SPATIO-TEMPORAL CHAOS IN A FINITE CHAIN OF DAMPED SPINS*

A. Jaroszewicz[a] and A. Sukiennicki[a,b,c]

[a]Institute of Physics, Polish Academy of Sciences
Al.Lotników 32/46, 02-668 Warsaw, Poland
[b]Institute of Physics, Warsaw University of Technology
Koszykowa 75, 00-662 Warsaw, Poland
[c]Department of Solid State Physics, University of Łódź
Pomorska 149/153, 90-283 Łódź, Poland

(Received September 16, 1997)

Deterministic chaos in a finite chain of coupled damped classical spins in the presence of external oscillating magnetic field is numerically investigated. The influence of a size of the chain is considered. Various routes to chaos are found. In some ranges of the control parameters the coexisting attractors are obtained.

PACS numbers: 05.50. +q, 05.45. +b

1. Introduction

Deterministic chaos in spatially extended nonlinear systems was usually investigated by use of simple models such as coupled maps lattices [1]. Various kinds of behaviour were observed: periodic, quasiperiodic and chaotic regimes, the intermittency, the coexistence of regular and chaotic regions, domain-like structures *etc.* The maps chosen for such investigations were usually of the mathematical character and they were not derived from the equations of motion of real systems. Ordinary differential equations (ODEs) seem to be more appropriate for the description of objects in real lattices. Such an aproach has been applied to various systems of coupled physical objects, *e.g.* Umberger *et al.* [2] considered the one-dimensional lattice of dissipative forced Duffing oscillators, and Geist and Lauterborn [3] investigated the dynamics of a periodically driven damped Toda chain.

* Presented at the X Symposium on Statistical Physics, Zakopane, Poland, 1997.

In our earlier paper [4] we investigated another nonlinear spatially extended system, namely, the one-dimensional system of N=100 classical Heisenberg spins in uniaxial anisotropy field with the presence of damping and external oscillating magnetic field. We obtained a rich variety of attractors as the control parameter (an external field amplitude) was varied.

Here we shall present some results on the influence of the size of the chain on its dynamics [5].

2. Equations of motion and method of calculation

We consider a one-dimensional chain of N classical damped spins $\boldsymbol{S}_i$ ($i = 1, 2, \ldots, N$) of constant length S, with uniaxial anisotropy, in the presence of external oscillating magnetic field $\boldsymbol{B}(t) = B_0 \cos(\omega_0 t)\, \boldsymbol{e}_x$. This system is described by the Hamiltonian:

$$H = -J \sum_{i=1}^{N-1} \boldsymbol{S}_i \boldsymbol{S}_{i+1} - \kappa \sum_{i=1}^{N} (S_i^z)^2 - B \sum_{i=1}^{N-1} S_i^x \,, \tag{1}$$

where J and κ are ferromagnetic exchange and anisotropy constants, respectively. The time evolution of the system is described by the Landau–Lifshitz equation of motion with the Landau damping term

$$\delta \boldsymbol{S}_i / \delta t = \boldsymbol{S}_i \times \boldsymbol{B}_{i,ef} - (\lambda/S)\, \boldsymbol{S}_i \times (\boldsymbol{S}_i \times \boldsymbol{B}_{i,ef}) \,, \tag{2}$$

where $\boldsymbol{B}_{i,ef} = -\delta H / \delta \boldsymbol{S}_i$ is an effective magnetic field and λ — damping coefficient. In the spherical coordinate system, with time t rescaled by (t/KS) and new parameters defined as follows $\varepsilon = J/K$, $h_x = B_0/KS$, we can transform the equation (2) to a system of N coupled pairs of ordinary differential equations. These equations were solved numerically using the predictor-corrector method. Damping coefficient $\lambda = 0.1$ was assumed.

