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BACKSCATTERING IN PROPAGATION OF SPHERICALLY SYMMETRIC MASSLESS SCALAR FIELDS*, **

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Dedicated to Andrzej Trautman in honour of his 64^{th} birthday

Propagation of scalar field waves interacting with a strong gravitational field exhibits effects of backscattering. Due to this the total flux of radiation can diminish if sources are placed in a region adjoining a compact body. Backscattering can be neglected in the case when the emitter is located at a distance much larger than the Schwarzschild radius. The effect should be detectable in some astrophysical sources of electromagnetic or gravitational radiation.

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1. Introduction

Backscattering is a classical phenomenon known in old mathematical literature [1] as a "breakdown of the Huyghens principle"; if that occurs then an energy carried by solutions of wave equations with variable coefficients *does not propagate along characteristic surfaces* but rather inside their whole interior. Backscattering causes a spread of energy — initial impulses that are sharp and strong can be weakened and dispersed over a volume much bigger than that occupied initially. Electromagnetic or gravitational radiation, for instance, reaching an asymptotic observer, can be weaker than expected, if their sources are located in a strongly curved spacetime.

I shall analytically evaluate that effect in the case of a gravitating massless scalar field, under the assumption of spherical symmetry. A more detailed and precise analysis of backscattering is presented in [2], but this paper gives some novel results concerning characteristics of tail terms.

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2. The spherically symmetric Einstein-massless scalar field system

I choose a spherically symmetric diagonal line element

$$ds^2 = -\beta(R,t)\delta(R,t)dt^2 + \frac{\beta(R,t)}{\delta(R,t)}dR^2 + R^2 d\Omega^2 \ , \tag{1}$$

where t is a time coordinate; R is a radial coordinate that coincides with the areal radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element on the unit sphere; $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi$. At spatial infinity δ and β go to $+1$. Standard convention is used that Greek letters change from 0 to 3 for spacetime objects while Latin indices range from 1 to 3 for space-like objects.

The above choice of coordinates implicitly assumes the polar gauge condition, which reads $\mathrm{tr}\, K = K^r_r$, in terms of the so-called extrinsic curvatures.

A massless scalar field ϕ satisfies the wave equation $\nabla_\mu \nabla^\mu \phi = 0$. The stress-energy is given by $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi/2$. The matter energy density reads $\rho = -T^0_0$ and the matter current density is $j = -T_{0R}/\beta$. A "radiation amplitude" $h(s)$, $s \in (-\infty, \infty)$, is defined as follows

$$\begin{aligned} h_+(R,t) &= h(R,t) = \tfrac{1}{2}\left(-\frac{1}{\delta}\partial_0 + \partial_R\right)(R\phi)\,, & (2)\\ h_-(R,t) &= h(-R,t) = \tfrac{1}{2}\left(\frac{1}{\delta}\partial_0 + \partial_R\right)(R\phi)\,. & (3)\end{aligned}$$

One can show, solving Einstein equations $G_{\mu\nu} + 8\pi T_{\mu\nu}$ [3] that

$$\beta(R) = \exp\left(-8\pi\left(\int_R^\infty - \int_{-\infty}^{-R}\right)\frac{dr}{r}(h-\hat{h})^2\right) \ , \tag{4}$$

where

$$\hat{h} = \frac{1}{2R}\int_0^R dr[h_+(r) + h_-(r)] = \frac{1}{2}\phi\,. \tag{5}$$

δ can be expressed in the following form [3]

$$\delta(R) = \frac{1}{R}\int_0^R \beta dr. \tag{6}$$

The scalar field equation can be written [3] as a single first order equation on a "symmetrized" domain $-\infty \le R \le \infty$,

$$(\partial_0 + \delta\partial_R)h = (h-\hat{h})\frac{\delta-\beta}{R} \ . \tag{7}$$

This equation, together with definitions of $h, \hat{h}, \beta$, and δ is equivalent to the full system of Einstein equations coupled to the scalar field. Initial data of compact support must satisfy the condition [3]

$$\int_{-\infty}^{\infty} dr\tilde{h}_0 = 0. \tag{8}$$

One can easily show, using relations between metric functions and their symmetry properties, that if (8) holds true then $\int_{-\infty}^{\infty} drh(r,t) = 0$ in the existence interval of a solution.

