ON AN INTERPRETATION OF NON-RIEMANNIAN GRAVITATION*

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Motivated by the invariance of actions under gauge symmetries the definitions of standard clocks in theories of gravitation are discussed. We argue that standard Einsteinian clocks can be defined in non-Riemannian theories of gravitation and that atomic clocks may be adopted to measure proper time in the presence of non-Riemannian gravitational fields. These ideas are illustrated in terms of a recently developed model of gravitation based on a non-Riemannian space-time geometry.

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1. Introduction

The experimental evidence for Einstein's theory received considerable enhancement with the recent observation of the rate of slowing of the binary pulsar PSR 1913+16. However it appears that certain other astrophysical observations do not rest so easily with classical gravitation. In particular some velocity distributions of stars in galaxies are hard to reconcile with the observed matter distributions if they follow from Newtonian dynamics.

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Since Einstein's theory reproduces Newtonian gravity in a non-relativistic weak-field limit this has led some to conjecture that such galaxies may contain significant amounts of dark matter [2, 3]. An alternative explanation is that Newtonian dynamics requires modifications in this context. If this alternative is taken seriously it invites one to consider alternatives to Einstein's metric theory that may be testable in an astrophysical domain.

Some of the earliest generalizations to Einstein's theory were entertained by Cartan and Weyl. The former suggested that the Levi–Civita connection used by Einstein remained metric-compatible but relaxed to admit torsion while the latter made an attempt to unify electromagnetism with gravity in terms of a theory based on a non metric-compatible connection with zero torsion. Although Weyl's efforts proved abortive, modern string-inspired low energy effective actions for gravity and matter can be formulated in terms of a non-Riemannian connection with prescribed torsion and non-metricity. [1, 6–13, 15]

2. Non-Riemannian geometry

Let us briefly recall that a non-Riemannian space-time geometry is defined by a pair $(\boldsymbol{g}, \boldsymbol{\nabla})$ where $\boldsymbol{g}$ is a metric tensor with Lorentzian signature and $\boldsymbol{\nabla}$ is a general linear (Koszul) connection. From this pair one can construct $\boldsymbol{S} = \boldsymbol{\nabla}\boldsymbol{g}$ the gradient of $\boldsymbol{g}$, $\boldsymbol{T}$, the torsion tensor and $\boldsymbol{R}_{X,Y}$, the curvature operator. If X, Y, Z are arbitrary vector fields on space-time then $\boldsymbol{T}(X,Y) = \boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X - [X,Y]$ and $\boldsymbol{R}_{X,Y}Z = \boldsymbol{\nabla}_X\boldsymbol{\nabla}_Y Z - \boldsymbol{\nabla}_Y\boldsymbol{\nabla}_X Z - \boldsymbol{\nabla}_{[X,Y]}Z$. If $\boldsymbol{\nabla}$ is chosen so that $\boldsymbol{S} = 0$ and $\boldsymbol{T} = 0$ the geometry is pseudo-Riemannian and the gravitational field is associated with the Riemann curvature tensor $\boldsymbol{R}$ where $\boldsymbol{R}(X,Y,Z,\beta) = \beta(\boldsymbol{R}_{X,Y}Z)$ for an arbitrary 1-form β. This tensor is then determined solely by the metric and the connection is called the Levi–Civita connection. Just as the Levi–Civita connection of a pseudo-Riemannian geometry can be expressed in terms of $\boldsymbol{g}$ alone a general $\boldsymbol{\nabla}$ can be expressed in terms of $\boldsymbol{g}, \boldsymbol{T}$ and $\boldsymbol{S}$:

