Vol. 29 (1998) ACTA PHYSICA POLONICA B No 6

ON THE "EXPERIMENTAL" WAY OF SHOWING SELF–SIMILARITY *

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(Received September 18, 1997)

A new "experimental" way of showing self-similarity or scale invariance of the solutions of one parameter, logistic map is presented. Depending on the value of parameter (R) four different solutions were obtained and analysed. Only for the chaotic region ($R = 4$) the obtained solutions were truly scale invariant. Some of the analytical operations commonly used in analysis of iterative maps were also discussed, and suitably alterated, were necessary.

PACS numbers: 87.10.+e

1. Introduction

For the realistic modelling of hierarchical structures like fractals and chaotic atractors, we need functional equations that generate so called self-similar solutions, *i.e.* solutions which are scale invariant. This is a very natural condition, especially for nonlinear phenomena of a fractal nature, where the self-similarity is a generic property of [1]. Consider, for example, a semilinear diffusion equation of a form [2]

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - |U|^{p-1} \cdot U = 0\,, \tag{1}$$

where U is real-valued function and $p > 1$. This equation has the scaling property, that is, if U solves Eq. (1) near $(x,t) = (0,0)$, then so do the rescaled functions

$$U_\lambda(x,t) = \lambda^{2\beta} U(\lambda x, \lambda^2 t)\,, \quad \beta = \frac{1}{p-1}\,,$$

* Presented at the Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 1–10, 1997.

for each $\lambda > 0$. If a solution U is invariant under this scaling, *i.e.*,

$$U(x,t) = U_\lambda(x,t) \text{ for } \lambda > 0\,, \tag{2}$$

U is called **a self-similar solution**. There are at least two types of self-similar solutions. If $U(x,t)$ solves Eq. (1) in $R \times (-\infty, 0)$ and satisfies the scaling invariancy (2) for all $x \in R^n$, $t \in (-\infty, 0)$, we say U is a backward self-similar solution to Eq. (1). Similarly, if $U(x,t)$ solves Eq. (1) in $R \times (0, +\infty)$ and has the property (2), $U(x,t)$ is called a forward self-similar solution to Eq. (1).

Another, slightly different, aspect of self-similarity of 1-D solutions is shown by a homogeneous function of order n

$$F(\lambda x) = \lambda^n F(x)\,, \tag{3}$$

which provides the power law scaling for

$$F(x) \equiv x^n \tag{4}$$

and, more complex scaling [3], when

$$F(x) \equiv x^n f\left(\frac{\log x}{\log \lambda}\right)\,, \tag{5}$$

where $f(1+x) = f(x)$, *i.e.* f is a periodic function of period 1.

In this paper we would like to show that the iterative logistic map of a form

$$x_{t+1} = Rx_t(1 - x_t)\,, \tag{6}$$

where $x_t \in (0,1)$, R is a constant, and $t \in N$, produces self-similar solutions as well. The way we are going to show it slightly differs from the one used in differential equations. Namely, instead of proving it, we will demonstrate it, in a way, experimentally, based on a suitably chosen definition of self-similarity.

2. The definitions of self-similarity

Although we will restrict our attention only to the logistic map (6), the whole analysis can be use for the general class of time series. To fulfil our task we need, at the moment, the self-similarity to be defined in a more descriptive way.

One of the most useful definitions says [4]: "Self-similar sets look the same, no matter the scale or resolution". Expressing this in a more rigorous way, we can say that self-similar sets are scale invariant [5]. At least one point