Another useful representation of δ is

$$\delta(R) = \left(1 - \frac{2m_0}{|R|} + \frac{2m_{\mathrm{ext}}(R)}{|R|}\right)\beta(R)\,, \tag{9}$$

where m_0 is the asymptotic mass and m_{ext} is a contribution to the asymptotic mass coming from the exterior of a sphere of a radius $|R|$ and

$$m_{\mathrm{ext}}(R) = 4\pi \int_{|R|}^{\infty} \frac{\delta}{\beta}\Big((h(r) - \hat{h})^2 + (h(-r) - \hat{h})^2\Big)dr\,. \tag{10}$$

Notice that $h(R)$, $R > 0$, represents outgoing radiation, while $h(-|R|)$ is an ingoing scalar wave.

After substituting Eq. (9) into (7) one obtains

$$(\partial_0 + \delta\partial_R)h = -(h - \hat{h})\frac{2m(R)R\beta}{|R|^3}\,, \tag{11}$$

where $m(R) = m_0 - m_{\mathrm{ext}}(R)$.

3. Solutions

Let me reformulate slightly the evolution problem. Equation (5) reads, taking into account (8),

$$\hat{h} = -\frac{1}{2R}\left(\int_{R}^{\infty} + \int_{-\infty}^{-R}\right) h(s)ds\,;$$

notice that $\hat{h}$ vanishes at spatial infinity. One can use also the representation (10) of $\delta(R)$; that in turn is an integral equation in the exterior. These facts, together with equations (8)–(11), clearly demonstrate that the evolution of the scalar field in the external region decouples entirely from what happens inside a domain that is not causally connected with the exterior. It is not suprising that there exists a global Cauchy evolution in the exterior part of a space-time ([3, 4]).

Consider *initial data* with a heavy central mass m_0. Assume that there exists a thin cloud of a massless field placed outside the central mass, or — if the mass is enclosed by an apparent horizon at R_A — between $R_A < R_0 \leq R \leq R_2$. As a parenthetical remark, let me point out that this way of dealing with backscattering is different from approaches adopted by other researchers [5], who investigate *stationary scattering* of massless scalar fields on a Schwarzschild background.

Let us choose an ε and an $\tilde{R}_A$ such that $m_{\mathrm{ext}}(\tilde{R}_A)/m_0 < \varepsilon \ll 1$ and $\tilde{R}_A > 2m_0(1+\varepsilon)$. Then one can safely neglect backreaction [2]. Thus I would track the evolution of this cloud on a fixed Schwarzschildean geometry. That setting resembles a picture known from astronomical observations. Compact objects, that are suspected to be black holes, have masses that range from a few solar massess (in the case of black holes of stellar origin) up to tens of billions of solar massess (inside some of known galaxies). Processess that are occuring nearby them usually involve only a tiny fraction of the total mass of the whole system.

Taking into account the above assumptions one infers that $\beta \approx 1$ and $\delta \approx (1 - \frac{2m_0}{R})$ outside a region bounded by a null ray outgoing from a radius $R > \tilde{R}_A$. Now define a new radial variable

$$r = R + 2m_0 \ln\left(\frac{R}{2m_0} - 1\right) . \tag{12}$$

In that case equation (11) reads

$$(\partial_0 + \partial_r)h = -(h - \hat{h})\frac{2m_0 R}{|R|^3} . \tag{13}$$

The outgoing flux is given by values of $h(R(t))$ with $R(t) > R_0 > \tilde{R}_A$ where R_0 describes the initial location of an outgoing null cone. Quantitative analysis of (13) is difficult due to the nonlocal character of $\hat{h}$, an important term that will be shown to be responsible for the backscattering. Its impact on the evolution is easy to predict in regions close to the external boundary of the radiation, but the analysis becomes more complicated elsewhere.