$$\begin{aligned} 2\boldsymbol{g}(Z, \boldsymbol{\nabla}_X Y) &= X(\boldsymbol{g}(Y,Z)) + Y(\boldsymbol{g}(Z,X)) - Z(\boldsymbol{g}(X,Y)) \\ &\quad -\boldsymbol{g}(X,[Y,Z]) - \boldsymbol{g}(Y,[X,Z]) - \boldsymbol{g}(Z,[Y,X]) \\ &\quad -\boldsymbol{g}(X,\boldsymbol{T}(Y,Z)) - \boldsymbol{g}(Y,\boldsymbol{T}(X,Z)) - \boldsymbol{g}(Z,\boldsymbol{T}(Y,X)) \\ &\quad -\boldsymbol{S}(X,Y,Z) - \boldsymbol{S}(Y,Z,X) + \boldsymbol{S}(Z,X,Y)\,. \end{aligned} \tag{2.1}$$

3. Weyl's theory

In 1918 Weyl [4, 5] proposed a theory based on an action functional $S[\boldsymbol{g}, \boldsymbol{A}]$ where $\boldsymbol{\nabla}$ was a non-Riemannian connection constrained to have

$\boldsymbol{T} = 0$ and $\boldsymbol{S} = \boldsymbol{A} \otimes \boldsymbol{g}$ for some 1-form $\boldsymbol{A}$. Since his action was invariant under the substitutions:

$$\boldsymbol{g} \to \mathrm{e}^{\lambda} \boldsymbol{g}\,, \tag{3.1}$$

$$\boldsymbol{A} \to \boldsymbol{A} + d\lambda\,, \tag{3.2}$$

for any 0-form λ, this theory determined a class of solutions $[\boldsymbol{g}, \boldsymbol{A}]$; elements being equivalent under what Weyl termed the *gauge transformations* (3.1), (3.2). Classical observables predicted by this theory should be gauge invariant.

In Einstein's pseudo-Riemannian description of gravitation a standard clock is modeled by any time-like curve C parameterized with a tangent vector $\dot{C}$ of constant length $\sqrt{(-\boldsymbol{g}(\dot{C}, \dot{C}))}$. Such a clock can be calibrated to measure *proper time* τ with a standard rate independent of C, by fixing the parameterization of C so that

$$\boldsymbol{g}(\dot{C}, \dot{C}) = -1\,, \tag{3.3}$$

(in a metric with signature $(-,+,+,+)$). The notion of a standard clock makes precise the notion of a freely falling observer, namely an affinely parameterized autoparallel (geodesic) integral curve of the Levi–Civita connection. Since such a connection is compatible with a prescribed metric, ($\boldsymbol{S} = 0$), the normalization of $\dot{C}$ is preserved for any C. Thus although the elapsed proper time between events connected by C is path dependent, any particular standard Einsteinian clock admits a proper time parameterization independent of its world line. In Weyl's geometry no particular $\boldsymbol{g}$ in the class $[\boldsymbol{g}, \boldsymbol{A}]$ is preferred so the identification of a clock as a device for *measuring* proper time requires more care. The condition (3.3) is not invariant under Weyl's gauge group. However in Weyl's geometry $\boldsymbol{T} = 0$ and it then follows from (2.1) that under the transformations (3.1), (3.2) Weyl's connection $\boldsymbol{\nabla}$ remains invariant. Thus if $\boldsymbol{g}$ and $\boldsymbol{A}$ determine $\boldsymbol{\nabla}^{[\boldsymbol{g}, \boldsymbol{A}]}$ and $\boldsymbol{g} = \mathrm{e}^{\lambda} \boldsymbol{g}_1$, $\boldsymbol{A} = \boldsymbol{A}_1 + d\lambda$ then for any X, Y, Z:

$$\boldsymbol{g}(Z, \boldsymbol{\nabla}_X^{[\boldsymbol{g}_1, \boldsymbol{A}_1]} Y) = \boldsymbol{g}(Z, \boldsymbol{\nabla}_X^{[\boldsymbol{g}, \boldsymbol{A}]} Y)\,. \tag{3.4}$$

