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DEFORMATIONS OF SPIN STRUCTURES AND GRAVITY*

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Dedicated to Andrzej Trautman in honour of his 64th birthday

New variables related to spin-structures are introduced with the aim of replacing the metric in the description of gravity. These new variables provide a general framework which allows one to deal with interactions between spinors and a *dynamical* gravitational field, thus generalizing the notion of spinors on curved spaces. In this framework there is no action of space-time diffeomorphisms on the configuration bundle, but there is covariance with respect to automorphisms of a suitable principal bundle, as it is standard in gauge theories. A concrete example of spinors interacting with gravity is considered as an application.

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1. Introduction

The standard attitude towards spinors on manifolds relies on the possibility of using the notion of spin structures with the purpose of overcoming globality problems in the definition of the Dirac operator (see, [1–7]).

This notion is usually defined by starting explicitly from a given pseudo-Riemannian structure (M, g) over a (spin)-manifold M; a spin structure on (M, g) is then a pair $(\Sigma, \tilde{\Lambda})$, where Σ is a spin bundle, *i.e.* a principal bundle having the appropriate spin group as fiber, while $\tilde{\Lambda}$ is a vertical principal morphism from Σ to the special orthonornal frame bundle on (M, g). This definition raises some interpretation problems about the role of the pseudo-Riemannian structure chosen on M. In fact, in order to define spinors following the classical definitions one has to *fix* the metric. This is viable if

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one is dealing with spinor physics on a (curved) background. However, in a truly dynamical theory of spinors interacting with gravity, the metric g *cannot* be fixed since it is the gravitational field and, as such, has yet to be determined through field equations themselves.

As we shall see, in fact, the definition of a spin structure $(\Sigma, \tilde{\Lambda})$ on (M, g) strongly depends in a far not canonical way on the pseudo-Riemannian structure chosen. As a result, we claim that the standard attitude allows a satisfactory description of spinors only if g is interpreted as a gravitational field acting on spinors but completely uneffected by them. The standard situation can be described in the following way:

$$(M, g) \quad \rightsquigarrow \quad \text{Dirac equations} \quad \rightsquigarrow \quad \text{Spinor fields on } (M, g)\,. \tag{1.1}$$

As we said, this picture is not completely meaningful, since we expect that also spinors (as every other physical field) should effect the gravitational field. Then, according to General Relativity, one would prefer to start from a manifold M without any metric structure and to determine, by means of field equations, *both* spinor fields *and* the metric structure; *i.e.* to have a picture of the following kind:

$$M \quad \rightsquigarrow \quad \begin{array}{l}\text{Einstein+Dirac}\\ \text{equations}\end{array} \quad \rightsquigarrow \quad \begin{array}{l}\text{Spinor fields } \textit{and}\\ \text{a metric } g \text{ on } M\,.\end{array} \tag{1.2}$$

Of course, the standard framework on (M, g) is physically important because it clarifies that spinors, even if dynamical, cannot exist *without* metrics and moreover they should be *compatible* (in a suitable sense) with the metrics themselves. However, our stringent physical requirement that the metric cannot be fixed before spinors are defined (since in this case the metric itself would remain *frozen* against General Relativity), obliges us to look for another perspective in which spinors and metrics are *both* and at the same time dynamically free. In other words, *metric cannot come before spinors.* Surprisingly enough, we have been able to realize that the viewpoint can in fact be completely overturned, in the sense that it is possible to coherently deal with spinors interacting dynamically with a dynamical metric by giving first spin-theoretical objects and *deduce* dynamical metrics as by-products.

To be more precise, we shall propose new variables related to spin structures rather than metrics as natural candidates to describe gravitational fields. As it happens in gauge theories, these new variables are not natural objects. This means that in general there is no preferred $\mathrm{Diff}(M)$-action on them, but just an action of automorphisms of a suitable principal bundle which encodes the symmetry structure and the conserved quantities of the theory itself. Our viewpoint will in fact belong to the so-called *gauge-natural* framework, which is mathematically analogous to the standard framework of Yang-Mills theories although it also encompasses theories which are *generally covariant* in the standard sense. Because of this, in our framework a question like: *how spinors transform under a diffeomorphism of space-time?* is more or less meaningless, at least in general, unless one decides to break the gauge-covariance by fixing some preferred (and *non-canonical*) way of extending the diffeomorphisms of M to automorphisms of the gauge bundle. However, this particular question will be shown to be still meaningful in some particular case: *e.g.*, on 4-dimensional, Lorentzian, non-compact spin manifolds, which, by the way, encompass most of the *reasonable* General Relativistic examples.