We chose initial conditions natural from the physical point of view: it was a state of the homogeneously magnetized chain with the spherical angles $\theta_i = \theta_0$, $\phi_i = \phi_0$ ($i = 1, 2 \ldots, N$) with a small random noise of an amplitude equal to 0.05. Most of the calculations were concerned with two initial states: magnetization along the easy axis z ($\theta_0 = \phi_0 = 0$) and magnetization along the x-axis ($\theta_0 = \pi/2, \phi_0 = 0$). Other values of θ_0, ϕ_0 and also the noise of much greater amplitudes did not yield any qualitatively different results. Periodic boundary conditions are used.

We observe the system dynamics stroboscopically with time-step equal to the period of the external field oscillation $T = 2\pi/\omega_0$.

3. Results

It is reasonable to present very shortly the main results obtained for the chain of 100 spins published in Ref. [4]. They are summarized by the Fig. 1(a), where ranges of existence of various types of solutions are depicted.

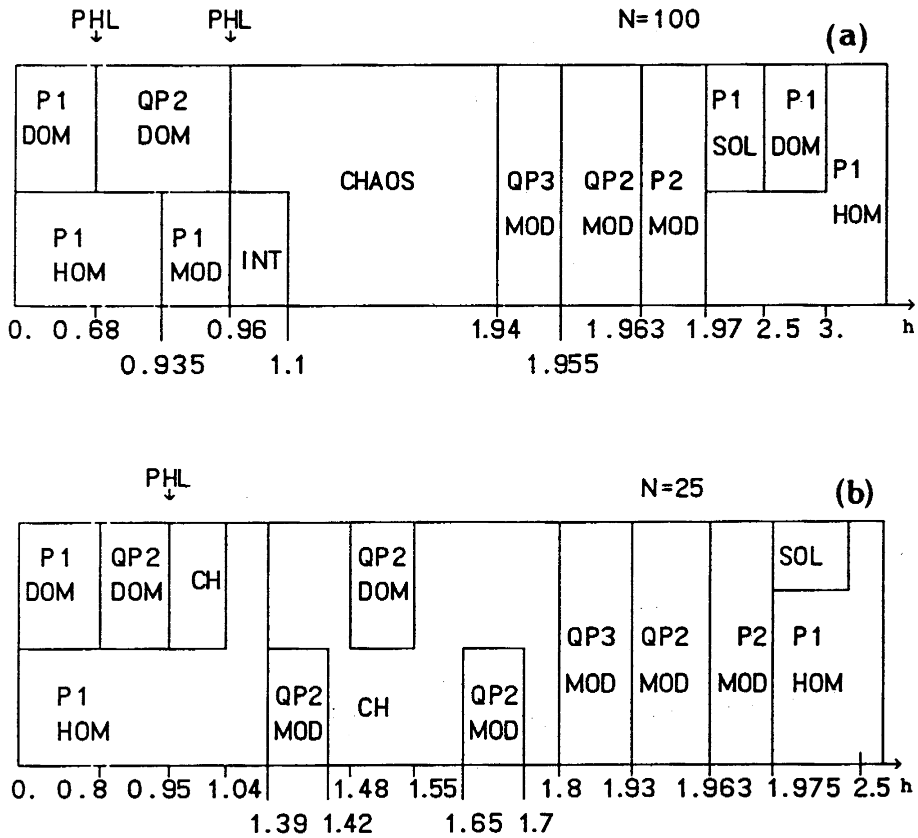

Fig. 1. Types of attractors for a chain of (a) $N = 100$ spins and (b) $N = 25$ spins. Upper rows correspond to the initial condition with magnetization along the x-axis and the lower — with magnetization along the z-axis. $P1$, $P2$ means attractors periodic and double-periodic. $QP2$, $QP3$ — attractors quasi-periodic on two- and three-dimensional tori. PHL — phase-locked, CH — Chaotic, HOM — homogeneous, DOM — domain-like, MOD — modulated, SOL — with solitons.

The chain with $N = 50$ spins exhibits basically the same sequence of attractors as that with $N = 100$ and only the interval of h_x for chaotic regime shrinks a little.