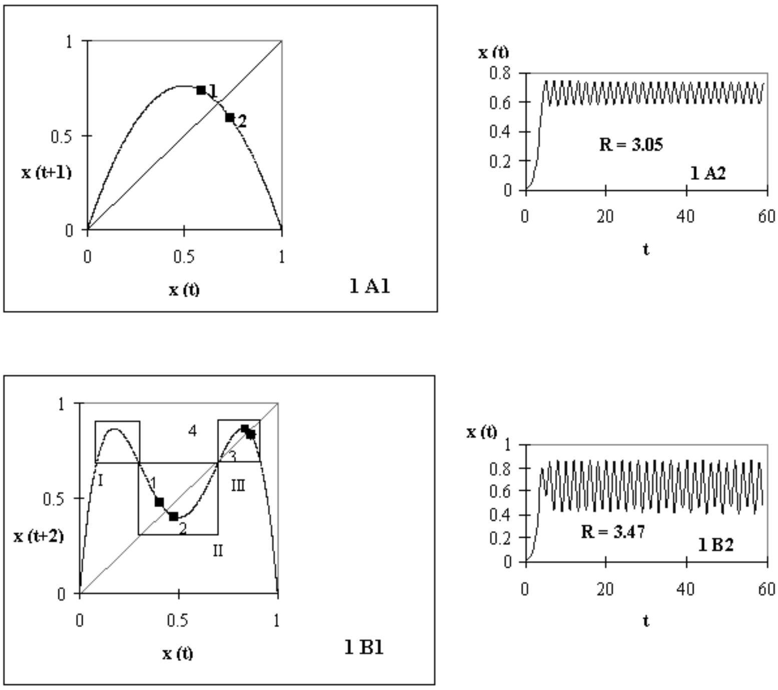

Fig. 1. Self-similarity of the logistic map in the pre-chaotic region. A1 — Pseudo–phase portrait of Eq. (6) for $R = 3.05$. Two–point attractor of a time series generated from map (6) for $R = 3.05$ and ONE, arbitrarily chosen initial condition is marked as 1 and 2; it belongs to the parabola whose formula is given by the right side of the map (6); A2 — The time series generated from the map (6) for $R = 3.05$; B1 — Second iteration of the quadratic transformation (6) for $R = 3.47$ of 1000 equally spaced initial conditions. The content of squares marked as I, II and III are similar to the parabola shown in Fig. 1 A1. Attractor of a time series generated from map (6) for one arbitrarily chosen initial condition is a set of four points marked as 1, 2, 3 and 4; B2 — The time series generated from the map (6) for $R = 3.47$.

in the above description needs clarification, namely the basis for expressions "the same" or "invariant". There are a few measures to choose from in this respect. First, by "eye" comparison: no matter what the scale, the same impression is generated (*c.f.* Hokkusai "The wave"). Second, very close values for constant R can be obtained for a rescaled time series as compared to the original one. A supporting measure to R value could be the power spectrum or Liapunov exponents [6]. It should be emphasised however, that we cannot expect from any of the above measures to be exactly the same for none of pairs of original and rescaled sets, except for some trivial cases (see the next Section).

The mathematical precision in showing self-similarity of one-dimensional maps is restricted to the case of their period-doubling route to chaos. The universal nature of period-doubling, or Feigenbaum sequence, can be understood through the renormalization group on the space of functions with a quadratic maximum [4]. To demonstrate one of these self-similarities, let us compare the behaviour of solution to Eq. (6) for two values of parameter R, namely $R = 3.05$ and $R = 3.47$ (see Fig. 1). After some renormalization procedure, *i.e.* rescaling and inverting, we can conclude that the previous (original) picture can be obtained from the next generation of the "growing structure" produced by the logistic map (*c.f.* Fig. 2).

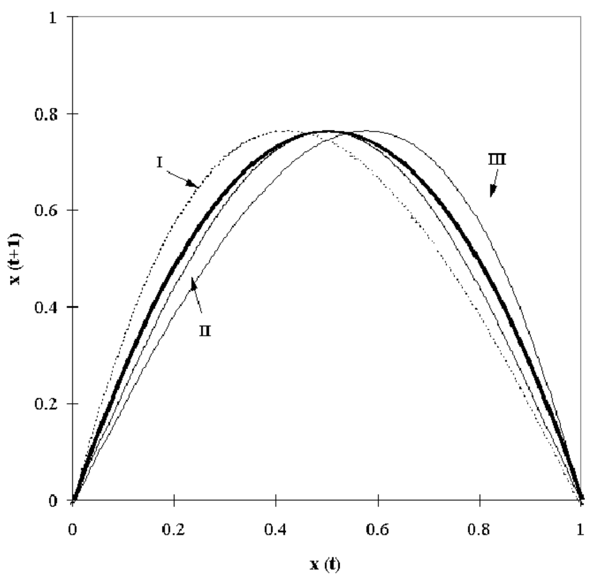

Fig. 2. The difference between rescaled curves in windows I, II and III in Fig. 1 B1 and parabola shown in Fig. 1 A1 — marked by bold line.