2. Standard formalism for spinors

First of all, let us recall the standard framework in which spinors are defined. Let M be a connected, paracompact, orientable, C^∞–manifold of finite dimension m. Let (M, g) be a pseudo-Riemannian structure of signature $\eta = (r, s)$ on M ($r + s = m = \dim(M)$). As is well known, some topological requirement has to be satisfied by M in order to allow the existence of pseudo-Riemannian metrics of the given signature η and the topological obstruction depends on the signature. In the strictly Riemannian signature, *i.e.* $\eta = (m, 0)$ or $\eta = (0, m)$, there is no obstruction at all since every paracompact manifold allows definite metrics; in all other cases a metric of signature $\eta = (r, s)$ is known to exist if and only if there is a reduction of the tangent bundle TM to the group $O(r) \times O(s) \subset \mathrm{GL}(m)$.

Throughout this paper, we shall assume that such a topological obstruction, if any, is satisfied by M, as it is *quite reasonable* from a physical viewpoint since we shall interpret M as describing the physical space-time.

For reasons which will be clear in the sequel, we say that M is a *spin manifold* if its second Stiefel-Whitney class vanishes, *i.e.* $w_2(M, Z_2) = 0$.

A *spin structure on* (M, g) is a pair $(\Sigma, \tilde{\Lambda})$, where:

(i) Σ is a *spin bundle*, namely a principal bundle having $\mathrm{Spin}(\eta)$ as structure group;

(ii) $\tilde{\Lambda} : \Sigma \longrightarrow \mathrm{SO}(M, g)$ is a vertical principal epimorphism from Σ to the special orthonormal frame bundle of (M, g), related to the group covering map $\ell : \mathrm{Spin}(\eta) \longrightarrow \mathrm{SO}(\eta)$. Namely, $\tilde{\Lambda}$ makes the following diagrams commutative:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\tilde{\Lambda}} & \mathrm{SO}(M,g) \\ \downarrow & \swarrow & \\ M & & \end{array} \qquad\qquad \begin{array}{ccc} \Sigma & \xrightarrow{\tilde{\Lambda}} & \mathrm{SO}(M,g) \\ {\scriptstyle R_S}\downarrow & & \downarrow{\scriptstyle R_{\ell(S)}} \\ \Sigma & \xrightarrow{\tilde{\Lambda}} & \mathrm{SO}(M,g) \end{array} \tag{2.1}$$

Here and hereafter if $\mathcal{P} = (P, M, \pi, G)$ is a principal bundle having G as structure group, then $R_h : P \longrightarrow P$ $(h \in G)$ denotes the canonical right action of G on P.

If we want to define a spin structure $(\Sigma, \tilde{\Lambda})$ on (M, g) we have to be careful in our choice of the spin bundle Σ. In fact, in general, there are spin bundles for which there is no global morphism $\tilde{\Lambda} : \Sigma \longrightarrow \mathrm{SO}(M, g)$: *e.g.*, if M is not parallelizable and Σ is chosen to be the trivial bundle $\Sigma = M \times \mathrm{Spin}(\eta)$, then obviously there is no such $\tilde{\Lambda}$. Whenever a spin bundle Σ admits a spin structure $(\Sigma, \tilde{\Lambda})$ on (M, g) we say that Σ is a *structure bundle (for the Riemannian structure* (M, g) *chosen)*.

It is however known (see, *e.g.* [1, 3, 8]) that spin structures always exist on spin manifolds, *i.e.*, if M is a spin manifold and (M, g) is a pseudo-Riemannian structure on it, then there exists a suitable spin bundle Σ on M such that global morphisms $\tilde{\Lambda} : \Sigma \longrightarrow \mathrm{SO}(M, g)$ exist. The proof of this result is, by some extent, constructive. One starts in fact from the pseudo-Riemannian structure (M, g) and defines the special orthonormal frame bundle $\mathrm{SO}(M, g)$ by relying on the hypothesis that M is orientable. Then one chooses a trivialization of $\mathrm{SO}(M, g)$ together with its transition functions $\gamma_{(\alpha\beta)} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(\eta)$ which have values in the special orthogonal group $\mathrm{SO}(\eta)$. To define the spin bundle, this $\mathrm{SO}(\eta)$-cocycle must be lifted to the spin group $\mathrm{Spin}(\eta)$, and this is shown to be possible if and only if the second Stiefel-Whitney class of M vanishes. Furthermore, there are many inequivalent ways of performing such a lift, namely one for each choice of a cocycle $\delta \in H^1(M, Z_2)$ in the Cěch cohomology of M. A principal bundle $\Sigma(M, g; \delta)$ then arises which has the lifted cocycle $G_{(\alpha\beta)}$ as transition functions (see, *e.g.* [1, 3, 8]).