One can show [3], however, that

$$|\hat{h}(R)| \leq \frac{\sqrt{m_{\mathrm{ext}}(R)}}{R^{1/2}\sqrt{8\pi(1 - \frac{2m_0}{R})}}. \tag{14}$$

Thus a solution of (13) can be estimated by solutions of following equations

$$(\partial_0 + \partial_r)h = -h\frac{2m_0}{R^2} - \frac{2m_0\sqrt{m_{\rm ext}(R)}}{R^{5/2}\sqrt{8\pi(1-\frac{2m_0}{R})}}\,, \tag{15}$$

$$(\partial_0 + \partial_r)h = -h\frac{2m_0}{R^2} + \frac{2m_0\sqrt{m_{\rm ext}(R)}}{R^{5/2}\sqrt{8\pi(1-\frac{2m_0}{R})}}\,, \tag{16}$$

where the minus sign gives a bound from below and the plus sign from above. These two equations are local, in contrast to the original Eq. (13), which is non-local.

Equations (15), (16) are solved by

$$h(r,t) = h_0(r-t)\frac{1-\frac{2m_0}{R'}}{1-\frac{2m_0}{R(r)}} \pm \sqrt{\frac{m_{\rm ext}(R)}{8\pi R}}\frac{2m_0}{1-\frac{2m_0}{R(r)}}\int\limits_{R'}^{R} dr\frac{1}{r^2\sqrt{r-2m_0}}\,, \tag{17}$$

where $R(r)$ is connected with R', a point lying on the initial hypersurface, by an outgoing null curve. The first term in Eq. (17) can be interpreted as coming entirely from redshift related damping. There would be no dipersion of energy if $\hat{h}$-related terms are ignored. In such a case there is a loss in intensity of the energy flux but the total energy would not change. One can easily see that by observing that in such a case initial data which are purely outgoing and concentrated between $r \in (r_1, r_2)$ give rise to a purely outgoing solution contained between two null cones $r = r_1 + t$ and $r = r_2 + t$).

Let initial data be $h(R) = 0$ for $R \leq -\tilde{R}_A$ and $\frac{2m_0}{R_0} < 1$. In that case the initially outgoing energy flux reads, ignoring terms of the order $\frac{m_{\rm ext}(R)}{R}$, $(h_0(r-t))^2\delta = (h_0(r-t))^2(1-\frac{2m_0}{R'})$.

Asymptotically

$$h(r,t) = h_0(r-t)\left(1-\frac{2m_0}{R'}\right) \tag{18}$$

and the flux density is $\langle h\rangle^2 = h^2$.

We see, comparing initial fluxes and the bound for the asymptotic flux that the latter diminishes by a factor $(1-\frac{2m_0}{R'})$. Total energy flux is constant, but its local values are decreased. A detailed calculation shows that this damping is due to the increase of the "equal time distance" between null cones. Two null cones that are separated at $t = 0$ by the areal radius distance Δ (with $\Delta/R_0 \ll 1$) shall be separated on the Cauchy slice $t \gg 1$ by a distance approximately equal to $\Delta/(1-\frac{2m_0}{R_0})$, where R_0 is the initial location of the inner null cone.

4. Backscattering

Formula (17) can be used to infer a crude estimation for backscattering. Asymptotically one arrives at an estimate for the radiation outgoing from $(R', t = 0)$

$$\begin{aligned}
&|h(r,t)| \leq \\
&\sup \left\{ \left| |h_0(r-t)| \left(1 - \frac{2m_0}{R'}\right) + \frac{4m_0\sqrt{m_{\mathrm{ext}}(R')}}{3R'\sqrt{8\pi(R'-2m_0)}} \right|, \right. \\
&\qquad \left. \left| |h_0(r-t)| \left(1 - \frac{2m_0}{R'}\right) - \frac{4m_0\sqrt{m_{\mathrm{ext}}(R')}}{3R'\sqrt{8\pi(R'-2m_0)}} \right| \right\} . \qquad (19)
\end{aligned}$$

In the last formula the asymptotic radiation amplitude is estimated entirely in terms of the initial profile $h_0(r-t)$ and the initial external contribution to the mass. In order to specify under what circumstances this backscattering is small, one should compare the asymptotic value of the outgoing flux density $j_r = \lim_{t\to\infty}(h - \hat{h})^2 = |h_0(r-t)|^2(1 - \frac{2m_0}{R'})^2$ with that part of the initial datum that can be backscattered, and which is bounded by the second term in (19). Thus one would demand

$$|h_0(r-t)| \left(1 - \frac{2m_0}{R'}\right)^{3/2} \gg \frac{4m_0\sqrt{m_{\mathrm{ext}}(R'(r))/8\pi}}{3R'^{3/2}}. \qquad (20)$$