The differences between the first period doubling parabola and three of rescaled ones of the next period doubling is quite small, confirming the self-similarity of Eq. (6) in the pre-chaotic region ($R = 3.05$). Note that the parabola consists of only few points if one initial condition is taken (for details see Concluding Remarks). The rescaling factor for the transition from period of length 2 to period of length 4 is equal to $-(2+2/R_1) = -(2+\gamma) = -2.72$, where γ is the golden mean [4] (the minus accounts for the upside-down orientation). The more spectacular evidence of self-similarity in the pre-chaotic region is obtained by drawing the bifurcation diagram of the cascade of solutions to Eq. (6) (see Fig. 3). In the next section we would like

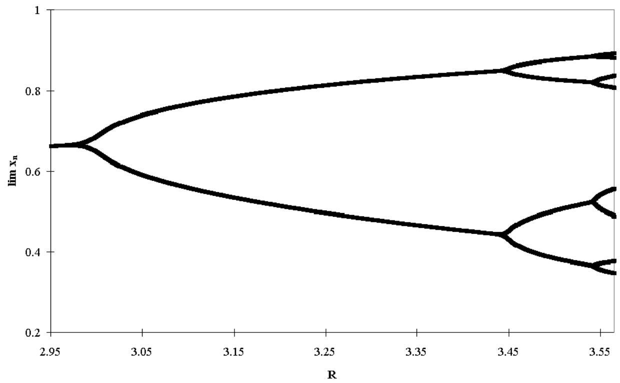

Fig. 3. The bifurcation diagram of the logistic map (6). $R = 3.05$ and $R = 3.47$, analysed in Fig. 1, lie after two bifurcation points.

to show the self-similarity in a more practical and direct way for the whole range of $R \in [0, 4]$ based on the measures mentioned above.

3. Results and discussion

As it was shown in the Introduction, self-similarity of a time series means that some property $L(t)$ measured at scale t is proportional to that property measured at scale "λt"

$$L(\lambda t) = kL(t)\,, \tag{7}$$

where $k = \lambda^{d_T - d_F}$ is a proportionality parameter [7]. For iterative maps the relation (7) holds for t, $\lambda \in N$. Our further analysis will go on the basis of Eq. (7) and the fact that the output generated by the map (6) can be treated as an experimental data. Consequently, we will select every n-th (odd or even) point of an original set, and draw it versus time as it were "measured" with different resolution. We will do it for different values of R along with the graphs of their measures (R, power spectrum, and k). When we compare some average measures of the two time series of the same length, like the area under the curve, power spectrum on logistic parameter R, we can see less difference when k tends to 1. We will start with $R \in (0, 1]$ for, as it was mentioned in the previous chapter, the map (6) is also defined for that region.

When $R \in (0, 1]$ the logistic map (6) generates a time series which monotonically reaches its stationary solution $x_s = 0$. Figure 4A shows first 14 iterations of the map (6) together with the time series created by choosing every second (frequency of probing $d = 2$), fifth ($d = 5$), and tenth ($d = 10$) point of the original set. It can be seen that the general pattern is preserved in all cases, showing slight differences in the speed of approaching the stationary solution. The value of logistic parameter R has been chosen as a first measure of self-similarity. The time series has been treated as experimental data which we want to fit with the map (6). We have used the standard nonlinear regression (performed numerically), namely minimalization of the sum of square differences between data and time series generated from Eq. (6) [6]. For all series R appeared to be the same, and equal to 0.5. Self-similarity has also been confirmed through the power spectrum and the area under the curve (integral of a time series) (see Figs 4B and 4C).

