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QUANTUM MECHANICS OF THE ELECTRIC CHARGE* **

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Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

The Coulomb field of the proton in a hydrogen atom is a completely classical object. We know it from the success of the Dirac equation in which the classical Coulomb field is put in. However, the proton's charge, which gives the scale of the Coulomb field, is quantized. Thus the Coulomb field behaves like Bohr's orbits in the old quantum theory: its spatio-temporal shape is classical but its magnitude is quantized. The Author explains this curious state of affairs. There are two distinct regimes of the electromagnetic field: the regime described by the standard quantum electrodynamics and zero-frequency regime, which is translationally invariant and has only the Lorentz group as its symmetry group. The electric charge is a part of the translationally invariant zero-frequency regime and as such can indeed be quantized.

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1. Introduction

The Coulomb field of the proton in a hydrogen atom is a completely classical object. We know it. The theory of the hydrogen atom is obtained by putting the classical Coulomb field of the proton into the Dirac equation. The result is in perfect agreement with observations, which shows that the purely classical treatment of the Coulomb field is indeed justified. Moreover, we have a quantitative measure of accuracy with which the purely classical treatment of the Coulomb field reproduces the observed spectrum. On the other hand, the total charge of the proton is equal to the elementary charge

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i.e. it is quantized; a certain parameter of a classical object, namely its scale, is quantized. This is a very curious state of affairs. Consider, for example, a car factory, which produces cars having only one length; the length is strictly correlated with a natural phenomenon apparently completely unrelated to the process of car making. Everyone would agree, I suppose, that such a fact calls for an explanation.

2. The inequality of Berestetsky, Lifshitz, and Pitaevsky

Berestetsky, Lifshitz, and Pitaevsky [1] say that the electromagnetic field $F_{\mu\nu}$ is approximately classical if ($\hbar = 1 = c$)

$$\sqrt{F_{01}^2 + F_{02}^2 + F_{03}^2}\, (\Delta x^0)^2 \gg 1\,,$$

where Δx^0 is the observation time over which the field can be averaged without being significantly changed. For a static field this time is obviously infinite and therefore, conclude Berestetsky, Lifshitz, and Pitaevsky, *a static field is always classical.* This conclusion explains the classical nature of the Coulomb field in atoms but, at the same time, makes the phenomenon of charge quantization even more mysterious: where the Planck constant $\hbar$ comes from into the expression for the elementary charge $e = \sqrt{\hbar c/137}$?

I have shown some time ago [2] that there is a way to bypass the Berestetsky, Lifshitz, and Pitaevsky inequality, at least as far as the total charge is concerned. The total charge "lives" at the spatial infinity, where the observation time Δx^0, formally infinite, is limited by the opening of the light cone:

$$|\Delta x^0| \leq 2r,\ r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\,.$$

Hence at the spatial infinity the B.L.P. inequality takes on the form

$$\frac{|Q|}{r^2}(2r)^2 \gg 1$$

i.e.

$$|Q| \gg \tfrac{1}{4}\,,$$

where Q is the total electric charge. In the natural units ($\hbar = 1 = c$) $e = 1/\sqrt{137}$ and therefore

$$|Q| \gg \tfrac{1}{4}\sqrt{137}\ e = 2.93\ e\,.$$

The total electric charge is approximately classical if it is substantially larger then three elementary charges. This condition looks eminently sensible when compared with the analogous condition for the harmonic oscillator; it is well known that the harmonic oscillator is pretty classical for $n \geq 5$, where n is the occupation number. This eminently sensible condition was obtained from the experimentally observed value of the fine structure constant, which is often found mysteriously small!

3. The difficulty of handling the total electric charge as a quantum object in quantum electrodynamics

It is unfortunately impossible to handle the total electric charge as a quantum object in the framework of the standard quantum electrodynamics. To see this it will be convenient to introduce the definition of zero-frequency part of classical electromagnetic field. This definition is due to Gervais and Zwanziger [3].

I say that there is no zero-frequency part in the field $F_{\mu\nu}$ if

$$\lim_{\lambda \to \infty} \lambda^2 F_{\mu\nu}(\lambda x) = 0\,.$$

If, however, this limit exists and is different from zero, as is the case for an arbitrarily moving Coulomb field, then the field $F_{\mu\nu}$ is said to have a zero-frequency part.

I will show that if the field $F_{\mu\nu}$ has a nonvanishing zero-frequency part then its total angular momentum as well as centre of mass motion are logarithmically divergent. This makes it impossible to investigate zero-frequency fields in the framework of a Poincaré invariant theory, such as the standard quantum electrodynamics is supposed to be.