On the other hand, it is reasonable also to present the dynamics of a single spin under the same condition as for the rest of our calculations: the spin evolves periodically for all values of the external field amplitude except the interval $1.465 < h_x < 1.775$, where quasiperiodic motion on a two-dimensional torus T^2 exists.

The chain of $N = 5$ spins exhibits time behaviour similar to that of a single spin. The following sequence of motions is observed:

— for $h_x < 1.385$ — periodic in time with homogeneous spatial structure,

— for $1.385 < h_x < 1.4$ — periodic with modulated spatial structure,

— for $1.4 < h_x < 1.78$ — quasiperiodic on a two dimensional torus T^2 with modulated spatial structure,

— for $1.78 < h_x < 1.802$ — intermittency,

— for $h_x > 1.802$ — periodic with homogeneous spatial structure.

Transition from quasiperiodic to periodic motion by intermittency is the main difference if compared with the single spin. However, the chains of $N = 4$ and $N = 6$ spins do not exhibit such an intermittent transition and all the attractors are of homogeneous spatial structure. The main result obtained for short chains is the appearance of only one kind attractor for one set of parameters (instead of two different attractors observed for various initial conditions ($\theta_0 = 0$ and $\theta_0 = \pi/2$) in the case of longer chains). The domain structure cannot appear in such a short chain, therefore, the spatial structure which we observe is homogeneous or modulated.

The most interesting dynamics of the chain with $N = 25$ has some features of the long chains and some of the chain with $N = 5$ spins. Fig. 1(b) depicts the types of attractors observed for the chain of $N = 25$ spins.

We shall first describe a more interesting case with the initial condition $\theta_0 = \pi/2$, $\phi_0 = 0$, when the spins are "frustrated" at the first moment. For weak driving amplitudes $h_x < 0.8$ we get the time period-1 behaviour with the spatial domain-like structure. For $0.8 < h_x < 0.875$ we still observe the domain structure but the spins evolve quasiperiodically (Fig. 2(a)), which corresponds to a trajectory that covers uniformly two-dimensional torus T^2. The trajectories in subspaces corresponding to different spins are slightly different in shape (*e.g.* Fig. 2(b)), however, the power spectra for all spins are almost identical. Fig. 2(c) presents the power spectrum of the time series of $\cos\theta_i$ with 2^{13} time steps (equal to $T_0/8$). Two incommensurate frequencies are visible: the first corresponds to the frequency of the external field ω_0 and the second ω_1 the value of which depends on the value of h_x (*e.g.* for $h_x = 0.825$: $\omega_1 = 0.054\omega_0$). Their linear combinations are also observed. For $0.875 < h_x < 0.901$ the trajectories of spins bifurcate from the torus T^2 to the two-dimensional double torus $2T^2$ (Fig. 3(a)) and subharmonics $\omega_1/2$ appear in the power spectrum (Fig. 3(b)). The domain-like spatial structure