The estimation of Liapunov exponent requires a wider comment. According to the definition given in [6], Liapunov exponent measures the average increase of initial condition error ϵ in propagation of the time series. It is equal to the average divergence of two trajectories starting at the points x_0 and $x_0 + \epsilon$, which can be expressed as the sum of differences calculated at every iteration step:

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} |x_i' - x_i| \,, \tag{8}$$

where x_i' and x_i are the i-th iterations of logistic map of two trajectories

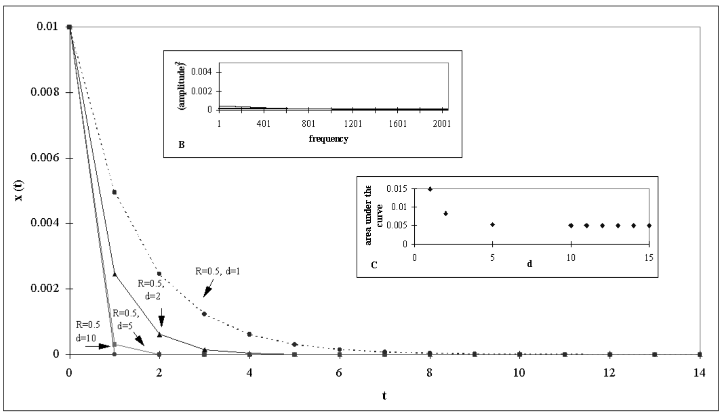

Fig. 4. Self-similarity of the logistic map (6) for $R = 0.5$. A — first iterations of logistic map (6) for $R = 0.5$ and $x_0 = 0.01$ together with time series formed by choosing every second ($d = 2$), fifth ($d = 5$) and tenth point ($d = 10$) of the original set. All series were fitted with the map (6) values of R found by nonlinear regression are given on curves; B — power spectrum of time series presented fragmentarily in Fig. 4A, hardly distinguishable from each other; C — area under the curves of time series shown fragmentarily in Fig. 4A (for 4000 iterations). For $d > 10$ area takes a constant value.

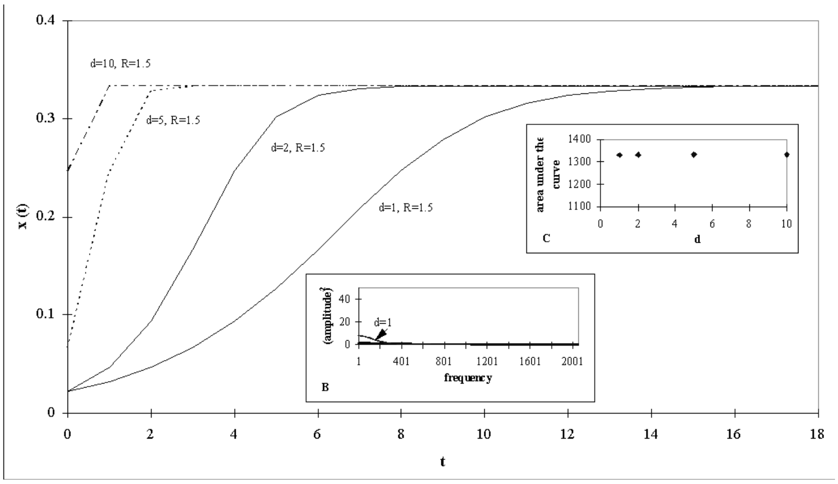

Fig. 5. Self-similarity of the logistic map (6) for $R = 1.5$. A — first iterations of logistic map (6) for $R = 1.5$ and $x_0 = 0.01$ together with time series formed by choosing every second ($d = 2$), fifth ($d = 5$) and tenth point ($d = 10$) of the original set. All series were fitted with map (6) values of R found by nonlinear regression have been given on curves; B — power spectrum of time series presented fragmentarily in Fig. 5A, hardly distinguishable from each other; the highest amplitude occurs in low frequency for $d = 1$; C – area under the curves for time series shown fragmentarily in Fig. 5A (for 4000 iterations).

starting at $x_0 + \epsilon$ and x_0, respectively. Because we do not consider here an exponential growth of error, which is usually present in the formula of Liapunov exponent [6], we will henceforth call λ a divergence parameter. For all series presented in Fig. 4A the value of λ has been found to be zero. On the basis of the above analysis one can conclude that the time series generated from the map (6) when $0 < R \leq 1$ is self-similar in a trivial way. The time series reaches zero value very quickly and the problem is reduced to similar behaviour of a constant, zero function on different time scales.