A definition of a Weyl standard clock should then refer to the gauge invariant connection rather than the gauge non-invariant condition (3.3). Thus one may model a Weyl standard clock to be a time-like (with respect to any $\boldsymbol{g}$ in $[\boldsymbol{g}, \boldsymbol{A}]$) curve C such that

$$\boldsymbol{g}(\boldsymbol{\nabla}_{\dot{C}} \dot{C}, \dot{C}) = 0\,. \tag{3.5}$$

This condition is manifestly gauge invariant under (3.1) and (3.2)İt follows that for each time-like curve there exists a standard clock parameterization of C that is unique up to the affine reparametrisation

$$\tau \mapsto a\tau + b$$

with real constants a and b [14]. However, if $d\boldsymbol{A} \neq 0$ one cannot choose $a = 1$ for all such curves. (If this were possible one could construct a Weyl parallel normalized tangent vector on any closed curve. That this is impossible with $d\boldsymbol{A} \neq 0$ follows by differentiating (3.3) with $\boldsymbol{\nabla}$.) Thus the relative *rates* of two such standard clocks depend on their relative histories in general. (This effect should not be confused with the dependence of elapsed time between events produced by the difference in paths linking such events.)

If one assumes that a standard clock in Weyl's geometry $[\boldsymbol{g}, \boldsymbol{A}]$ corresponds to an atom emitting light of a definite frequency then two identical atoms that diverged from a unique space-time event and returned to any later event, could not have the same frequency at such an event if $\int_\Sigma d\boldsymbol{A} \neq 0$, where Σ is any world sheet bounded by the world lines of the two atoms. Weyl attempted to identify $\boldsymbol{F} = d\boldsymbol{A}$ with the Maxwell electromagnetic field before the U(1) nature of the coupling to charged fields was recognized. Hence the spectra emitted by atoms in an ambient electromagnetic field would be predicted to depend on their histories contrary to observation. This was the reason that his unified theory of gravitation and electromagnetism fell prey to the early criticisms by Einstein and Pauli. Note however that such criticisms remain valid whether or not $\boldsymbol{F}$ is identified with the electromagnetic field. They rely only on the gauge invariant definition of the time parameterization of a Weyl standard clock and the assumed correspondence of an atomic spectral line with the rate associated with such a clock [18].

Thus the criticism of Weyl's theory is essentially based upon the notion used for identifying standard clocks. By contrast to Einstein's theory which works with a well defined metric, the necessity of making observables class invariant necessitates a definition of a standard clock based on the identification of atomic clocks with parameterized curves defined by (3.5). The criticisms made by Einstein, Pauli and others remain in force however one identifies the metric-gradient field $\boldsymbol{S}$ in Weyl's theory. We stress that the essence of these criticisms lies in the fundamental gauge symmetry associated with Weyl's action principle not with the identification of $\boldsymbol{F}$ with the Maxwell field. This symmetry, in turn, follows from Weyl's particular choice of a non-Riemannian geometry having zero torsion and $\boldsymbol{S} = \boldsymbol{A} \otimes \boldsymbol{g}$. However by relaxing the constraint that leads to Weyl's particular geometry and demanding that the metric be uniquely determined one may use Einsteinian clocks without ambiguity. Motivated by a rather remarkable simplification that occurs in the variational equations from a broad class of actions for a non-Riemannian geometry we have explored [16, 17] certain astrophysical consequences of non-Einsteinnian components of the gravitational field.

4. Non-Riemannian gravitational fields

Given the success of the gauge description of the Yang–Mills interactions in which the connection associated with any Yang–Mills gauge group is unconstrained in a variational principle, a more natural approach to a non-Riemannian description of gravitation is to seek a purely gravitational action $S[\boldsymbol{g}, \boldsymbol{\nabla}]$ that gives field equations determining a unique metric $\boldsymbol{g}$ without constraining $\boldsymbol{T}$ and $\boldsymbol{S}$.

