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# ON DUAL LATTICES IN COMPACTIFIED PHASE SPACE*

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*Dedicated to Andrzej Trautman in honour of his 64$^{th}$ birthday*

It is conjectured that space-time and momentum space may be both conformally compactified and correlated by conformal inversion, rendering *a priori* impossible the empirical realization of the concept of both infinity and infinitesimal. It appears that in such a world momentum space is appropriate for the description of quantum mechanics in spinorial form. An exactly soluble, two-dimensional model is presented and discussed.

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## 1. Introduction

The concept of infinity (and of infinitesimal), despite its old age ($\alpha\pi\varepsilon\iota\rho o\nu$: Anaximander, Pythagoras, Aristotle) and its central role both in mathematics and in philosophy has not yet found an universally accepted, selfconsistent, definition and keeps being one of the main sources of difficulties both in mathematics and in physics. According to D. Hilbert, infinity should not exist since there is no empirical evidence of it. Here we will try to bring some arguments in favour of this conjecture.

Some cosmological arguments seem to favour the model of the Universe with positive spatial curvature ($\Omega \geq 1$) corresponding to a closed Universe well represented by the Robertson–Walker space-time:

$$M_{\mathrm{RW}} = S^3 \times \mathbb{R}\,, \tag{1}$$

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compatible with the cosmological principle [1], in which infinite space intervals would not be conceivable. But even stronger arguments in favour of the absence of infinity in space-time may be obtained from the study of conformal symmetry.

In a similar way as Maxwell's equations Lorentz-covariance suggested the familiar structure of Minkowski space-time: $M = \mathbb{R}^{1,3}$, their conformal covariance suggested [2] its conformally compactified structure: $M_c = (S^3 \times S^1)/Z_2$ compatible with $M_{\rm RW}$ of Eq. (1), if $\mathbb{R}$ is interpreted as an infinite covering of $S^1$, subsequently adopted by several authors [3]. Mathematically $M_c$ may be defined as an homogeneous space:

$$M_c = \frac{\mathfrak{G}}{H_1} = \frac{S^3 \times S^1}{Z_2}\,, \tag{2}$$

where $\mathfrak{G} = L \otimes D \otimes\!\!\!\!\!\!\supset P \otimes\!\!\!\!\!\!\supset S$ represents the conformal group (L, D, P, S meaning Lorentz, Dilatation, Poincaré, Special conformal transformations, respectively) and $H_I = L \otimes D \otimes\!\!\!\!\!\!\supset /S$ represents its eleven parameter subgroup. $\mathfrak{G}$ has another eleven parameter subgroup $H_{II} = L \otimes D \otimes\!\!\!\!\!\!\supset /T$ and the homogeneous space

$$P_c = \frac{\mathfrak{G}}{H_I} = \frac{S^3 \times S^1}{Z_2} \tag{3}$$

has the same geometrical structure as $M_c$.

As is well known conformal inversion $I$ transform $T$ in $S$ and vice versa and, therefore

$$I M_c = P_c I\,. \tag{4}$$

If $\mathfrak{G}$ is extended to $G$ by inclusion of $I$, then $M_c$ and $P_c$ are two copies of the same [4] homogeneous space $H$:

$$H = \frac{G}{H_I} = \frac{G}{H_{II}}\,. \tag{5}$$

In previous papers [5] the conjecture was proposed that if the copy $M_c$ of $H$ represents conformally compactified space-time of the closed universe (of which $M_{\rm RW}$ represents a particular realization) then the copy $P_c$ of $H$ represents conformally compactified momentum space.

In this paper we try to bring some further arguments in favour of the conjecture which, if true, would not only eliminate *a priori* the necessity but also the possibility of concrete realization of infinity in space-time in agreement with Hilbert's conjecture, but also in momentum space, or, in the frame of quantum mechanics, it would eliminate the concept of infinitesimal in space-time, which, in fact, would have to be expected in a conformal world without infinity.