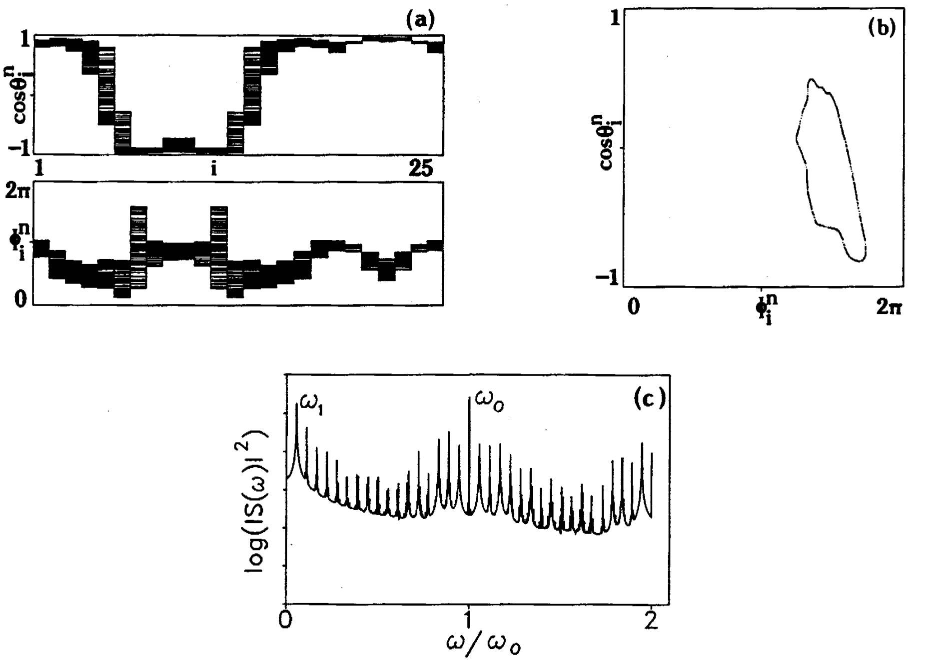

Fig. 2. Quasiperiodic motion at $h_x = 0.825$. (a) Plot of $\cos\theta$ and ϕ for $850 < n < 1450$ (600 profiles superimposed); (b) Poincare sections for spin $i = 5$; (c) power spectrum of $\cos\theta$.

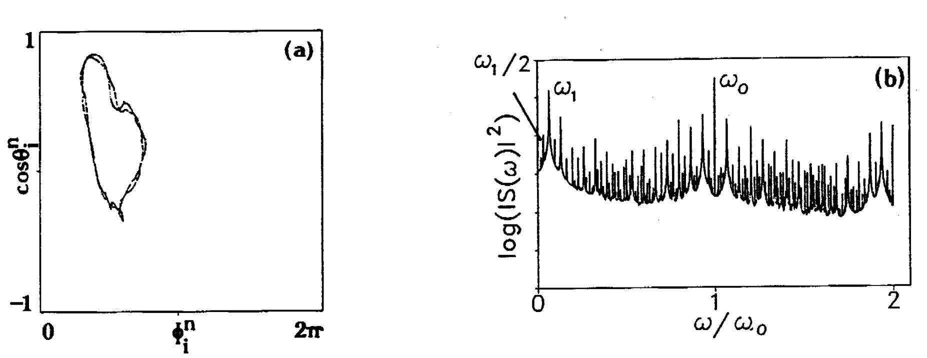

Fig. 3. Quasiperiodic motion on doubled torus at $h_x = 0.9$. (a) Poincare section; (b) corresponding power spectrum of $\cos\theta$.

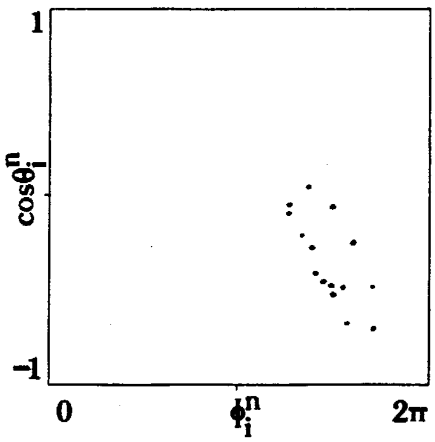

Fig. 4. Periodic motion on a phase-locked torus at $h_x = 0.91$: Poincare section of the trajectory of one spin.