Based on the reduction of the non-Riemannian action to a theory of gravity in terms of the standard Levi–Civita torsion free, metric compatible connection $\overset{\circ}{\nabla}$, we have investigated [17] a model of gravity and matter that gives rise to a Proca field in the gravitational sector. As befits its origin in terms of purely geometrical concepts the Proca field is regarded as a gravitational vector field that is expected to modify the gravitational effects produced by the tensor nature of Einsteinian gravity.

In general we consider matter to be composed of ordinary matter defined to have zero coupling to the Proca field and "dark matter" defined to have a non-zero coupling. We call this coupling Proca charge and denote the basic unit of Proca charge by q. For our discussion of cosmology we model both types of matter by fluids with standard stress tensors $\mathcal{T}_0$ and $\mathcal{T}_q$ respectively. Denoting the standard Levi–Civita Einstein tensor by $\overset{\circ}{\mathbf{Ein}}$ and the contribution of the Proca potential α to the Einstein equation by $\sigma\Sigma$, $\sigma = \pm 1$ we have

$$\overset{\circ}{\mathbf{Ein}} + \sigma\,\Sigma = \frac{8\,\pi\,G}{c^4}\,(\mathcal{T}_0 + \mathcal{T}_q)\,, \tag{4.1}$$

where

$$\begin{aligned}
\mathcal{T}_0 &= (c^2\,\rho_0 + P_0)\,V_0 \otimes V_0 + P_0\ \ \mathrm{g}\,,\\
\mathcal{T}_q &= (c^2\,\rho_q + P_q)\,V_q \otimes V_q + P_q\ \ \mathrm{g}\,,\\
\mathrm{g}(V_0, V_0) &= -1\,,\\
\mathrm{g}(V_q, V_q) &= -1\,.
\end{aligned}$$

The mass densities ρ_f, $f = 0, q$ of ordinary and dark matter respectively are functions of the particle densities n_f and the entropies s_f per particle:

$$\begin{aligned}
\rho_0 &= \rho_0(n_0, s_0)\,,\\
\rho_q &= \rho_q(n_q, s_q)\,.
\end{aligned}$$

The pressure P_f and temperature T_f may be derived from Gibb's relation

$$c^2\, \mathrm{d}\rho_f = c^2\, \mu_f\, \mathrm{d}\, n_f + n_f\, T_f\, \mathrm{d}s_f\,, \tag{4.2}$$

where $\mu_f = \frac{\rho_f + P_f/c^2}{n_f}$ is the associated chemical potential. Denoting the Weyl 1-form $\mathbf{A}$ by the Proca potential α the modification to the Einstein tensor is

$$\Sigma = \left(\frac{c\, m_\alpha}{\hbar}\right)^2 \left(\alpha \otimes \alpha - \tfrac{1}{2}\, \alpha(\widetilde{\alpha})\ \ \mathrm{g}\right) + \left(\mathrm{i}_c\, F \otimes \mathrm{i}^c\, F - \tfrac{1}{2}\, \star^{-1} (F \wedge \star F)\ \ \mathrm{g}\right)$$

where $F = \mathrm{d}\alpha$ is the Proca field strength and $\alpha(\widetilde{\alpha}) = g(\widetilde{\alpha}, \widetilde{\alpha})$. Equation (4.1) for the metric must be supplemented by field equations for the Proca field α and the fluid variables together with their equations of state. We adopt as matter field equations

$$\overset{\circ}{\nabla} \cdot \mathcal{T}_0 = 0 \tag{4.3}$$

and

$$\overset{\circ}{\nabla} \cdot \left(\frac{8\,\pi\, G}{c^4}\, \mathcal{T}_q - \sigma\, \Sigma\right) = 0 \tag{4.4}$$

which are certainly compatible with the Bianchi identity $\overset{\circ}{\nabla} \cdot \mathbf{Ein} = 0$ and give rise to the expected Lorentz forces on charged matter due to vector fields. Since the Proca field couples to a current $\boldsymbol{j}_q$ of Proca charged matter we have

$$\mathrm{d} \star F + \left(\frac{c\, m_\alpha}{\hbar}\right)^2 \star \alpha + \sigma\, \boldsymbol{j}_q = 0\,. \tag{4.5}$$

The Proca charge current will be assumed to take the convective form

$$\boldsymbol{j}_q = q\, n_q \star \widetilde{V_q} \tag{4.6}$$

with constant Proca charge, q (of dimension length).