$$\begin{array}{ccc}
(M,g) & & \\
\downarrow & & \\
\mathrm{SO}(M,g) & \longrightarrow & \gamma_{(\alpha\beta)} : U_{\alpha\beta} \longrightarrow \mathrm{SO}(\eta) \\
 & & \Big\downarrow \begin{smallmatrix} w_2(M,Z_2)=0 \\ \delta\in H^1(M,Z_2) \end{smallmatrix} \\
\Sigma(M,g;\delta) & \longleftarrow & G_{(\alpha\beta)} : U_{\alpha\beta} \longrightarrow \mathrm{Spin}(\eta)
\end{array} \tag{2.2}$$

Building $\Sigma(M, g; \delta)$ in this way, a global morphism $\tilde{\Lambda}_g : \Sigma(M, g; \delta) \longrightarrow \mathrm{SO}(M, g)$ is automatically defined, which acts on fibers by the covering map ℓ. Thus $(\Sigma(M, g; \delta), \tilde{\Lambda}_g)$ is a spin structure on (M, g) and of course it depends strongly both on g and on the possible choice of δ.

If we want now g to be deformed, as it is necessary in the calculus of variations, one obtains in principle inequivalent spin bundles $\Sigma(M, g; \delta)$, different morphisms and, of course, different orthonormal frame bundles (unless the deformation allowed is of very special character). Thus, if general deformations of spin structures are to be defined, one should first say what is meant by continuous (or, even better, C^∞-) deformations of a bundle, which is a difficult task and, above all, it cannot generally be done in a canonical way.

The general attitude in current literature is thence to somehow restrict the possible deformations of spin structures, so that the bundles involved do not actually change, leading to *trivial deformations* of the underlying metric (*e.g.*, Killing symmetries).

3. Spin frames on Σ

The framework we propose relies instead on the possibility of "changing the game-rules" so that, letting bundles unchanged, one still has non-trivial deformations of spin structures which correspond to *actual* deformations of the underlying metric.

From now on we shall fix once for all a signature $\eta = (r, s)$ and assume that M is a spin manifold which satisfies the topological conditions which ensure the existence of a metric of signature η (shortly, a *η-manifold*). Since M is a spin manifold we can always choose a spin bundle Σ so that at least one morphism $\tilde{\Lambda} : \Sigma \longrightarrow \mathrm{SO}(M, g)$ exists for some metric g of the given signature η, *i.e.* that there exists at least a pseudo-Riemannian structure (M, g) for which Σ is a structure bundle. We call such a bundle a *structure bundle for M* and we *forget* the metric. Let us then fix *any* structure bundle Σ for M, without assuming any particular pseudo-Riemannian structure.

Definition: a *spin frame on Σ* (see also [9, 10]) is a vertical principal morphism $\Lambda : \Sigma \longrightarrow L(M)$, *i.e.* a morphism for which the following diagrams commute:

$$\begin{tikzcd} \Sigma \arrow[r, "\tilde{\Lambda}"] \arrow[d] & L(M) \arrow[dl] \\ M & \end{tikzcd} \qquad \begin{tikzcd} \Sigma \arrow[r, "\tilde{\Lambda}"] \arrow[d, "R_S"'] & L(M) \arrow[d, "R_{\hat{\ell}(S)}"] \\ \Sigma \arrow[r, "\tilde{\Lambda}"] & L(M) \end{tikzcd} \tag{3.1}$$

where $\hat{\ell} = i \circ \ell : \mathrm{Spin}(\eta) \longrightarrow \mathrm{SO}(\eta) \longrightarrow \mathrm{GL}(m)$.