still exists. A little higher field amplitude causes the system to evolve on the torus T^2 again. Further increase of h_x ($0.909 < h_x < 0.95$) leads to the complex system with coexisting attractors. For the same values of h_x and only the noise in the initial conditions changed slightly, three types of attractors can be found: quasiperiodic tori $QP2$ (the trajectory covers the two-dimensional torus), phase-locked tori $P21$ and $P15$ (two fundamental frequencies in the power spectrum are locked in the ratio $\omega_1/\omega_0 = 21$ and 15 (Fig. (4)), respectively, and chaotic. Starting from one initial condition with fixed values of all the parameters and changing only the small noise one can enter the basin of a qualitatively different attractor. We must emphasise here that these small differences in initial conditions are finite (two orders lower than the values of angles). The sensitivity to the initial conditions we are talking about corresponds to crossing over the basins of attraction and should not be mistaken with the sensitivity to the initial conditions that characterizes chaotic motion. The situation becomes more clear when one starts with one initial condition and increase gradually h_x. One can observe *e.g.* the transition to chaos from torus T^2 via phase-locked torus (see Ref. [6]). Fig. 5 presents a destabilizaed torus T^2 obtained for $h_x = 0.92$: its Poincare section (Fig. 5(a)), power spectrum (Fig. 5(b)), and exponential divergence of two initially close trajectories (Fig. 5(c)). The last figure enables us to estimate the value of the maximal Lyapunov exponent to $\lambda = 0.11$. We calculated also the correlation dimension [6] of obtained attractors. The calculations performed in the subspace of the phase space corresponding to 10 spins give the increase from $D_c = 2$ for tori T^2 at $h_x < 0.905$ to $D_c = 2.7$ for the strange attractor at $h_x = 0.92$ and $D_c = 6.7$ at $h_x = 1.0$. The domain-like spatial structure is still observed but it evolves irregularly. As

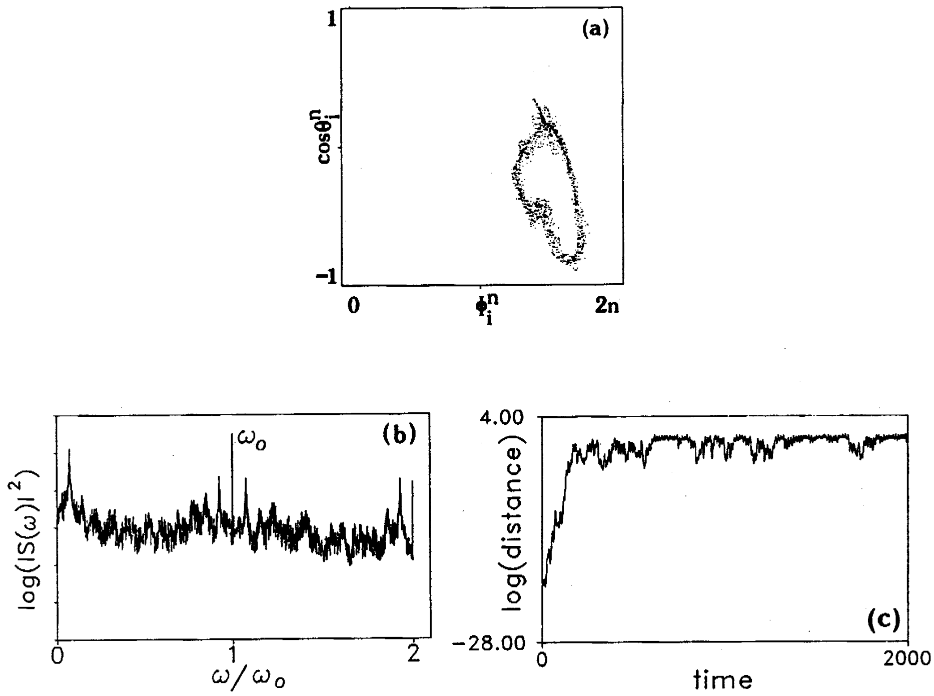

Fig. 5. Chaotic motion at $h_x = 0.9$. (a) Poincare section for one spin; (b) corresponding power spectrum of $\cos\theta$; (c) divergence of two initially close trajectories.

one goes further from the transition point, irregularity becomes stronger. Although the chaotic behaviour is observed for $0.91 < h_x < 1.795$, the periodic and quasiperiodic windows can be found. For $1.04 < h_x < 1.39$ there is the period-1 in time and homogeneous in space window and for $1.48 < h_x < 1.55$ there is quasiperiodic window with a domain spatial structure.