The second case considers the range of $R \in (1, 3]$. Figure 5A shows first iterations of map (6) for $R = 1.5$ together with time series generated by picking every 2nd, 5th and 10th point of the original set. All series are increasing, however, the difference in time of reaching the plateau value can be observed. The three generated sets were subjected to a fitting procedure by map (6). Logistic parameter R is equal to 1.5 for all series. Fourier transform, the next measure of self-similarity, shows minute difference in the amplitude only at low frequency (see Fig. 5B). Values of the area under the curve change insignificantly with d (Fig. 5C). Divergence parameter, defined by Eq. (8), is close to zero for all cases. The above analysis shows that similarly to the previous case, the time series produced by the logistic map is trivially self-similar in the range of $R \in (1,3]$. When $R \in (3, 3.57)$,

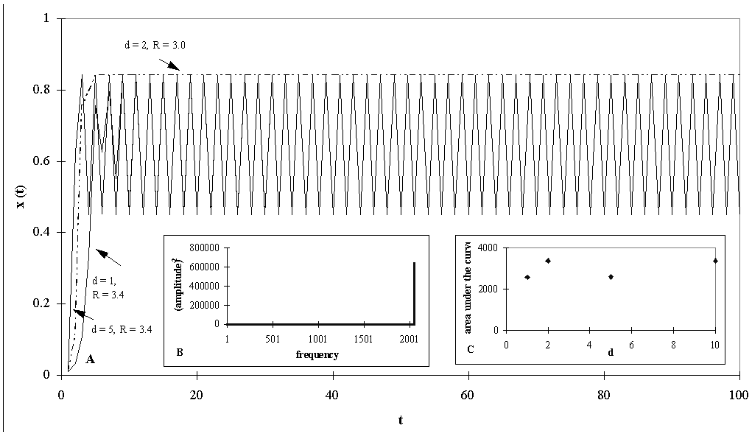

Fig. 6. Self-similarity of the logistic map (6) in the pre-chaotic region, for $R = 3.4$. A — first iteration of the map (6) for $R = 3.4$ and $x_0 = 0.01$ together with time series created by choosing every second ($d = 2$) and fifth ($d = 5$) point of original set. Values of R given in figures were found by fitting presented series with the map (6), by means of nonlinear regression; B1–B3 — power spectrum of time series presented fragmentarily in Fig. 6A for $d = 1$ and $d = 5$, indistinguishable from each other. Power spectrum for $d = 2$ is a zero function; C — area under curves (for 4000 iterations) shown fragmentarily in Fig. 6A.

time series generated from map (6) behaves periodically with period 2^n; n is a natural number increasing with R. At $R = 3.57$, n reaches infinite value and that point is considered as the onset of chaos. Using the procedure of picking every second and fifth point (as described previously) of the time series generated from map (6) for $R = 3.4$, we have got series presented in Fig. 6A. It can be seen that for even values of d a monotonically increasing function is obtained, while for odd d we get the periodic behaviour similar to that of original set. "Visual" difference has been confirmed by fitted value of R, which is equal to 3.4 for $d = 2n + 1$, and 3.0 for $d = 2n$. In Fourier transform spectrum the characteristic peak at the highest frequency is observed only for odd frequency of probing. The area under the curves fluctuates around the average value of 3000 (Fig. 6C).

When we identify the scaling factor λ in Eq. (7) as equal to d and $L(t)$ is an area under the curve, then for odd probing frequency the parameter $k = 1$, while for even d, k equals approximately 1.3.

