STOCHASTIC RESONANCE IN ON-OFF INTERMITTENCY*

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Stochastic resonance (SR) is studied in chaotic systems exhibiting on-off intermittency (OOI). They include a discrete-time system — the logistic map with the control parameter varying randomly in time and a continuous-time system — chaotic oscillators just below the synchroniation threshold. As a weak additive or multiplicative periodic forcing is added to such systems, the signal-to-noise ratio (SNR) exhibits a maximum as a function of the intermittency control parameter. In all cases SNR shows dependence on the forcing frequency. In the case of additive periodic forcing in continuous-time systems a distinct minimum of SNR is observed when the periodic forcing frequency is close to the characteristic frequency of chaotic oscillations of the system. In the case of multiplicative periodic forcing this dependence retains even for very small frequencies; this is a result of a very long characteristic time scale, typical of systems with OOI.

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1. Introduction

1.1. Stochastic resonance

The primary signature of SR is that addition of random (stochastic) noise can improve SNR at the output of a periodically modulated nonlinear system [1-3]. The power spectrum density (PSD) $S(f)$ of the output signal in systems with SR usually consists of peaks at the multiples of the periodic forcing frequency f_s superimposed on a broad noise background $S_N(f)$. SNR (in dB) for the first peak is defined as SNR$=10\log\left[S_P(f_s)/S_N(f_s)\right]$, where $S_P(f_s) = S(f_s) - S_N(f_s)$ is the first peak height. SNR exhibits a maximum as a function of the input noise power in systems with SR. So far,

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SR has been observed *e.g.* in bistable [4] and monostable [5] systems and in both dynamical [6] and non-dynamical [7] threshold-crossing systems.

The occurence of SR in systems in which chaotic rather than stochastic dynamics was used to improve SNR was also reported in [8-11]. Investigation of SR in chaotic systems offers a possibility to observe noise-free SR in which appropriate properties of internal dynamics of a system are used to amplify SNR instead of an external noise. In this contribution some results are presented, concerning the influence of additive or multiplicative periodic forcing on systems with OOI [12-14] and attractor bubbling [15-17]. The results were obtained by means of numerical simulations, but the possibilities of their experimental verification are also discussed.

1.2. On-off intermittency and attractor bubbling

This kind of chaos-chaos intermittency is a phenomenon occuring in systems with a chaotic attractor contained inside an invariant subspace. For certain control parameter values a sequence of laminar phases, during which the system trajectory remains close to the invariant subspace, and bursts, during which it departs far from this subspace is observed [12]. For example two identical interacting chaotic systems, described by vector variables x, x' may possess an invariant subspace $x = x'$; if this subspace is stable, the two systems are synchronized [18, 19] and if not, OOI appears just below the synchronization threshold in the time series of $x - x'$ [20]. A generic model for OOI is the logistic map with the control parameter varying randomly in time [13]

$$y_{n+1} = ax_n y_n (1 - y_n) + \xi x'_n . \tag{1}$$

Here, x_n, x'_n are random uncorrelated variables with uniform distribution at [0,1), a is the system control parameter and ξ is the thermal noise amplitude. If $\xi = 0$ and $a < a_c = e = 2.71\ldots$ then y_n always approaches the invariant subspace $y_n = 0$, independently of the choice of initial conditions. Inside this surface there is a stable, noisy attractor $0 \le x_n < 1$. If $a > a_c$ but still $a \approx a_c$ the so-called blowout bifurcation [21] occurs and the system exhibits OOI: during long laminar phases y_n remains practically equal to zero but occasionally increases rapidly and a chaotic burst appears. The probability that the laminar phase has length τ obeys a power-law scaling $P(\tau) \propto \tau^{-3/2}$ and the mean laminar phase length decreases according to another power law $\langle\tau\rangle \propto (a - a_c)^{-1}$ [13]. Before the blowout bifurcation occurs the subspace $y_n = 0$ loses the asymptotic stability for $a > a_b = 1$. Then under the influence of any small perturbation destroying the invariant subspace, *e.g.* thermal noise $\xi > 0$, the bursts characteristic of OOI appear already for $a > a_b$, below the OOI threshold [14]. This phenomenon is called attractor bubbling [15-17].