Now, let us describe the route from a chaotic to periodic regime for higher values of the external field amplitude. Figures 6 and 7 present Poincare sections (Figs. 6(a) and 7(a)), power spectra (Figs. 6(b) and 7(b)), and divergence of initially close trajectories (Figs. 6(c) and 7(c)) for the chaotic regime (at $h_x = 1.77$) and quasiperiodic one (at $h_x = 1.8$ — just above the transition point), respectively. The value of the maximal Lyapunov exponent at $h_x = 1.77$ can be estimated from Fig. 6(c) as equal to $\lambda = 0.23$ and the correlation dimension of the attractor considered is $D_c = 6.9$. The results indicate that the motion at $h_x = 1.77$ corresponds to the chaotic attractor. On the other hand, at $h_x = 1.8$ one finds slower than exponential (linear or,

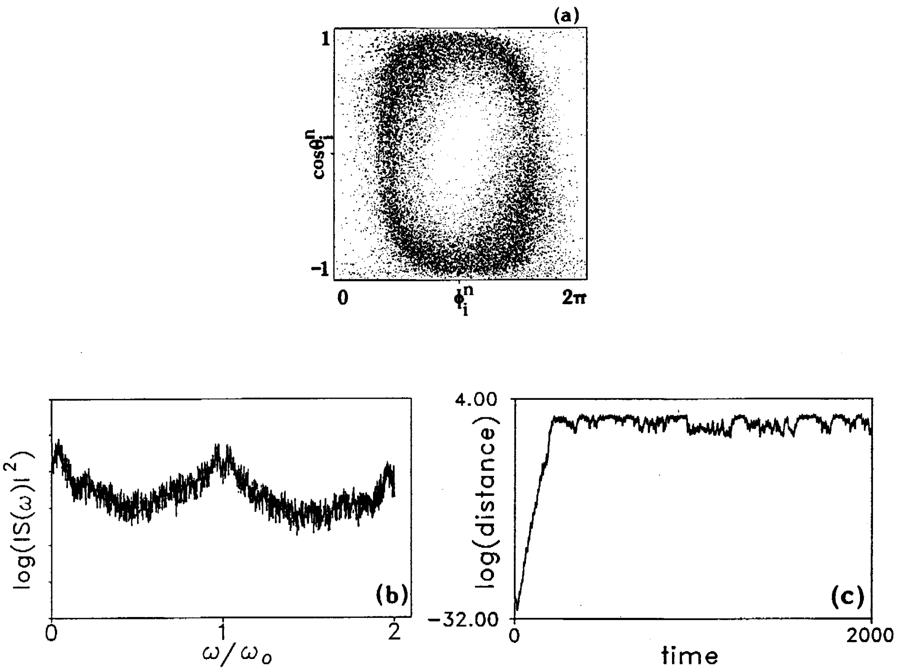

Fig. 6. Chaotic motion at $h_x = 1.77$. (a)–(c) — the same as for Fig. 5

most probably, power) divergence of initially close trajectories (Fig. 7(c)) which corresponds to the maximal Lyapunov exponent $\lambda = 0.0$. In the power spectrum in Fig. 7(a) one can find three incommensurate frequencies: ω_0, $\omega_1 = 0.008\omega_0$ and $\omega_2 = 0.042\omega_0$. The calculated correlation dimension equals to $D_c = 3.0$. These facts let us classify the attractor appearing at $h_x = 1.8$ as a three-dimensional torus T^3. Fig. 7(d) presents the snapshot of the spatial structure of the quasiperiodic attractor — it is a wave propagating along the chain.

For $1.93 < h_x < 1.963$ system evolves quasiperiodically on a two-dimensional torus T^2 with the spatially modulated structure.

For $1.963 < h_x < 1.975$ the period-2 evolution with spatially modulated structure is observed, and for $h_x > 1.975$ one enters the period-1, homogeneous in space regime. Stable, moving solitons observed for $N = 100$ in this case [4] are very rare.