Let us remark that the Definitions (2.1) and (3.1) are very similar; indeed if $(\Sigma, \tilde{\Lambda})$ is a spin structure on (M, g) and $i_g : \mathrm{SO}(M, g) \longrightarrow L(M)$ is the canonical immersion, then $\Lambda_g = i_g \circ \tilde{\Lambda} : \Sigma \longrightarrow L(M)$ is a spin frame on Σ. Conversely, if $\Lambda : \Sigma \longrightarrow L(M)$ is a spin frame on Σ, there is one and only one metric $g(\Lambda)$, called the *metric induced by Λ*, such that the frames in $\mathrm{Im}(\Lambda) \subset L(M)$ are the only $g(\Lambda)$-orthonormal frames. Then Λ factorizes through the canonical immersion $i_{g(\Lambda)} : \mathrm{SO}(M, g(\Lambda)) \longrightarrow L(M)$ as follows:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\Lambda} & L(M) \\ & \underset{\tilde{\Lambda}}{\searrow} & \Big\downarrow i_{g(\Lambda)} \\ & & \mathrm{SO}(M, g(\Lambda)) \end{array} \tag{3.2}$$

and $(\Sigma, \tilde{\Lambda}(\Lambda))$ is a spin structure on $(M, g(\Lambda))$.

Accordingly, spin structures and spin frames on the same Σ are in one-to-one correspondence; however, they are deeply different as far as their relations to the pseudo-Riemannian structures are concerned. In fact, spin structures are defined *with respect to* a given pseudo-Riemannian manifold (M, g); while a spin frame Λ is defined with respect to a given spin bundle Σ, which, in a sense, is a much weaker structure than a metric, but it *induces* uniquely a metric, which is then automatically *compatible* with the spin structure $(\Sigma, \tilde{\Lambda}(\Lambda))$ induced as in (3.2).

It is precisely because of these differences that one should decide to use spin frames instead of metrics to describe gravity. However, if we want to use spin frames as dynamical fields they must have some *value* at each space-time point, *i.e.* they have to be identified with sections of some suitable bundle over M. This can be done by considering the following action on $\mathrm{GL}(m)$:

$$\rho : \mathrm{Spin}(\eta) \times \mathrm{GL}(m) \times \mathrm{GL}(m) \longrightarrow \mathrm{GL}(m) : (S, J, e) \mapsto J \cdot e \cdot \ell(S^{-1}) \tag{3.3}$$

and using it to construct the bundle Σ_ρ which is associated through the representation ρ to the principal bundle $\Sigma \times_M L(M)$. It can be shown that for each structure bundle Σ there is a $1-1$ correspondence between spin frames on Σ and sections on Σ_ρ, as well as a canonical vertical epimorphism $g : \Sigma_\rho \longrightarrow \mathrm{Met}(M;\eta)$ onto the bundle $\mathrm{Met}(M;\eta)$ of all metrics having signature η on M. This morphism is called *the inducing metric morphism.*

Local coordinates on Σ_ρ are (x^μ, e^μ_a), with $e^\mu_a \in \mathrm{GL}(m)$. If we choose a family of local sections $\sigma^{(\alpha)}$ of Σ and a family of local sections $\partial^{(\alpha)}$ of $L(M)$, both inducing trivializations on Σ and $L(M)$ respectively, then a point in Σ_ρ is of the form $[\sigma^{(\alpha)}, \partial^{(\alpha)}, e^\mu_a]_\rho$. Being $(x^\mu, g_{\mu\nu})$ local coordinates on $\mathrm{Met}(M;\eta)$ the inducing metric morphism is then given by:

$$g_{\mu\nu} = \bar{e}^a_\mu \, \eta_{ab} \, \bar{e}^b_\nu \,, \tag{3.4}$$

where $\bar{e}^a_\mu$ is the inverse matrix of e^μ_a and η_{ab} is the canonical diagonal matrix of signature η.

This construction of the bundle Σ_ρ is a particular case of a more general construction defining so-called *gauge-natural bundles associated to a* *principal bundle* (see [11]); as in the general theory, if $\Phi \in \mathrm{Aut}(\Sigma)$ is an automorphism of Σ that projects over a diffeomorphism $f : M \longrightarrow M$, then it induces an automorphism of Σ_ρ defined by:

$$\Phi_\rho : \Sigma_\rho \longrightarrow \Sigma_\rho : [\sigma^{(\alpha)}, \partial^{(\alpha)}, e^\mu_a]_\rho \mapsto [\Phi(\sigma^{(\alpha)}), L_f(\partial^{(\alpha)}), e^\mu_a]_\rho \,, \tag{3.5}$$

where $L_f : L(M) \longrightarrow L(M)$ is the natural lift of f to the frame bundle.