That condition cannot be met when R' is close to the apparent horizon; left hand side of (20) is approaching zero while the right hand side is approximately constant. On the other hand at the external part of the initial profile, *i.e.*, $R' \to R_2$ the right hand side becomes arbitrarily small and (20) can be satisfied. Thus from that cursory analysis one draws a conclusion that backscattering can be important in strongly curved regions of spacetime and deep inside the region filled with an outgoing radiation, while it is expected to be small in regions adjacent to the exterior boundary of the outgoing flux.

5. Mass transfer through outgoing null cones

I will show in this Section that the $\hat{h}$-related term can be interpreted as responsible for the change in the amount of energy that gets to infinity. From the exact evolution equation (11) one obtains that the contribution to the asymptotic mass changes as follows

$$(\partial_0 + \delta\partial_R) m_{\mathrm{ext}}(|R|) = -8\pi\delta^2 \Big(h(-|R|) - \hat{h}(R) \Big)^2 , \qquad (21)$$

that is, it depends on the ingoing radiation $h(-|R|)$ and $\hat{h}$. Notice that even if initially $h(-|R|)$ vanishes then it still can become nonzero due to the presence of $\hat{h}$ in the evolution equation (13). Estimates of the former Section give a bound for this backscattered ingoing radiation

$$
\begin{aligned}
|h(-|R|)| &= \sqrt{\frac{m_{\mathrm{ext}}(R)}{8\pi}}\frac{2m_0}{1-\frac{2m_0}{R}}\int\limits_{R'}^{R} dr\frac{1}{r^2\sqrt{r-2m_0}} \\
&\leq \sqrt{\frac{m_{\mathrm{ext}}(R)}{8\pi}}\frac{4m_0}{3(R-2m_0)^{3/2}}\,. \qquad (22)
\end{aligned}
$$

The last inequality follows from the fact that for an ingoing null ray the initial position R' is bigger than its actual location R.

Now, since $\delta h(-|R|) = (\partial_0 + \delta\partial_R)(R\hat{h})$, we obtain, in the approximation specified hitherto,

$$
|\hat{h}(R(t),t)| \leq |\hat{h}(R_0,t=0)|\frac{R_0}{R(t)} + \sqrt{\frac{m_{\mathrm{ext}}(R)}{8\pi}}\frac{8m_0}{3R\sqrt{(1-\frac{2m_0}{R_0})}R_0^{1/2}}\,; \qquad (23)
$$

here $R_0 = R(t=0)$. Take a piece of initial hypersurface that is placed inside the outgoing cloud; thus $\hat{h}$ initially vanishes, $h(R_0,t=0)=0$, and along an outgoing null cone

$$
|\hat{h}(R(t),t)| \leq \sqrt{\frac{m_{\mathrm{ext}}(R)}{8\pi}}\frac{8m_0}{3R\sqrt{(1-\frac{2m_0}{R_0})}R_0^{1/2}}. \qquad (24)
$$

This estimate holds true, in particular, alongside the inner boundary $\partial\Omega$ of the flash of radiation. Integration of equation (21) yields a bound for the amount of matter that can diffuse through a null cone $\partial\Omega$ (that starts from R_0 at $t=0$)

$$
\Delta m < m_{\mathrm{ext}}\frac{64}{9}\left(\frac{2m_0}{R_0}\right)^2\left(\frac{1-\frac{m_0}{R_0}}{1-\frac{2m}{R_0}}\right)\,. \qquad (25)
$$

Thus the backscattering becomes negligibly small at distances big in comparison with the Schwarzschild radius $R_{\mathrm{S}} = 2m_0$ of the central mass. The above estimates can be improved by a factor ≈ 10 [2].