What seems to be especially interesting in the dynamics of a chain of $N = 25$ spins is that similarly as for $N = 100$ the transition from the regular behaviour to chaos takes place in a different way than the transition from chaos to the regular behaviour for higher values of h_x.

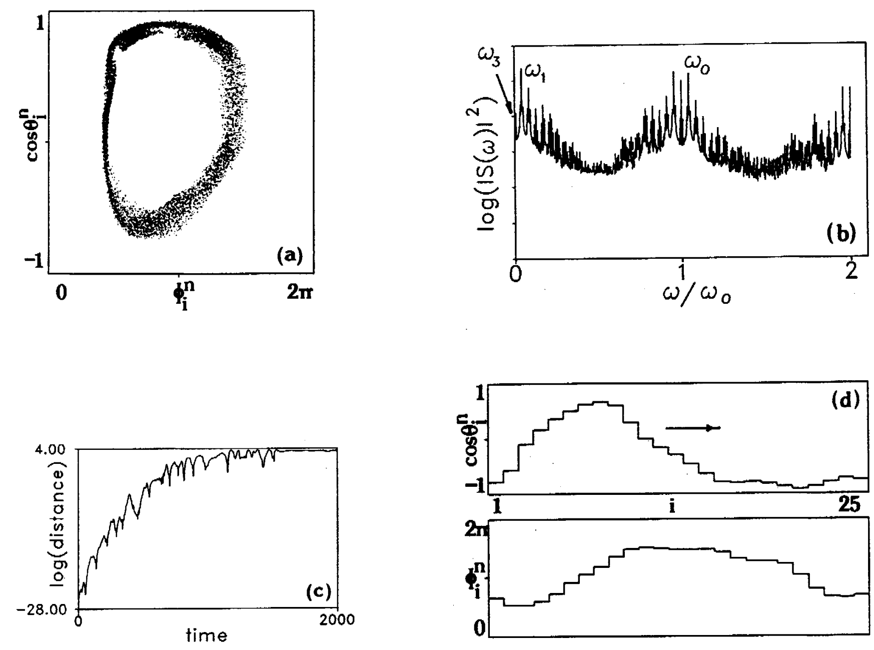

Fig. 7. Quasiperiodic motion on three-dimensional torus at $h_x = 1.8$. (a)–(c) — the same as in Fig.5; (d) snapshot of the attractor, indicated direction of motion of a structure.

Situation for the initial condition $\theta_0 = \phi_0 = 0$ differs from that described above mainly for smaller values of h_x. For $h_x < 1.395$ the period-1, spatially homogeneous attractor appears, for $h_x = 1.395$ the transition to the quasiperiodic $QP2$ regime with spatially modulated structure takes place, and then destabilization of two-dimensional torus T^2 leads directly to the chaotic regime. Inside the chaotic region one quasiperiodic $QP2$ window exists. Scenario of the transition from chaos to the regular behaviour for higher field amplitudes is the same as for the case considered previously.

4. Conclusions

Our results show that the size of the system of spins has an important influence on the dynamics of this system. Due to the periodic boundary conditions, very short chains exhibits basically the quasiperiodic route to chaos, although intermittency also appears, but only in a very narrow range of control parameters. The domain structure cannot appear in such a short chains, therefore, the spatial structures are usually homogeneous or slightly modulated.

On the other hand, the longer chains exhibit behaviour similar to that found in Ref. [6] for the chain of 100 spins. The most interesting case of the chain of 25 spins shows some features of very short chains and some of the long chains. In some ranges of the control parameters the very complex behaviour is obtained with coexisting attractors. Starting from one set of system parameters and changing only the small noise in initial conditions one enters the basin of a qualitatively different attractor. The interesting property in dynamics of the system is that the transition from the regular behaviour to chaos for smaller values of the external field amplitude appears in a different way than the transition from chaos to the regular behaviour for high values of this amplitude.

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