It is obvious that this programme, if realizable at all, would imply the necessity of drastic changes in the traditional way of dealing with field theories in such a compact conformal world. In fact, not only integrations extended up to infinity both in space-time and in momentum space world have to be eliminated, but also the classical methods of differential calculus and differential geometry would have to be changed. Furthermore, the traditional infinite sums representing completeness relations would have to be substituted by finite sums.

## 2. Dual lattices

The duality of space-time and momentum space may be derived from Fourier transforms correlating functions taking values in $M = \mathbb{R}^{1,3}$, and $P = \mathbb{R}^{1,3}$ which, in our case have to be substituted by $M_c$ and $R_c$ copies of $H$ as defined in Eq. (5). Therefore, our first task will be to search for transforms between functions taking values on them such that they identify with the standard Fourier transforms in the flat limit (when the radii of $S^3$ and $S^1$ go to infinity). We will show that this programme can indeed be performed but, for the moment, only in the particular, exactly soluble, case of two-dimensional space-time $M = \mathbb{R}^{1,1}$, when $M_c$ and $P_c$, where fields may be defined, restrict to two dual lattices $M_L$ and $P_L$ in two toruses $S^1 \times S^1$, where, however, some of the features anticipated in the previous paragraph already appear, and may be discussed.

For $M = \mathbb{R}^{1,1}$ and $P = \mathbb{R}^{1,1}$ the corresponding conformally compactified spaces are:

$$M_c = \frac{S^1 \times S^1}{Z_2} \text{ and } P_c = \frac{S^1 \times S^1}{Z_2}$$

copies of $H = G/H_I = G/H_{II}$. Suppose now that $R$ is the radius of the $S^1$ in $M_c$ and $K$ is the radius of $S^1$ in $P_c$. If $R$ is a length $K$ is the inverse of a length (remember that $IM_c = P_cI$). Now inscribe in each $S^1$ a regular polygon with

$$2N = 2\pi RK \tag{6}$$

vertices, these define in $M_c$ and $P_c$ two lattices: $M_L \subset M_c$ and $P_L \subset P_c$.
**Proposition 1**. The lattices $M_L$ and $P_L$ are Fourier dual.

In order to prove it we may start by restricting to $S^1$, one point compactification of $x \in \mathbb{R}^1$. On it we may define the arc coordinate $x_n = n/K$ of the lattice defined by the mentioned polygon with $2N$ vertices and indicate a point of it by

$$U_n = Re^{\frac{i}{R}x_n} = Re^{i\frac{n}{RK}} = Re^{i\frac{\pi n}{N}} = R\varepsilon^n ,$$

where $\varepsilon = e^{\frac{i\pi}{N}}$ represents the $2N$-root of unity. The normalized eigenfunctions on the lattice may be defined to be:

$$\langle \rho n \rangle = (2N)^{-\frac{1}{2}} \varepsilon^{n\rho} \tag{7}$$

satisfying the orthonormality condition

$$\sum_{\rho=-N}^{N-1} \langle n\rho \rangle \langle \rho m \rangle = \delta_{nm} \,. \tag{8}$$

If we now introduce $S^1$ compactification of $k$, dual of $x$, and the lattice coordinate $k_\rho = \rho/R$ we easily obtain:

$$f(x_{mn}) = (2\pi)^{-1} R^{-2} \sum_{\rho,\tau=-N}^{N-1} \varepsilon^{(n\rho - m\tau)} F(p_{\rho\tau}) \,,$$

$$F\,(p_{\rho\tau}) \; = (2\pi)^{-1} K^{-2} \sum_{m,n=-N}^{N-1} \varepsilon^{-(n\rho - m\tau)} f(x_{mn}) \,, \tag{9}$$

where

$$x_{mn} = Re^{\frac{i}{R}(t_n - x_m)} \in M_L ; \;\; p_{\rho\tau} = Ke^{\frac{i}{K}(\epsilon_\rho - p_\tau)} \in P_L$$

and $f(x_{mn})$ and $F(p_{\rho\tau})$ indicate functions taking values on $M_L$ and $P_L$, respectively. Eqs (9) are true because of (8) and Proposition 1 is proved.