The fourth case considers the range $R \in [3.57, 4]$. We have examined two values of R, namely $R = 3.6$, being at the beginning of the chaotic range, and $R = 4.0$, where chaos is fully developed. Figure 7A1–A3 presents time series

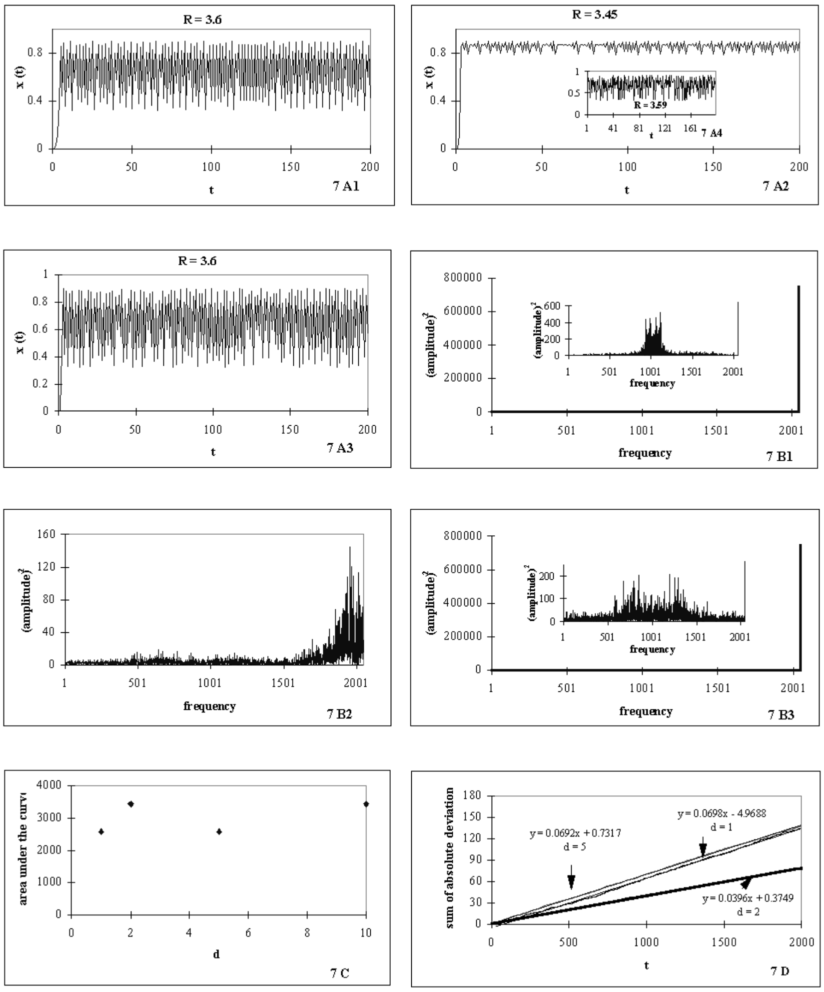

Fig. 7. Self-similarity of the logistic map (6) in the chaotic region, for $R = 3.6$. A1 — first iterations of the map (6) for $R = 3.6$, $x_0 = 0.01$; A2–A3 — sets obtained from the series shown partly in Fig.7 A1, by choosing every second ($d = 2$) and fifth point ($d = 5$), respectively. Values of R were found by the standard fitting; A4 — rescaled A2 series. B1–B3 — power spectrum of time series whose fragments are presented in Figs. A1–A3, together with "blown-up" substructure; C1–C3 — area under the curves (for 4000 iterations), presented fragmentarily in Figs. A1–A3; D — Change of the divergence parameter (equal to the slope) of two trajectories developed by the map (6) for $x_0 = 0.01$ and $x_0 = 0.01 + 10^{-7}$.

generated from map (6) for $R = 3.6$ and sets formed by picking its every second and fifth point. The striking "visual" difference in behaviour for odd

