\right)\omega^\gamma \\ +\frac{1}{2}\sum_k\left(\Phi_{ijk}-\Phi_{jik}-\Phi_{kij}\right)\varphi^k\end{pmatrix}\otimes e_i\otimes\hat{e}_j. \qquad (40)
\end{aligned}$$

The Levi–Civita connection θ^L is $\theta^{(1,1,1)}$.

3. The scalar curvature

The curvature 2-form $\Omega^{(a,b,c)}\in\Omega^2\left(P,\mathfrak{so}\left(\mathfrak{g}\oplus\mathbb{R}^n\right)\right)$ of $\theta^{(a,b,c)}$ with respect to the global coframe field ϖ is given by

$$\Omega^{(a,b,c)} = d\theta^{(a,b,c)} + \theta^{(a,b,c)}\wedge\theta^{(a,b,c)}. \qquad (41)$$

Ultimately, $\Omega^{(a,b,c)}$ can be written as

$$\Omega^{(a,b,c)} = \frac{1}{2}\sum_{H,I,J,K}R^{(a,b,c)}_{HIJK}v_H\otimes\hat{v}_I\otimes\varpi^J\wedge\varpi^K. \qquad (42)$$

We have actually computed all of the components $R^{(a,b,c)}_{HIJK}$, and we will gladly furnish them along with the Ricci tensor (symmetric if $(a,b,c)=(1,1,1)$) via email. However, here we provide the scalar curvature.

Theorem 1 *The scalar curvature for the metric connection $\theta^{(a,b,c)}$ on P is*

$$\begin{aligned}
R^{(a,b,c)} &= \sum_{H,I}R^{(a,b,c)}_{HIHI} \\
&= \frac{1}{2}a\left(1-\frac{1}{2}a\right)\sum_{\alpha,\beta,\gamma}c_{\alpha\beta\gamma}c_{\alpha\beta\gamma} + \left(b\left(1-c\right)+c\right)\sum_{\alpha,i,j}\tau_*\left(u_\alpha\right)_{ij}\Omega_{\alpha ij} \\
&\quad + \frac{1}{2}\left(c\left(b-1\right)-\frac{1}{2}b^2\right)\sum_{i,j,\alpha}\Omega_{\alpha ij}\Omega_{\alpha ij} + 2c\sum_{i,j}e_i^*\left[\Phi_{jji}\right] \\
&\quad - \frac{1}{2}c^2\sum_{l,i,j}\left(\Phi_{lij}\Phi_{ilj} + 2\Phi_{iil}\Phi_{jjl}\right) - \frac{1}{2}c\left(1-\frac{1}{2}c\right)\sum_{l,i,j}\Phi_{lij}\Phi_{lij}. \qquad (43)
\end{aligned}$$

We remark that to simplify as much as possible, we used the Bianchi identities

$$0 = d\Omega^{\omega} + [\omega, \Omega^{\omega}] \quad \text{and} \quad 0 = d\Phi + \tau_* (\omega) \wedge \Phi - \tau_* (\Omega^{\omega}) \wedge \varphi, \tag{44}$$

which are easily derived from the definitions $\Omega^{\omega} = d\omega + \frac{1}{2}[\omega, \omega]$ and $\Phi = d\varphi + \omega \wedge \varphi$ by exterior differentiation. In particular, although one expects to see derivatives of $\Omega_{\alpha ij}$ in $R^{(a,b,c)}$, none appear, due to the Bianchi identities. Moreover, the only sum $\sum_{i,j} e_i^* [\Phi_{jji}]$ involving derivatives vanishes in the case where the torsion is traceless, as is often the case in applications (e.g., when Φ_{ijk} is totally antisymmetric). For example, in U_4 and ECSK theories with Dirac fields, Φ_{ijk} is proportional to the spin-angular momentum of the Dirac field, which is a 3-form or axial vector (see [2] or [5]).

To compare the result (43) with the Kaluza–Klein result (4), first assume that $G = G_0 \times \mathrm{O}(n)$, and P is the fibered product of a principal bundle P_0 with group G_0 with the bundle $F(M)$ of orthonormal frames of a Riemannian n-manifold M with metric g_M. The $\mathfrak{g}_0 \oplus \mathfrak{so}(n)$-valued connection form ω splits into two components, say $\omega = \omega_0 \oplus \omega_M$. Next assume that φ is the lift of the canonical 1-form on $F(M)$ to P and that ω_M is the lift to P of the Levi–Civita connection of g_M. We then have $\Phi = D^{\omega}\varphi = D^{\omega_0}\varphi = 0$. Note that the terms in the sum $\sum_{\alpha,i,j} \tau_* (u_\alpha)_{ij} \, \Omega_{\alpha ij}$ vanish for $\alpha \leq \dim (G_0)$, and this sum is the scalar curvature R_{g_M} of g_M. In what follows, the constant K is the scale that one chooses for the scalar product $k_{\mathfrak{so}}$ on $\mathfrak{so}(n)$, namely $k_{\mathfrak{so}} (A, B) = K A_{ij} B^{ij} = -K \,\mathrm{trace}(A \circ B^T)$, and $R_{\mathrm{O}(n)}$ is the scalar curvature of $\mathrm{O}(n)$ when K is chosen to be 1. With these choices,