It is easily seen that in the limit $R \to \infty$ and $K \to \infty$ Eq. (9) become the standard Fourier transforms in two-dimensional Minkowski space-time. This then justifies the name of Fourier transforms adopted for Eq. (9) and Fourier dual for lattices $M_L$ and $P_L$.

Observe that Eq. (8) implies the finite completeness relation:

$$\sum_{\rho=-N}^{N-1} |\rho\rangle\langle\rho| = \mathbb{1}. \tag{10}$$

It is obvious that any field theory on $M_L$ and its dual $P_L$ will be free from both infrared and ultraviolet divergences, one could call them the "physical spaces" to be distinguished from the mathematical space $H = G/H_I = G/H_I$ in which they are contained.

### 3. Role of conformal inversion

In quantum field theory dynamical variables and fields are preferably represented in momentum space $P = \mathbb{R}^{1,3}$ (see $S$-matrix theory). This representation is obtained through Fourier integrals in Minkowski space-time $M = \mathbb{R}^{1,3}$. As such the space $P$ obtained in this way should be more properly called wave-number space. (The actual momentum space is obtained through the adoption, *ad hoc*, of the De Broglie relation between wave number and momentum.)

In our case, besides the Fourier transforms given by Eq. (9) we have also a direct correlation between the dual lattices $M_L$ and $P_L$ (which in the limit will become the usual, continuous spaces $M$ and $P$). In fact, we know that $IM_c = P_c I$, and conformal inversion $I$, represented, in the quotient rational formalism, by

$$I = \begin{vmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{vmatrix} , \tag{11}$$

with $\alpha = (RK)^{1/2}$, transforms every point of $M_L$ in a point of $P_L$ as follows:

$$M_L \ni x_{mn} = Re^{\frac{i}{R}(t_n - x_m)} \to I(x_{mn}) = K^{\frac{-i}{K}(\epsilon_n - p_m)} = p_{mn} \in P_L\,. \tag{12}$$

We have then that to each function $f(x_{mn})$ taking values on $M_L$, there corresponds its Fourier dual $F(p_{mn})$ taking values on $P_L$ defined by Eq. (9), and vice versa, and this is in agreement with the familiar case of flat spaces $M$ and $P$. But now we have also a direct transformation from the configuration lattice $M_L$ to the momentum lattice $P_L$ operated by conformal inversion $I$ by which every point $x_{mn} \in M_L$ is brought to a point $p_{mn} \in P_L$ as shown in (12) and this is new[1]. It may have a meaning that we will try to interpret.

Let us remind that in ordinary Minkowski space-time conformal inversion $I$ operates as follows:

$$x_\mu \to I(x_\mu) = \frac{x_\mu}{x^2}, \qquad \mu = 0, 1, 2, 3\,, \tag{13}$$

which is usually interpreted by affirming that, for $x_\mu$ space-like, every point $P(x)$ inside of the sphere $S^2 : x^2 = 1$, centered in 0, is transformed to a point $P(x^{-1})$, on the same ray form 0, outside $S^2$ at a distance $[x]^{-1}$ from 0. One could then be tempted to call the "inside" and "outside" sectors of $S^2$ in $\mathbb{R}^3$ as "small" and "large", respectively, if it were not that $x_\mu$ has to be dimensionless for (13) to have a meaning (alternatively those words have a

[1] In terms of functions this simply means that, in general, to every function $f$ on $M_L \in M_c$ there corresponds an (identical) function $f$ on $P_L \in P_c$ which is possible since $M_c$ and $P_c$ are two copies of the same homogeneous space (unless $f$ represents a tensor quantity to be transformed by $I$).

meaning only with respect to an arbitrary unit of length $\alpha^2 = L$ appearing in $I$ given in Eq. (11)).

It is easy to see that in our two-dimensional case the above conformal inversion obtained from $I$ given in (11) with $\alpha = 1$ becomes:

$$x \to I(x) = \frac{1}{x} = k \quad \text{and} \quad k \to I(k) = \frac{1}{k} = x \tag{14}$$

which is compatible with (13). This means that conformal inversion transforms every point of configuration space to a point of wave number space.