6. Tail radiation

A pulse of outgoing radiation produces at a point R' some signal directed inward; its intensity $h(-|R|)$ is estimated by formula (19). That ingoing impulse backscatters again on the spacetime at a point R_0 and creates an outgoing radiation, called tail term. One gets, after several calculations, a crude bound for the tail term generated within an initially vacuous region [2]

$$|h_+(\infty,\infty)| \le \frac{4}{9}\sqrt{\frac{m_{\text{ext}}}{2\pi m_0}}\frac{\left(\frac{m_0}{R_0}\right)^{5/2}}{\sqrt{1-\frac{2m_0}{R_0}}}\,. \tag{26}$$

This estimate can be strengthened to

$$|h_+(\infty,\infty)| \le \alpha\frac{2m_0}{R_0}\,\frac{2m_0}{R'}\sqrt{\frac{m_{\text{ext}}}{2\pi(R_0-2m_0)}}\,; \tag{27}$$

here α is a constant of the order of 1. (27) gives a natural bound for the strength of the double backscatter — the radiation is damped by a factor of a form $2m/R$ during each of the two constituent backscatterings.

In order to get some understanding of the consequences of the formulae, let us invoke to an appaling, from the point of view of mathematical rigour, reasoning. Let the initial support of the scalar field be contained within $(R_0, R_0+\Delta)$. One shows that $\sqrt{m_{\text{ext}}/R_0}$ is of the order of $\langle h\rangle\sqrt{\Delta/R_0}$. The assumption that $\Delta/R_0 \approx 1$ yields $\sqrt{m_{\text{ext}}/R_0} \approx \langle h\rangle$. Comparing asymptotics of (26) with (18) one arrives to the conclusion that the tail term starting from a region that is far away from an apparent horizon is much smaller (in the average) than the main peak h_0; it decreases essentially by the factor $(4m_0^2)/R_0R'$, in accordance with (27). In contrast with that, tail terms generated in regions closer to apparent horizons may be substantially bigger, since the factor $1/(1-\frac{2m_0}{R_0})^{1/2}$ would enhance them.

The reasoning applied hitherto can be used in order to demonstrate that black holes can shine by reflecting a small fraction of ingoing radiation; notice a purely classical nature of that phenomenon.

7. Numerical results

In order to estimate quantitatively the effect of backscattering one needs to solve numerically Eqs. (13) and calculate $m_{\text{ext}}(R)$ using Eq. (10). Below I shall report some of numerical results obtained by Chmaj; they are presented in full detail in [2]).

Assume the following form of initial data:

$$h_+(R) \;=\; A_0(R-R_0)\exp(-20(R-R_0)^2/s^2)\,, \tag{28}$$
$$h_-(R) \;=\; 0\,. \tag{29}$$

where A_0 — amplitude, R_0 — position of peak center, s — the measure of its width.

The mass flux ratio

$$r_m(T) = \frac{(m_{\text{ext}}(R,T) - m_{\text{ext}}(R_i,0))}{m_{\text{ext}}(R_i,0)} \ . \tag{30}$$

describes quantitatively effects of backscattering; above $(R_i, 0)$ and (R, T) are connected by outgoing future radial null ray.

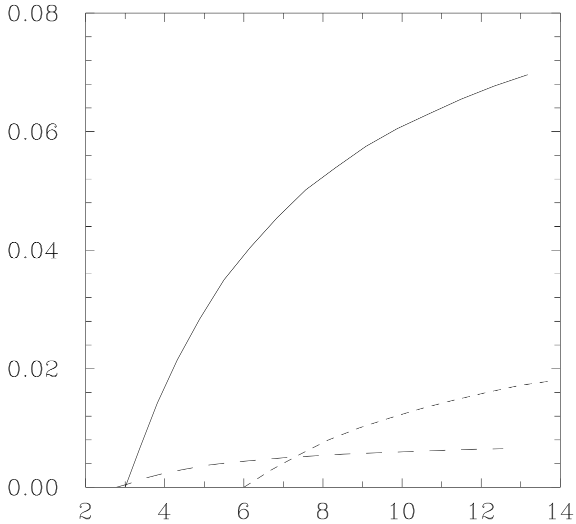

Fig. 1. Examples of backscattering effects — mass flux ratio $r_m(T)$ through different outgoing future radial null rays as a function of position along the ray $R(T)$. The curves correspond to the following sets of parameters: solid line — $R_i = R_0 = 3, s = 1, A_0 = 0.2, m_0 = 1$; short dashed line — $R_i = R_0 = 6, s = 1, A_0 = 0.2, m_0 = 1$; long dashed line — $R_i = 2.776, s = 1, A_0 = 0.2, m_0 = 1$.