In a similar manner we assume that the Proca neutral particle current is given by

$$\boldsymbol{j}_0 = n_0 \star \widetilde{V_0}\,.$$

We postulate conservation of Proca charged particles

$$\mathrm{d}(n_q \star \widetilde{V}) = 0 \quad \Leftrightarrow \quad \mathrm{d}\boldsymbol{j}_q = 0 \quad (\text{since } q \text{ is constant})\,. \tag{4.7}$$

If the neutral Proca matter is also conserved (as befits behaviour in the late post inflationary epoch)

$$\mathrm{d}\boldsymbol{j}_0 = 0 \tag{4.8}$$

and one may then interpret the matter field equations as equations of motion for the fluid flows.

5. Discussion

Motivated by the structure of a class of actions that involve (in addition to the generalized Einstein–Hilbert action) terms including the torsion and metric gradient of a general connection on the bundle of linear frames over space-time, the consequences of Einstein–Proca gravitation coupled to matter have been examined. This theory may be written entirely in terms of the traditional torsion free, metric compatible connection where all the effects of torsion and non-metricity reside in a single vector field satisfying the Proca equation. In such a theory the weak field limit admits both massless tensor gravitational quanta (traditional gravitons) and massive vector gravitational quanta. The mass of the Proca field is determined by the coupling constants in the parent non-Riemannian action. The interaction mediated by the new Proca component of gravitation is expected to modify the traditional gravitational interaction on small scales. In order to confront this expected modification with observation we have constructed an Einstein–Proca–Fluid model in which the matter is regarded as a perfect thermodynamic fluid. We have suggested that in addition to ordinary matter that couples gravitationally through its mass the conjectured dark matter in the Universe may couple gravitationally through both its mass and a new kind of gravitational charge. The latter coupling is analogous to the coupling of electric charge to the photon where the analogue of the Maxwell field is the Proca field strength (the curl of the Proca field). If one assumes that the amount of dark matter dominates over the ordinary matter in the later phase of evolution of the Universe, that the Proca field mass is of the order of the Planck mass and the appropriate coupling to the dark matter is of the same order as the fine structure constant then one finds that such hypotheses are consistent with both the inflationary scenario of modern cosmology as well as the observed galactic rotation curves according to Newtonian dynamics. The latter follows by assuming that stars, composed of ordinary (as opposed to dark matter), interact via Newtonian forces to an all pervading background of massive gravitationally charged cold dark matter in addition to ordinary matter. The novel gravitational interactions are predicted to have a significant influence on pre-inflationary cosmology. For attractive forces between dark matter charges of like polarity the Einstein–Proca–matter system exhibits homogeneous isotropic eternal cosmologies that are free of cosmological curvature singularities thus eliminating the horizon problem associated with the standard big-bang scenario. Such solutions do however admit dense hot pre-inflationary epochs each with a characteristic scale factor that may be correlated with the dark matter density in the current era of expansion.

The Einstein–Proca–Fluid model offers a simple phenomenological description of dark matter gravitational interactions. It has its origins in a geometrical description of gravitation and the theory benefits from a variational formulation in which the connection is a bona fide dynamical variable along with the metric. The simplicity of the model is a consequence of the structure of a class of non-Riemannian actions whose dynamical consequences imply that the new physics resides in a component of gravitation mediated by a Proca field. It will be of interest to confront the theory with other aspects of astrophysics such as localized gravitational collapse, the nature of the inflation mechanism and the origin of dark matter.

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