If we consider a 1-parameter subgroup Φ_t of automorphisms of the structure bundle Σ which projects onto a flow f_t on M, then we obtain a 1-parameter subgroup of automorphisms $(\Phi_\rho)_t$ which can be used to drag a section $\Lambda : M \longrightarrow \Sigma_\rho$ by the following prescription:

$$\Lambda_t = (\Phi_\rho)_t \circ \Lambda \circ f_t^{-1} \tag{3.6}$$

which defines a *deformation* of the spin frame corresponding to Λ. It corresponds to the natural deformation of the induced metric along the flow f_t which is, in general, a non-trivial deformation.

4. Dynamics for spin frames

Choosing dynamics, *i.e.* a Lagrangian, for spin frames, we restrict ourselves to consider Lagrangians which are covariant with respect to automorphisms of Σ acting on the configuration bundle Σ_ρ via the action (3.5). This is the reason why the gauge-natural structure of Σ_ρ is essential for our purposes.

Our formalism has been defined to be as *conservative* as possible. One can in fact pull-back the usual dynamics for a metric g along the inducing metric morphism up to Σ_ρ. Since usual metric Lagrangians, such as the Hilbert Lagrangian, are generally covariant, *i.e.* covariant with respect to $\mathrm{Diff}(M)$[1], then the pull-back Lagrangian on Σ_ρ will be automatically $\mathrm{Aut}(\Sigma)$-covariant.

The inducing metric morphism is canonical, so that this mechanism provides a canonical way to describe General Relativity by means of spin frames. Since symmetries of the theory are completely encoded by the structure of the group $\mathrm{Aut}(\Sigma)$ and its action (3.5) on the configuration bundle Σ_ρ, then, by Nöther's theorem, also conserved currents are completely determined. In this framework, as it happens in fact in every gauge-natural theory, currents are forms which are not only conserved, *i.e.* they are closed along solutions, but they are even exact along solutions; thus they always admit superpotentials in the sense of [12] and [13].

Turning then back to our case, as it will be shown below vertical automorphisms of Σ do not contribute to conserved quantities at all, since they

[1] Here the natural structure of $\mathrm{Met}(M;\eta)$ is used.

give as a result *conserved* currents which vanish identically, not only along solutions but along *every* section of Σ_ρ. This is an important coherence check since if they generated non-trivial conserved quantities, we would not be able to interpret them in a classical pure gravity framework.

5. Choosing the structure bundle Σ

Our approach can thence be schematically described as follows:

$$(M,\Sigma) \rightsquigarrow \begin{array}{l}\text{Einstein+Dirac}\\ \text{equations}\end{array} \rightsquigarrow \begin{array}{l}\text{Spin frames} \rightsquigarrow \text{a metric on } M\\ \text{Spinors}\end{array}$$

which is something in between the standard framework (1.1) and what we were looking for, namely (1.2). However, the choice of the bundle Σ is essential to us, and we aim here to discuss the implications of our choice. An analysis can be carried over with respect to the metrics on M which can be obtained as metrics induced by some spin frame on Σ. A metric g is called *Σ-admissible* if there exists a spin frame Λ on Σ which induces g. Then there are two mutually exclusive *a priori* possibilities:

(A) all metrics on M are Σ-admissible;

(B) there is at least one metric on M which is not Σ-admissible.

We remark that under our assumptions, the spin bundle is chosen so that at least one spin frame Λ on Σ exists; thence there is always at least one metric, namely $g(\Lambda)$, which is Σ-admissible.

Both possibilities correspond in fact to real situations, namely depending on the topology of M and on the signature η either (A) or (B) can hold true.

For example, on a (strictly) Riemannian manifold M, one can use Gram-Schmidt orthonormalization procedure to build a vertical principal isomorphism of orthonormal frame bundles $\mathrm{SO}(M,g) \longrightarrow \mathrm{SO}(M,g')$ of any two (Riemannian) metrics g and g'. This shows that, if g is Σ-admissible, then also g' is Σ-admissible; thence every metric on M is Σ-admissible. Hence (A) holds in the positive (or negative) definite signature.

Another example for case (A) is represented by the class of all 4-dimensional, Lorentzian, non-compact spin manifolds (M,g). Under these hypotheses, in fact, a theorem of Geroch holds (see [14]), stating that, for every spin structure $(\Sigma,\tilde{\Lambda})$ on (M,g), the structure bundle Σ is compelled to be the trivial bundle and thence also $\mathrm{SO}(M,g)$ is trivial, thus implying that M is parallelizable, too. Again we have an isomorphism $\mathrm{SO}(M,g) \longrightarrow \mathrm{SO}(M,g')$ for any two (Lorentzian) metrics g and g' on M, so that if g is Σ-admissible then also g' is Σ-admissible.