In order to obtain momentum space, of interest for physics, we need only to adopt $I$ given by (11) with $\alpha = \hbar^{1/2}$ and we obtain

$$x \to I(x) = \hbar k = p \tag{15}$$

and (extended to the four-dimensional case) the sphere $S^2$ has radius $\hbar$ and the above words "large" and "small" would then refer to the product $xp$ referred to the Planck action $\hbar$; but then the "large" would refer to the classical mechanics-world while the "small" to the quantum mechanics-world. In the above hypothesis then conformal inversion would connect the two and since, as astronomy suggests, configuration space is the appropriate space for the description of classical mechanics, momentum space should be appropriate space for the description of quantum mechanics.

We will now try to bring some more arguments, from spinor geometry, on why momentum space should be appropriate for dealing with quantum mechanics.

## 4. Quantum mechanics in momentum space

According to Cartan [6] Euclidean geometry, one of the pillars of classical mechanics, may be bilinearly derived from spinor geometry which should then be considered as the most elementary form of geometry, an opinion shared by many authors.

Consider the $2n$-dimensional pseudo Euclidean space $V = \mathbb{R}^{n-1,n+1}$ and the corresponding Clifford algebra $Cl(n-1, n+1) = \mathrm{End}\, S$, where $S$ is the space of spinors $\varphi \in S$. If $\gamma_\mu (\mu = 1, 2, \ldots 2n)$ are the generators and $\gamma_{2n+1} = \gamma_1, \gamma_2 \ldots \gamma_{2n}$ the volume element of $Cl(n-1, n+1)$ the Cartan equation:

$$p_\mu \gamma^\mu (1 \pm \gamma_{2n+1}) \varphi = 0\,, \tag{16}$$

where $p_\mu \in V$, define the Weyl spinors $\frac{1}{2}(1 \pm \gamma_{2n+1})\varphi = \varphi_\pm$ associated with $V$. For $\varphi \neq 0$ the vectors $p_\mu \in V$ are null and define a projective quadric:

$$PQ(p_\mu p^\mu = 0) = \frac{S^n \times S^{n-1}}{Z_2}\,. \tag{17}$$

For $n = 2$ Eq. (16) gives the most elementary equation of quantum mechanics: the one for massless neutrinos in momentum space, and also Maxwell's equation for the electromagnetic tensor $F_{\mu\nu}$ bilinearly represented in terms of spinors [7] if $\varphi_{\pm}$ are identified as minimal left ideals.

For $n = 3$ Eq. (16) represents twistor's equation [8] in momentum space and (17) the copy $P_c$ of $H$ given by Eq. (3) and representing, in our hypothesis, compactified momentum space. Furthermore one may easily obtain from Eq. (16): the Dirac equation for massive fermions, and, for $n = 4$, the equation for the proton–neutron doublet interacting with the (pseudoscalar) pion triplet [7]. All of them in momentum space.

Now these are notoriously some of the equations which were postulated *ad hoc* for the description of some elementary phenomena of quantum mechanics.

The remaining ones (like Schrödinger's one) may be derived as particular cases of them. Here they are all naturally derived from the fundamental Cartan equation (16) of spinor geometry if $V$ is interpreted as momentum space, Fourier dual of configuration space. This fact might be not an accidental coincidence and, as a consequence of the above arguments, might instead suggest the following scenario:

I. Quantum mechanics is the elementary form of mechanics apt to describe physical systems involving actions comparable with the Planck constant $h$, and its fundamental, elementary equations for fermions (or fermion-fields), may be identified with those of Cartan for spinor geometry in momentum space. Bosons (or boson fields), as well as vectors of momentum space, may be bilinearly constructed from spinors representing fermions (or fermion fields) and in this way also Maxwell equations naturally derive from the elementary ones for fermions.

II. For actions much larger then the Planck constant $h$ Maxwell's equations bring us, through optical geometry, to Euclidean geometry of space-time (bilinearly derivable from spinor geometry), where classical mechanics of macroscopic systems may be properly described (as derived from quantum mechanics).

The correlation between the two forms of mechanics and between the two dual spaces is operated by conformal inversion.

This would then support the conjecture, formulated at the end of the previous paragraph, that momentum space is the appropriate one for the description of (spinor) quantum mechanics.