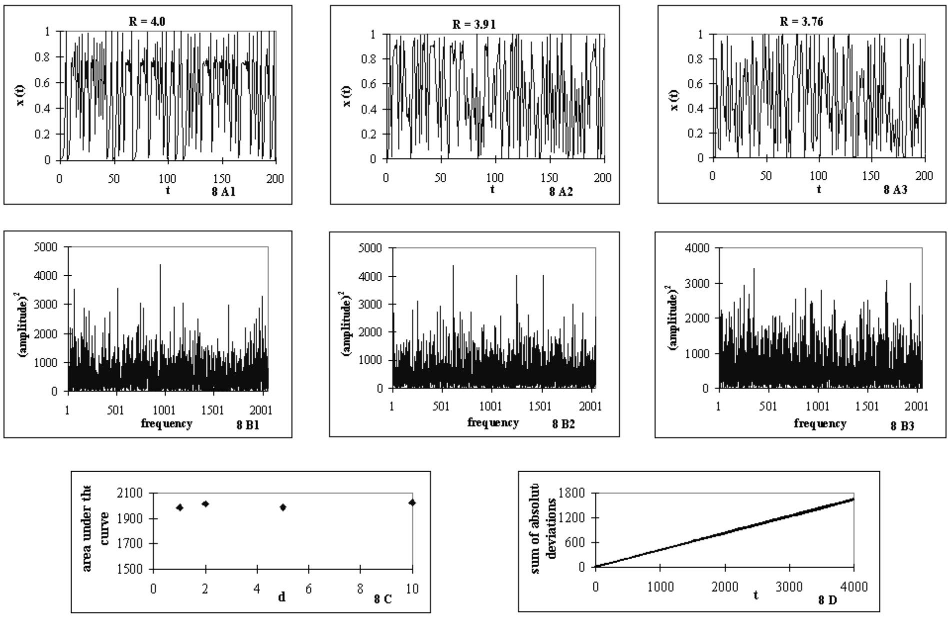

Fig. 8. Self-similarity of the logistic map (6) in the chaotic region. A1 — Logistic map (6) iterations for $R = 4.0$; A2, A3 — time series received from the set shown in Fig. A1 by picking every second ($d = 2$) and fifth point ($d = 5$) of the original set. Values on figures were found by fitting them with the map (6), using the criterion of square error minimalization; B1–B3 — power spectrum of time series presented fragmentarily in Figs A1–A3; C — area under the curves (for 4000 iterations) shown fragmentarily in Figs A1–A3; D — Increase of the divergence of two trajectories developed by the map (6) for $x_0 = 0.01$ and $x_0 = 0.01{+}10^{-7}$. Divergence parameter λ is equal to the slope of obtained straight line, estimated by linear regression. Average λ for $d = 1$, $d = 2$ and $d = 5$ is equal to $\langle\lambda\rangle = 0.411 \pm 0.005$.

and even d is observed and revealed by all measures of self-similarity used. The logistic parameter fitted by nonlinear regression equals to $R = 3.45$ for d even, and 3.6 for odd ones. Time series also differ in the character of their behaviour: it is more chaotic for odd probing frequency (Fig. 7D). If L in Eq. (7) is an area under the curve, $k = 1$ for odd frequency of probing and k is close to 1.3 for even d values. After linear scaling of series presented in Fig. 7A2 a new set, similar to the original one (Fig. 7A1), has been obtained (see Fig. 7A4). The same analysis has been performed for $R = 4.0$. All series obtained by rescaling the set generated from map (6) is similar to the original series, which has been confirmed by all self-similarity measures used above (Figs 8A–D).

We would like to emphasise that in the chaotic region we cannot expect the self-similarity measures to be exactly the same for all series, *i.e.* the the original and rescaled ones. The identical values of R, divergence parameter and the same power spectrum patterns have been obtained only in the region $R \in (0, 3.0)$, where self-similarity takes, more or less, an exact, or better yet, trivial form. Let us conclude the above results in a table form (Table I).

TABLE I

Self–similarity dependence on parameter R

Range of R	0–1	1–3	3–3.57	3.57–4
self–similarity	exact in a sense of Eq. 3	exact in a sense of Eq. 2	dual *i.e.* present, and exact in a sense of Eq. 2 for d-odd; absent for d-even	present, and exact in a sense of Eq. 2 for d-odd; present, exact in a sense of Eq. 3 for d-even

4. Final remarks

To start with, let us notice that the iterative map is a very specific mathematical object very often confused with the finite difference representations of differential equations [10]. A full discussion of the above, rather serious problem is provided elsewhere [9].