But there are also simple η-manifolds in which (B) holds. Consider as an example the manifold $M = \mathbb{R}^3 - \{0\} \simeq \mathbb{R}^+ \times S^2$, viewed as a submanifold of $\mathbb{R}^3$ through the canonical injection $j : M \longrightarrow \mathbb{R}^3$. Choose $\eta = (1, 2)$ and consider the following two metrics on M:

- the first metric g_1 is the pull-back through j of the Minkowski metric on $\mathbb{R}^3$;

- the second metric is $g_2 = dr \otimes dr - r^2 d\Omega$, where r is the radial coordinate and $r^2 d\Omega$ is the standard metric on spheres of constant radius. In this metric the *radial* versors are timelike versors while the versors *tangent* to spheres of constant radius are spacelike.

Let us also choose Σ to be the trivial structure bundle $\Sigma_T = M \times \mathrm{Spin}(1, 2)$. Then it can be easily proved that g_1 is Σ_T-admissible, while g_2 is not by simply remarking that S_2 has a non-trivial Euler characteristic.

From the physical viewpoint our formalism is very similar to the framework of gauge theories; and it is precisely this analogy which shows the way to interpret theories based on spin frames. When (B) holds we have more than one available choice for Σ, each one selecting a class of Σ-admissible metrics on M. If we want to obtain a metric which does not belong to this class, we simply have to change Σ. It is exactly the same thing that happens in gauge theories; imagine we are interested in a Maxwell theory on the 2-sphere. There we can choose several possible $U(1)$-bundles as gauge bundles, *e.g.* the trivial bundle $B_T = S^2 \times U(1)$ or the Hopf bundle $B_H = S^3$. If we choose B_T as *structure bundle*, our solutions have then to satisfy certain boundary conditions, which allow the extension to conformal infinity $\infty \in S^2$; on the other hand, monopole solutions satisfy *other* boundary conditions which correspond instead to the choice of B_H as *structure bundle*.

6. Comparison with another framework

For the sake of completeness we shall discuss here an alternative perspective on deformations of spin structures due to Dąbrowski and Percacci (see [15, 16]), which relies on the double coverings of the whole frame bundle $L(M)$. Let us denote by $\tilde{L}(M) \longrightarrow L(M)$ any one of such 2-fold coverings. They exist if M is an orientable spin manifold, but uniqueness is not achieved in general, the construction depending on the choice of a cocycle $\delta \in H^1(M, Z_2)$ in the Cěch cohomology of M, as it also happens for ordinary spin structures.

Now, if metrics of signature η are required to exist on M, then for each such metric g one can define an ordinary spin structure by pulling back along the canonical immersion $i_g : \mathrm{SO}(M,g) \longrightarrow L(M)$:

$$\begin{tikzcd} \tilde{L}(M) \arrow[d] & \Sigma_g \arrow[l] \arrow[d, "\tilde{\Lambda}_g"] \\ L(M) & \mathrm{SO}(M,g) \arrow[l, "i_g"'] \end{tikzcd} \tag{6.1}$$

It is then clear that the choice of a covering $\tilde{L}(M) \longrightarrow L(M)$ corresponds to a recipe to associate (somehow *canonically*) a spin structure to a metric. In fact, metrics correspond to equivalence classes of spin structures under a suitable definition (see [15, 16]).

This is thence another way to overcome the problems in defining the deformations of spin structures. In fact, since we now have a canonical recipe to build a spin structure out of a metric on M, whenever we have a deformation of the metric, a deformation of spin structures is induced. This approach is, in some sense, more general than ours because it allows diffeomorphisms to act on spin structures. In fact, if $f : M \longrightarrow M$ is a diffeomorphism and g a metric on M, then we can drag g along f obtaining a new metric $g' = f_* g$. Having fixed $\tilde{L}(M) \longrightarrow L(M)$ once for all, we have a canonical way to associate a spin structure to any metric and, in particular, we have commutative diagrams:

$$\begin{tikzcd} \tilde{L}(M) \arrow[d] & \Sigma_g \arrow[l] \arrow[d, "\tilde{\Lambda}_g"] \\ L(M) & \mathrm{SO}(M,g) \arrow[l, "i_g"'] \end{tikzcd} \qquad \begin{tikzcd} \tilde{L}(M) \arrow[d] & \Sigma_{g'} \arrow[l] \arrow[d, "\tilde{\Lambda}_{g'}"] \\ L(M) & \mathrm{SO}(M,g) \arrow[l, "i_{g'}"'] \end{tikzcd} \tag{6.2}$$