There is nothing absurd about a system having finite mass but infinite angular momentum. This is seen in the following example taken from the Newtonian Theory of Gravity. Imagine a planetary system, which consist of a large central body like the Sun and a number of small planets moving along circular orbits in the same plane and in the same direction. The n-th planet has the mass m_n, the kinetic energy $\frac{1}{2}m_n v_n^2 = \frac{1}{2}GMm_n/r_n$ and the angular momentum $m_n v_n r_n = \sqrt{GM}m_n\sqrt{r_n}$, where G is the Newtonian constant, M is the mass of the central body and r_n is the radius of the orbit of the n-th planet. It is clearly possible to choose positive numbers m_n and r_n so that

$$\sum_{n=1}^{\infty} \frac{m_n}{r_n} < \infty, \quad \sum_{n=1}^{\infty} m_n < \infty$$

but

$$\sum_{n=1}^{\infty} m_n\sqrt{r_n} \; = \infty\,.$$

The above argument depends on the assumption of infinite divisibility of matter. For ponderable matter this assumption is not true, as the electron seems to be the lightest particle. For light, however, the assumption is true: a photon can have an arbitrarily small energy while its spin is always equal to $\hbar = 1$. This allows to construct a packet of classical electromagnetic radiation, which has an arbitrarily small mass but infinite angular momentum. Moreover, such packets cannot be declared "unphysical", as, for example, plane waves having infinite energy are, quite correctly, declared unphysical, because such packets necessarily arise during scattering of electrically charged particles.

Let us perform the Gervais–Zwanziger rescaling

$$\lim_{\lambda\to\infty} \lambda^2 F_{\mu\nu}\left(\lambda x\right)$$

directly in the Noether constant of motion

$$M_{\mu\nu} = \int\limits_{x^0=0} \left[dx^1 dx^2 dx^3\right] \left(x_\mu T^0_\nu - x_\nu T^0_\mu\right) .$$

Assume that

$$A_\mu(x) = \frac{1}{2\pi} \int\limits_{kk=0,k^0>0} dk\, a_\mu(k) e^{-ikx} + \text{c.c.}\,,$$

where $dk = (1/k^0)[dk^1 dk^2 dk^3]$ and $k^\mu a_\mu(k) = 0\,$.
Then

$$M_{\mu\nu} = \frac{1}{i} \int dk (\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)\,,$$

where

$$\nabla_{\mu\nu} = k_\mu \frac{\partial}{\partial k^\nu} - k_\nu \frac{\partial}{\partial k^\mu}\,.$$

Perhaps several comments will be useful. The amplitude $a_\mu(k)$ depends on three variables only, namely three internal coordinates on the light cone $kk = 0, k^0 > 0$. Nevertheless the formally four-dimensional derivative $\nabla_{\mu\nu}$ is well defined because the operator $\nabla_{\mu\nu}$ acts within the light cone $kk = 0$; in particular $\nabla_{\mu\nu} kk = 0$.

The content of the above covariant expression for $M_{\mu\nu}$ is identical with noncovariant expressions usually given in the literature, for example by Källén on page 25 of his book [4].

$M_{\mu\nu}$ is gauge invariant; it does not change if the amplitude $a_\mu(k), k^\mu a_\mu(k) = 0$, is replaced by $a_\mu(k) + k_\mu f(k)$, where $f(k)$ is an arbitrary function on the light cone $kk = 0, k^0 > 0$. This, of course, should be the case since the original x-space expression for $M_{\mu\nu}$ is manifestly gauge invariant.

Most authors have some more complicated factors in front of the Fourier transform, typically $(2\pi)^{-3/2}$. This results from several unfortunate conventions, one of them being the use of so called rationalized units. There is nothing rational about rationalized units. They were introduced by Heaviside and Lorentz to simplify several formulae of minor importance at the expense of complicating the Coulomb law, which is clearly of fundamental importance. Besides, there is only one truly rational system of electrical units, namely the one in which the elementary charge is taken for the unit of charge. As long as we cannot arrive theoretically at this system it is preferable to use the traditional *i.e.* unrationalized Gaussian units. These units are used in this paper.

Now, the Gervais–Zwanziger limit

$$\lim_{\lambda\to\infty} \lambda^2 F_{\mu\nu}(\lambda x)\,,$$

if different from zero, gives a field which is homogeneous of degree -2. This means that the amplitude $a_\mu(k)$ is homogeneous of degree -1:

$$a_\mu(\lambda k) = \lambda^{-1} a_\mu(k) \quad \text{for each } \lambda > 0\,.$$

The invariant volume

$$dk = \frac{[dk^1 dk^2 dk^3]}{k^0}\,,$$

where [] denotes the outer product, can be written in the form

$$dk = \frac{[dk^1 dk^2 dk^3]}{k^0} = \left[\frac{dk^0}{k^0} d^2k\right]\,,$$

where d^2k is the Lorentz invariant volume of the set of null directions [5],

$$d^2k = \frac{k^1[dk^2 dk^3] + k^2[dk^3 dk^1] + k^3[dk^1 dk^2]}{k^0}\,.$$

In this way the constant $M_{\mu\nu}$ takes on the form

$$M_{\mu\nu} = \int\limits_0^\infty \frac{dk^0}{k^0} \frac{1}{i} \int d^2k(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)\,.$$

It is seen to be a product of an infinite constant $\int_0^\infty dk^0/k^0$ and of a perfectly well defined, Lorentz an gauge invariant expression

$$\frac{1}{i} \int d^2k(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)\,,$$

in which the amplitude $a_\mu(k)$ is homogeneous of degree -1,

$$a_\mu(\lambda k) = \lambda^{-1} a_\mu(k) \quad \text{for each } \lambda > 0\,;$$

this makes this integral Lorentz invariant. In this way we have proved that for a field having a generic zero-frequency part the Noether constant $M_{\mu\nu}$ is actually infinite *i.e.* does not exist in the usual sense.