2. Models

In this contribution, SR was investigated in systems with OOI and with both discrete and continuous time. The discrete-time model is the map (1) with either additive (amplitude δ) or multiplicative (amplitude ε) small periodic forcing

$$y_{n+1} = [ax_n + \varepsilon (1 + \cos 2\pi f_s n)] \, y_n (1 - y_n) + \delta (1 + \cos 2\pi f_s n) \ . \qquad (2)$$

The measured variable is y_n. The continuous-time model consists of a set of two Rössler oscillators with two-way coupling via the y variable (coupling strength k) and with a small additive periodic term (amplitude δ) added to the coupling term in one of the oscillators

$$\begin{aligned} \dot{x} &= -(y+z), \quad \dot{y} = x + ay + k\left(y' - y\right) , \\ \dot{x'} &= -\left(y' + z'\right), \quad \dot{y'} = x' + ay' + k\left(y - y' + \delta \cos 2\pi f_s t\right) , \\ \dot{z} &= b + z(x - c) , \\ \dot{z'} &= b + z'\left(x' - c\right) . \end{aligned} \qquad (3)$$

The parameters are $a = b = 0.2$ and $c = 10$. For $\delta = 0$ the two systems synchronize if $k > k_c = 0.12$ and if $k < k_c$ OOI appears. The measured variable is $\Delta y(t) = y(t) - y'(t) + \delta \cos 2\pi f_s t$ (to make connection with the recently developed methods of secure commmunication [22]). A continuous-time model with multiplicative periodic forcing is under investigation yet.

The sequence of laminar phases and bursts in the time series of y_n in (2) and $\Delta y(t)$ in (3) suggests that SR should appear in these systems in a similar way as in dynamical threshold-crossing devices [6]. Thus, in the case of system (2) instead of y_n a two-state approximation Y_n of the full signal was analyzed: $Y_n = \Theta(y_n - y_{\rm thr})$, where $\Theta(\cdot)$ is the Heaviside unit step function and $y_{\rm thr} = 0.01$ was an arbitrarily chosen threshold for a burst. In the case of system (3) a three-state approximation was analyzed: $\Delta Y(t) = {\rm sign}(\Delta y(t)) \, \Theta(|\Delta y(t)| - \Delta y_{\rm thr})$, where$\Delta y_{\rm thr} = 0.1$. However, the similarity to threshold-crossing systems is not perfect. The time series consist of a sequence of pulses of unit height and various lengths rather than of short pulses with equal lengths as in typical threshold-crossing systems exhibiting SR [6, 7]. This resembles the situation investigated previously in a system with Pomeau–Maneville type-III intermittency [11], where the phases of periodic and chaotic motion play a role of the two states in a bistable (in general, asymmetric) system and under the influence of external periodic forcing their sequence has a strong periodic component.

3. Results

3.1. Additive periodic forcing. The discrete-time model

The results from the system (2) have been already discussed elsewhere [23], thus only a summary is given here. If $\varepsilon = 0$ then setting $0 < \delta \ll y_{\rm thr}/2$ is equivalent to switching on a periodic perturbation transverse to the invariant subspace $y_n = 0$ and for $a > a_b = 1$ the noisy attractor contained inside this surface bubbles. In Fig. 1(a) the SNR *vs.* a curves at the first