Thus we can define the action of f on spin structures to be:

$$f_* : (\Sigma_g, \tilde{\Lambda}_g) \mapsto (\Sigma_{g'}, \tilde{\Lambda}_{g'})\,. \tag{6.3}$$

To recover our formalism from this framework, we can consider spin structures that can be obtained on a fixed structure bundle Σ and, by composition with the immersion, consider the spin frame $\bar{\Lambda}_g = i_g \circ \tilde{\Lambda} : \Sigma \longrightarrow L(M)$

$$\begin{tikzcd} \tilde{L}(M) \arrow[d] & \Sigma \arrow[l] \arrow[d, "\tilde{\Lambda}"] \arrow[dl] \\ L(M) & \mathrm{SO}(M,g) \arrow[l, "i_g"'] \end{tikzcd} \tag{6.4}$$

With these restrictions, we are now able to make automorphisms of Σ act on spin frames, as shown above. Then, in some sense, our framework is not included in this one, since the gauge-natural structure cannot be derived from the action of diffeomorphisms, but has to be super-imposed.

On the other hand, the gauge-natural structure is essential to solve another peculiarity of Dirac theory of spinors, namely the fact that one needs a vielbein to write down the Dirac Lagrangian in order to transform *world indices* into *vielbein indices* and viceversa. But, from a geometrical point of view, a vielbein is a local object being a local section of $L(M)$. Thence, by definition, a global vielbein exists just on parallelizable manifolds. Since a vielbein has to be considered as a dynamical field, *i.e.* it should be varied to get field equations, and thence has to be continuous, one is then compelled to work just on parallelizable manifolds, which is a far too strong requirement for space-times.

As we have seen, thanks to the gauge-naturality of the spin frame bundle, global spin frames exist instead on every spin manifold and they can be used as well to transform *world indices* into *vielbein indices* (and viceversa) and to write down the Dirac Lagrangian. Our framework is thence suitable to describe interactions between gravity and spinors, without any limitations on dimension and signature, and on fairly general (*i.e.* not necessarily parallelizable) spin manifolds.

As we have seen, there are *lights and shades* in both approaches, which unfortunately do not seem to us to be encompassable by a larger theory which includes both.

7. Conclusions

If one analyses General Relativity as a standalone theory without any interaction with spinors (possibly allowing also Bosonic matter and gauge theories) our framework is an exactly equivalent way of describing an older matter; in fact, it can be shown that, because of covariance requirements, the Lagrangian dependence on spin frames factorizes through the inducing metric morphism.

Differences arise instead when coupling gravity in a dynamical way to spinors, as spinors Lagrangians explicitly depend on spin frames as well as on the associated metric. In these cases (and we believe this is a general feature) spin frames are *physically different* variables and we claim they are geometrically and globally well defined candidates to replace a vielbein.

Appendix A

An example

Let us consider a 4-dimensional manifold M which allows metrics of signature $\eta = (1,3)$. We choose Dirac matrices as

$$\gamma_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{A.1}$$

which induce a representation λ of $\mathrm{Spin}(1,3)$ on $\mathbb{C}^4$. Set $\Sigma_\lambda = \Sigma \times_M \mathbb{C}^4$ to be the associated vector bundle. Local coordinates on Σ_λ are (x^μ, v^i), while (x^μ, e^μ_a) are local coordinates on Σ_ρ. We can define covariant derivatives as follows

$$\nabla_a v = e^\mu_a(d_\mu v - \tfrac{1}{8}[\gamma_c, \gamma_d] v \Gamma^{cd}_\mu)\,. \tag{A.2}$$

We also set $\bar{v} = v^\dagger \gamma_0$, where $v^\dagger$ denotes the transpose conjugate matrix.

Let us consider the following Lagrangian

$$L = \Big(\frac{1}{2\kappa} R + \frac{i}{2}\bar{v}\gamma^a \nabla_a v - \frac{i}{2}\nabla_a \bar{v}\gamma^a v - m\bar{v}v\Big)\sqrt{g}\, ds = (\mathcal{L}_H + \mathcal{L}_D)\sqrt{g}\,\boldsymbol{ds}, \tag{A.3}$$

where $\boldsymbol{ds} = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ is the local volume of space-time, R is the scalar curvature of the associated metric $g_{\mu\nu}$ and $\sqrt{g}$ is the square root of the absolute value of the determinant of the metric.