Backscattering happens to be of significance only close to the horizon in the regions where the $\hat{h}$ function is significant. There is no major flux through the outgoing null cone starting inside inner parts of an impulse — the leakage through the inner boundary of a signal defined as a radius which contains initially 90 % of m_{ext} is much less than 1 % (Figure, long dashed line). This effect contributes, however, to a more impressive shift of flux inside the main impulse of radiation, up to 10 % for $R_i = R_0$, where R_0 — is the position of the peak center for configuration starting from $R_0 = 1.5R_{\text{hor}}$ (see Figure, solid line). This effect decreases with the distance from the horizon. For the parametr $R_0 = 3R_{\text{hor}}$ the effect is less then 2 % (see Figure, short dashed line). Thence the backscattering phenomenon can change a shape of the pulse of radiation, in addition to the overall weakening of the signal.

8. Concluding remarks

I would like to stress out that this paper gives strict results only for the backscatter of the massless scalar field. The propagation of the latter has some pecularities (see Section II in [2]), but I believe that the backscattering of electromagnetic or gravitational waves will conform to the picture outlined hitherto. There are two different astrophysical situations that backscattering of electromagnetic or gravitational radiation can manifest itself:

(i) in sporadic bursts;

(ii) in continuously emitting sources.

Below I will evaluate the predicted impact of backscattering in some specific cases. If electromagnetic fields propagate similarly to the massless scalar field, then the forthcoming estimates give the order of measurable effects. A good measure of the strength of the gravitational field is the ratio $(2m/R)^2$, where R is the areal radius of the boundary of a compact object and m is its mass. That ratio is circa 10^{-12} for the Sun, 10^{-6} for white dwarfs, 0.01-0.16 for neutron stars and 1 for Schwarzschildean black holes.

For sources located close to the Sun the backscattered part of a burst of radiation is at least 10^{12} times weaker than the main peak while a corresponding figure for a heavy neutron star would be 10^2. Weak tail terms (probably two or three orders weaker than the main signal) would contribute to afterglows of X-ray bursts, which are believed to originate at surfaces of some neutron stars. They are expected to decay quickly, decreasing by a factor 10^4 (at least) in one second. Tail terms can persist for much longer in the case of a radiation produced close to a black hole. Flashes of radiation coming from a vicinity of a black hole of a mass 10^9 of solar masses can produce a tail whose intensity decreases by 10^3 during a couple of months. Gamma-ray bursts are believed to form in collisions or mergers of black holes with other compact bodies [6], therefore they may reveal imprints of backscattering.

Backscattering may diminish an overall efficiency of continuous radiation produced by black holes interacting with clouds of gas. Its effect can be of significance in the description of relatively thin clouds of gas surrounding heavy black holes of masses exceeding tens of millions of solar masses. The importance of that would require further investigation. A problem that is probably worth to study, is the mysteriously faint radiation generated by of a black hole hidden inside the elliptical galaxy M87 [7].

REFERENCES

[1] J. Hadamard, *Lectures on Cauchy's problem in linear partial differential equations*, Yale University Press, Yale, New Haven 1923.

[2] E. Malec, N. O' Murchadha, T. Chmaj, Spherical scalar waves and gravity — red shift and backscattering, gr-qc/9712022, to appear in *Classical Quantum Gravity.*

[3] E. Malec, *J. Math. Phys.* **38**, 3650 (1997).

[4] E. Malec *Classical Quantum Gravity* **13**, 1849 (1996).

[5] see, for instance: N. Sanchez, *J. Math. Phys.* **17**, 688 (1976) and The Black Hole: Scatterer, Absorber and Emitter of Particles, hep-th/9711068; A. Bachelot, *J. Math. Pures. Appl.* **76**, 155 (1997); B.C. Nolan, *Classical Quantum Gravity* **14**, 1295 (1997).

[6] For a review on gamma-ray bursts, see B. Paczyński *Publ. Astron. Soc. Pac.* **167**, 1167 (1995) and GRBs as Hypernovae, astro-ph/9712123;

[7] C.S. Reynolds *et al.*, *Mon. Not. Astron. Soc.* **283**, L111 (1996).