$$\begin{aligned} R^{(a,b,c)} &= \frac{1}{2} a \left(1 - \frac{1}{2} a\right) \sum_{\alpha,\beta,\gamma} c_{\alpha\beta\gamma} c_{\alpha\beta\gamma} + (b (1-c) + c) \sum_{\alpha,i,j} \tau_* (u_\alpha)_{ij} \, \Omega_{\alpha ij} \\ &\quad + \frac{1}{2} \left(c (b-1) - \frac{1}{2} b^2\right) \sum_{i,j,\alpha} \Omega_{\alpha ij} \Omega_{\alpha ij} \\ &= \frac{1}{2} a \left(1 - \frac{1}{2} a\right) \left(R_{G_0} + K^{-1} R_{O(n)}\right) + (b (1-c) + c) R_{g_M} \\ &\quad + \frac{1}{2} \left(c (b-1) - \frac{1}{2} b^2\right) \left(\sum_{i,j,\alpha} \Omega^{\omega_0}_{\alpha ij} \Omega^{\omega_0}_{\alpha ij} + K \sum_{h,k,i,j} \left(R^M_{hkij}\right)^2 \right) \end{aligned} \tag{45}$$

When $(a, b, c) = (1, 1, 1)$, we obtain

$$\begin{aligned} R^{(1,1,1)} &= \left(R_{G_0} - \frac{1}{4} (k_{\mathfrak{g}_0} \otimes g_M) (\Omega^{\omega_0}, \Omega^{\omega_0}) + R_{g_M} \right) \\ &\quad + K^{-1} R_{\mathrm{O}(n)} - \frac{1}{4} K \left\| \mathrm{Riem}_M \right\|^2 . \end{aligned} \tag{46}$$

The terms in parentheses constitute the scalar curvature of P_0 in standard Kaluza–Klein theory. Note that had a been chosen to be 0 or 2 instead of 1, then the constant terms R_{G_0} and $R_{\mathrm{O}(n)}$ in $R^{(a,b,c)}$, which ultimately yield a cosmological constant in Einstein's equations, would be absent. Generally speaking, the scalar curvature function on P can be naturally altered, not only by scaling the scalar product on the group-like fibers, but also by using different G-invariant connections on P. In particular, contrary to popular belief, tiny curled-up fibers do not necessary have to produce huge cosmological constants. There are also the additional terms in $K^{-1}R_{\mathrm{O}(n)} - \frac{1}{4}K\,\|\mathrm{Riem}_M\|^2$. One would not have $K^{-1}R_{\mathrm{O}(n)}$, if a was chosen to be 0 or 2. People have found uses for terms such as $-\frac{1}{4}K\,\|\mathrm{Riem}_M\|^2$, such as in regularizing quantum gravity, but the consensus seems to be that they are a mixed blessing at best.

4. Further directions and speculations

One of the attractive features of working on a soldered principal bundle $\pi : P \to M$ is that it has a natural parallelization depending only on the choice of basis for $\mathfrak{g} \oplus \mathbb{R}^n$. Thus, one has a trivial spin structure, say $\mathrm{Spin}(P) \to F(P)$, associated with this trivialization of the frame bundle $F(P)$, even if the base M has no spin structure. One may then consider spinor fields on P, and we are investigating the harmonic analysis of spinor fields on P. The space of such spinor fields decomposes into invariant subspaces under the group action of G, and these subspaces can be identified with various types of particle fields on M (sections of vector bundles over M, associated with various representations of G). A program of this sort in the Kaluza–Klein context was started in [6]. Recently, a harmonic analysis of spinor fields on circle bundles was carried out in [1].

One tantalizing prospect occurs in the special case where $P = F(M)$, the oriented orthonormal frame bundle of an oriented Riemannian 4-manifold M. Then, $\dim(P) = 4 + \dim(\mathrm{SO}(4)) = 10$, and the group of the bundle $F(P) \to P$ is SO(10), while the group of the composed bundle $\mathrm{Spin}(P) \to F(P) \to P$ is Spin(10). One of the most elegant of the grand unified theories (GUTs) is the SO(10) theory which neatly "explains" some mysterious features of the original SU(5) theory [9]. Any one of the three generations of 16 fundamental fermions (including a right-handed neutrino) fits perfectly in 16-dimensional fundamental spinor representation of Spin(10). Actually, SO(10) GUT is a misnomer, since the fundamental spinor representation of Spin(10) does *not* descend to a single-valued representation of SO(10). Although the group for the frame bundle $\pi : P \to M$ is SO(4) which is regarded as an external symmetry group, it would be nice if the group Spin(10) for the composed bundle $\mathrm{Spin}(P) \to F(P) \to P$ *could* be interpreted as an internal symmetry group. Note that the further composition $\mathrm{Spin}(P) \to F(P) \to P \to M$ is a soldered principal bundle with group $\mathrm{Spin}(10)\times\mathrm{SO}(4)$ (regarded as "internal $\times$ external"). This is exactly what one wants in a grand unified euclidean field theory. Have we GUEFT or goofed?

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