## 5. Higher spaces

For the realistic case of Minkowski space-time $M = \mathbb{R}^{1,3} = P$ the compactified spaces are $M_c$ and $P_c$ copies of $H$ given by (2) and (3), which densely contain (but for two points) $M$ and $P$. For them a Fourier transform may only be approximated [9]. Exact Fourier transform may be instead obtained if we adopt the compact form:

$$M_c = \frac{S^1 \times S^1 \times S^1 \times S^1}{Z_2} = P_c \tag{18}$$

which also densely contain (but for 4 points) $M$ and $P$ and which may serve for computations since, for realistic radii $R$ and $K$, the lattices $M_L$ and $P_L$ contained in $M_c$ and $P_c$, building up the "physical spaces", should be very dense.

Otherwise, in order to search for a rigorous definition of Fourier transforms in $M_c$ and $P_c$ one should adopt some drastic change of the standard mathematical algorithms. Anyhow, both infrared and ultraviolet convergence in "physical spaces" will be granted, in accordance with Hilbert's conjecture.

## 6. Further consequences

There are several consequences of the scenario proposed above, we will briefly mention a few of them.

Some derive from conformal symmetry, the starting point of our study, conceived as SO(2,4) symmetry. Its maximal compact group of symmetry SO(4) should manifest its presence in both classical- and quantum-non relativistic systems and in fact it does so in planetary orbits and in the hydrogen atom, respectively, and the first in space-time, while the second in momentum space, as well known and, as it should, according to our conjecture.

But there is another fascinating possibility. Recent observations on the distribution of distant galaxies have revealed periodicities (eleven peaks in the direction of North and South galactic poles equally spaced by $\sim 400$ million lights years) [10] which seem to violate the cosmological principle. In fact they could manifest its spontaneous violation due to an eigenvibration of $S^3$ of Robertson–Walker space given by Eq. (1) in consequence of the mentioned SO(4) symmetry. In fact, the up to now known astronomical observations are well explained by an eigensolution of the Laplace–Beltrami equation for $S^3$: the most symmetric Gegenbauer polynomial:

$$Y_{n,\mathrm{o},\mathrm{o}} = C_n^1(\cos\rho) = k_n \frac{\sin(n+1)\rho}{\sin\rho}; \quad n = 46\,, \tag{19}$$

where $\rho$ is the geodesic distance from the center of the eigenvibration on $S^3$ [11].

Should further observations confirm this interpretation then it could constitute a strong support to our hypothesis. In fact, it was shown long ago by Fock [12] that Gegenbauer polynomials given by Eq. (19) are precisely the most symmetric eigenfunctions of the H-atom in stationary states, however, in momentum space. And this is precisely in agreement with our model which implies that conformal inversion correlates configuration space of classical systems and momentum space of quantum systems, and then the Universe and the H-atom would constitute a natural realization of this correlation.

Should conformal symmetry have the important role in physics as conjectured in this paper, then one should expect that both conformal reflections implied by O(2,4) will be of importance; that is in the Clifford Algebra language, both $\gamma_5$ and $i\gamma_6$ generators (remember that if the sixth axis is time-like $i\gamma_6$ is the generator of reflections since the square of a reflection must be the identity). In fact, together with the volume element $\gamma_7$ they build up the algebra SU(2). Now, it appears from the equation representing the nuclear isodoublet interacting with the pion isotriplet obtained from Cartan — equation (16), that those are at the origin of isospin internal symmetry algebra [7].

It was recently shown by Bandyopadhyay and his group [13] that a conformal-reflection origin of internal symmetry can be extended and applied to a soliton model of baryons with results in very good agreement with experimental data.

Notoriously quantum mechanics is very well experimentally verified while its interpretation in configuration space presents well known paradoxical aspects.

Perhaps in momentum space some of those antinomies could be eliminated. As an example a plane wave (or an eigenfunction) is represented by a well defined point (or a function) in momentum space. A position measurement naturally destroys that knowledge in momentum space (the Fourier transform of a delta function is a constant) while in configuration space one has to introduce the concept of wave function-reduction, source of several antinomies.

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