We have not been able to show the self-similarity of logistic map solutions in the similar way as was shown for solutions to Eq. (1), *i.e.* by substitution. Instead, we have followed the classical, period-doubling, route to chaos [11], and shown an "experimental" way to do it. Using the renormalization group technique we have analysed the properties of the logistic, quadratic transformation, and shown the self-similarity of its geometrical representation, as it is presented in Fig. 1. The parabola shows the first iteration of a set of initial conditions (abscissa) so we can treat it as a sort of transient behaviour of the system. One should be aware of the fact that the asymptotic behaviour of time series for $R = 3.05$ and $R = 3.47$ is different — for a single arbitrarily chosen initial condition it consists of two and four points, respectively. Generally, for the regular and pre-chaotic region the attractor (shown in a pseudo-phase portrait) consists of finite number of points. The more chaotic behaviour, the more points belong to the attractor, and only for $R = 4.0$, when the chaos is fully developed, it forms a continuous parabola (see Fig. 9). The fractal dimension d_F of a pseudo-phase portraits starts, therefore, with the $d_F = 0$ for one point (in a regular region), passes through fractional values (the portrait consists of finite number of points) and reaches 1 for $R = 4$ (continuous parabola). In the light of the above we would like to rise a question about the possibility of performing the local analysis of maps [4], which is based, from the very definition, upon the infinitesimal calculus. For the map we have fixed $\Delta t = 1$ and one can hardly see the existence of an infinitesimal distance between two subsequent states of a system.

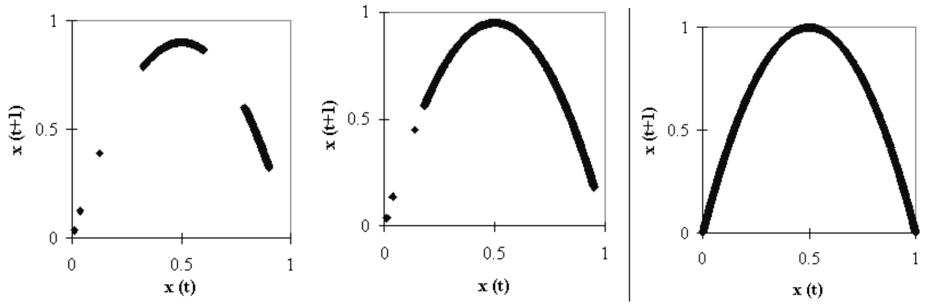

Fig. 9. Pseudo-phase portraits of a logistic map for $x_0 = 0.01$ and $R = 3.6$ (A), $R = 3.8$ (B) and $R = 4.0$ (C). Note their uncontinuous character for $R < 4.0$.

An "experimental" way of showing the self-similarity of solutions to the logistic map is based upon looking at the time series with different resolution and treating new obtained series as experimental data and fitted them with the map (6). Time series were compared on the basis of R, power spectrum and divergence parameter.

In the whole range of logistic parameter R solution to map (6) is self-similar, however, for regular and periodic regions its self-similarity is trivial.

A fractal dimension of a time series generated from the logistic map, and calculated on the basis of integral scaling in time, has been found to be approximately equal to 1. It seems to be consistent with a generic property of maps, which generate only next step from the previous one. The integral of a map solution scales linearly in time for the whole range of R, however, for more chaotic behaviour of a time series, a slower increase of area is observed. If x is identified as ionic current flowing through a narrow channel [9], then a smaller integral (charge) may indicate that ions meet more obstacles, which hamper their movement down to the potential difference.

It is worth mentioning that the presented way of demonstrating self-similarity may also be used in case of stochastic approach to the modelling of ionic transport through biological cell membranes [11, 12].

Careful reading of the manusctipt, and valuable suggestions by Jacek Stolarczyk are greatly acknowleged. This research was supported in Part by the STU grant BK-292/RCH4/97.

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