4. The weaker transversality condition

The Noether constant

$$M_{\mu\nu} = \frac{1}{i}\int dk(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)$$

is known to be gauge invariant if $k^\mu a_\mu(k) = 0$. The tensor

$$\frac{1}{i}\int d^2k(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)\,,$$

in which the amplitude $a_\mu(k)$ is homogeneous of degree -1, remains invariant under gauge transformations which replace $a_\mu(k)$ by $a_\mu(k) + k_\mu f(k)$, where $f(k)$ is an arbitrary function homogeneous of degree -2, if $k^\mu a_\mu(k)$=const. The mechanism of this relaxation of gauge invariance is this: the set of null directions with the invariant measure d^2k is a two-dimensional sphere. On a sphere the integral of each gradient field vanishes identically; this allows to weaken the transversality condition to the form $k^\mu a_\mu(k)$=const. The real part of the constant $k^\mu a_\mu(k)$ is the total electric charge Q divided by 2π. In this way, simply by relaxing the transversality condition to the form $k^\mu a_\mu(k) = Q/2\pi$ *we can build in the total electric charge into the generators of proper, ortochronous Lorentz transformations i.e. to give a purely kinematical meaning to the total electric charge.*

Interpreting the tensor

$$\frac{1}{i}\int d^2k(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)$$

as a generator of proper , ortochronous Lorentz transformations I have dropped the infinite constant $\int_0^\infty dk^0/k^0$. This cannot be called “renormalization”, even allowing for the rather vague meaning of this term. It is rather an independent normalization in a subspace and may be best illustrated by the following elementary example. Take a particle in a spherically symmetric potential. The energy eigenfunctions may be normalizable or not, depending on the shape of the potential. However, the angular part of the wave function can always be normalized and the usual quantum mechanics of angular momentum can be constructed. Many prominent authors, Condon and Shortley and Landau and Lifshitz among them, use such an independent normalization of the angular part of the wave function although a logical purist could insist that only normalization of the whole wave function is physically meaningful. My dropping of the infinite constant $\int_0^\infty dk^0/k^0$ is in no way different from normalization of spherical functions to 1 and certainly does not deserve the name of “renormalization”.

5. The principle of charge quantization

The generator

$$\frac{1}{i}\int d^2k(\overline{a}_\lambda \nabla_{\mu\nu} a^\lambda + \overline{a}_\mu a_\nu - \overline{a}_\nu a_\mu)$$

contains two parts, the electric part and the magnetic part, which can be completely and Lorentz invariantly separated from each other. I have shown in [2] that the magnetic part has a property which is completely harmless classically but fatal in quantum mechanics. It can be summarized by saying that the magnetic counterpart of the fine structure constant is negative. This is inconsistent with several important physical principles, in particular with unitarity of Lorentz transformations. Thus there is an important physical reason to put the magnetic part equal to zero *i.e.* to exclude magnetic monopoles. However, I do not wish to elaborate upon this point here because, as I said, the electric part and the magnetic part are completely independent and I can consider the electric part alone.

The really important point which I wish to make is this. The total electric charge is a *linear* functional of the field while the angular momentum and centre of mass motion are *quadratic* functionals of the field. However, they cannot contain Q^2 because the coefficient of Q^2 would have to be an invariant, antisymmetric c-number tensor. Such a tensor does not exist. Therefore the tensor $M_{\mu\nu}$ which contains the total charge Q must be of the form

$$M_{\mu\nu} = M^{\rm tr}_{\mu\nu} + Q(a^+_{\mu\nu} + a_{\mu\nu})\,,$$

where $M^{\rm tr}_{\mu\nu}$ contains only transversal degrees of freedom in the usual sesquilinear form while $a^+_{\mu\nu}$ (respectively $a_{\mu\nu}$) contains only transversal creation (respectively annihilation) operators. The operator $a^+_{\mu\nu} + a_{\mu\nu}$ cannot annihilate a normalizable state. Therefore the only way to have a Lorentz invariant vacuum state is to put

$$Q|0\rangle = 0\,, \quad \langle 0|Q = 0\,.$$

If, however, the vacuum state $|0\rangle$ is to be normalizable, the coordinate canonically conjugate with the total charge Q must be periodic. This gives immediately quantization of the total electric charge in terms of a *single* universal constant.

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