Under these assumptions we have the following field equations:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2\kappa H_{\mu\nu} \\ i\gamma^a \nabla_a v = m v\,, \end{cases} \tag{A.4}$$

where the tensor density $H_{\mu\nu}$ is the Hilbert stress tensor. It controls gravitational sources (see [5]) and it takes here the following form:

$$\begin{aligned} H_{\rho\sigma} \;=\; & -\frac{1}{4}(i\gamma^a \nabla_a \bar{v} + m\bar{v})\Big(\gamma_\rho\gamma_\sigma - 3g_{\rho\sigma}\Big)v \\ & +\frac{1}{4}\bar{v}\Big(\gamma_\sigma\gamma_\rho - 3g_{\sigma\rho}\Big)(i\gamma^a\nabla_a v - mv) \\ & +\frac{i}{2}\Big(\nabla_{(a}\bar{v}\gamma_{b)}v - \bar{v}\gamma_{(b}\nabla_{a)}v\Big)e^a_\rho e^b_\sigma - \frac{m}{2} g_{\rho\sigma}\bar{v}v\,, \end{aligned} \tag{A.5}$$

where we have set $\gamma_\mu = \gamma_a \bar{e}^a_\mu$.

If we consider a family of automorphisms of Σ given by

$$\begin{cases} x' = f(x) \\ S' = \varphi(x) \cdot S \end{cases} \tag{A.6}$$

and we denote by Ξ its infinitesimal generator, then we get the following conserved current

$$\begin{aligned} \mathcal{E}(L,\Xi) &= \mathcal{E}_1 + \mathcal{E}_2 \,, \\ \mathcal{E}_1 &= \frac{\sqrt{g}}{2\kappa}\Big[\Big(\frac{3}{2}R^{\lambda}_{\cdot\,\nu} - R\delta^\lambda_\nu\Big)\xi^\nu + \Big(g^{\sigma\mu}\delta^\lambda_\nu - g^{\lambda(\mu}\delta^{\sigma)}_\nu\Big)\nabla_{\sigma\mu}\xi^\nu\Big]\, \boldsymbol{ds}_\lambda \,, \\ \mathcal{E}_2 &= \Big[\Big(\frac{i}{2}\nabla_a\bar{v}\gamma^\sigma v - \frac{i}{2}\bar{v}\gamma^\sigma\nabla_a v - \mathcal{L}_D e^\sigma_a\Big)\bar{e}^a_\lambda\xi^\lambda \\ &\quad + \frac{i}{4}\bar{v}\Big(-\gamma^\lambda\gamma^{[\mu}\gamma^{\sigma]} + 2g^{\lambda[\mu}\gamma^{\sigma]}\Big)v\nabla_\mu\xi_\lambda\Big]\sqrt{g}\,\boldsymbol{ds}_\sigma \,, \end{aligned} \tag{A.7}$$

where $\boldsymbol{ds}_\lambda = i_{\partial_\lambda}\boldsymbol{ds}$ and i_{∂_λ} denotes the contraction with the vector ∂_λ.

This form is not only conserved, *i.e.* closed along solutions, but also exact along solution. It allows then superpotentials

$$\mathcal{U}(L,\Xi) = \frac{1}{2}\Big[\frac{1}{2\kappa}(\nabla^\nu\xi^\mu - \nabla^\mu\xi^\nu) + \frac{i}{8}\bar{v}(\gamma^{[\mu}\gamma^{\nu]}\gamma^\rho + 2g^{\rho[\mu}\gamma^{\nu]})\xi_\rho v\Big]\sqrt{g}\,\boldsymbol{ds}_{\mu\nu} \,, \tag{A.8}$$

where $\boldsymbol{ds}_{\mu\nu} = i_{\partial\nu}\,\boldsymbol{ds}_\mu$. The gravitational part of $\mathcal{U}(L,\Xi)$ reproduces the well known superpotential of Komar (see [12]) while the additional part is responsible for spinorial currents.

In a region D of space-time, conserved quantities along a solution σ are thence defined as follows:

$$Q_D(L,\Xi,\sigma) = \int_{\partial D} (j^1\sigma)^*\,\mathcal{U}(L,\Xi) \,, \tag{A.9}$$

where $j^1\sigma$ denotes the prolongation of σ.

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