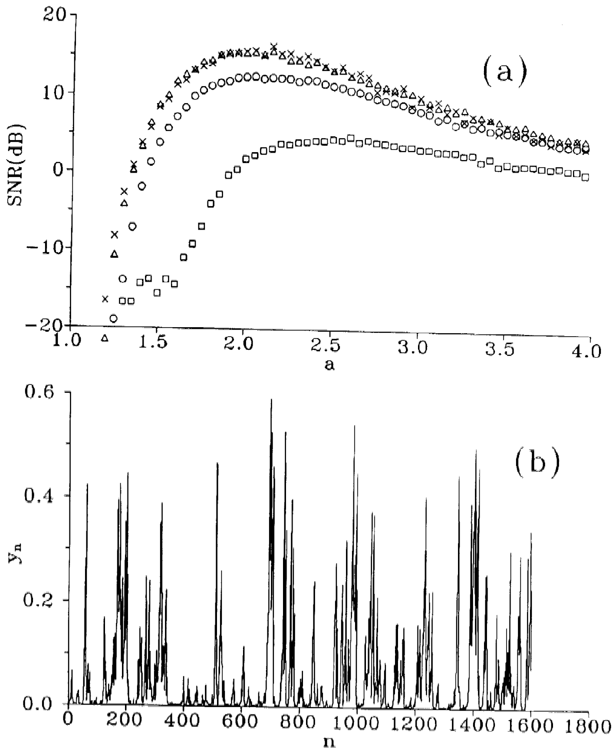

Fig. 1. SR in the map (2) with additive periodic forcing. (a) — SNR at the first harmonic vs. a, $\delta = 4 \cdot 10^{-4}$, $f_s = 1/8$ Hz (□), 1/16 Hz(○), 1/32 Hz(△), 1/128 Hz(×); (b) — Time series for y_n at $a = 2.5$, $\delta = 4 \cdot 10^{-4}$, $f_s = 1/8$.

harmonic of f_s are depicted for $\delta = 4 \cdot 10^{-4}$ and various f_s. The shape of the SNR curve is characteristic of SR. The maximum of SNR is located above $a = a_b$ but below $a = a_c$ for a wide range of frequencies f_s, thus here SR in attractor bubbling is obtained. It should be emphasized that y_n during the bursts can exceed $y_{\rm thr} = 0.01$ by more than one order of magnitude (Fig. 1(b)), thus the dynamics of the system (2) plays an important role in the occurence of SR in this case. The maximum value of SNR and its location strongly depend on the frequency of periodic forcing for high f_s, an effect rather characteristic of SR in bistable systems, but saturate for low frequencies. For $f_s = 2^{-3}$ a small additional peak occurs at $a \approx 1.4$ (hardly visible) which disappears for decreasing f_s, an effect also typical of bistable systems [24].

3.2. Additive periodic forcing. The continuous-time model

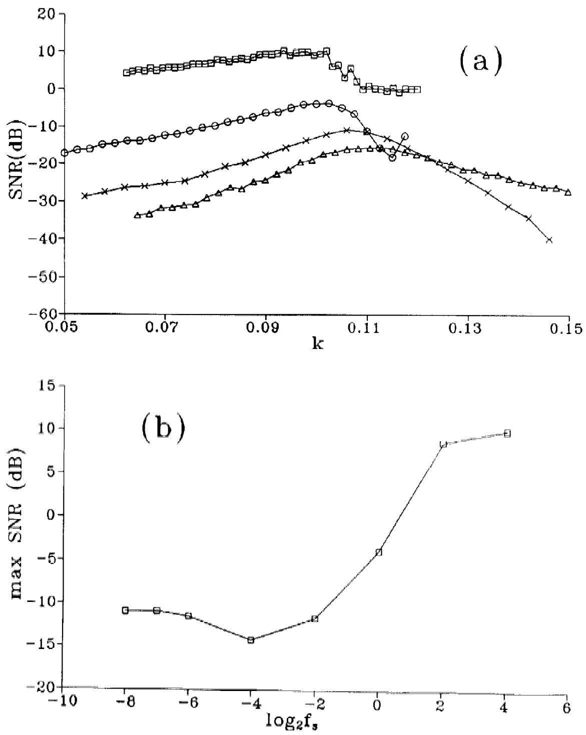

Fig. 2. SR in the system (3). (a) — SNR at the first harmonic*vs.* k, $\delta = 4 \cdot 10^{-2}$, $f_s = 16$ Hz (□), 1 Hz (○), 1/16 Hz (△), 1/256 Hz (×) (the solid lines are guides to the eyes); (b) — Maximum value of SNR as a function of f_s.

The discrete-time model enables us to observe SR only for f_s on the order of, or smaller than, the characteristic frequency of chaotic oscillations of a system with OOI. The effect of fast periodic forcing may be investigated using the system (3). The SNR vs. k curves at the first harmonic of f_s for $\delta = 0.04$ and a wide range of f_s are depicted in Fig. 2(a). The behaviour of these curves for small f_s resembles this in Fig. 1(a). For high f_s, however, SNR increases and thus the maximum value of SNR vs. k curves, measured as a function of frequency f_s, has a distinct minimum (Fig. 1(b)). This is the opposite of what can be inferred from the word "resonance" and what is really observed in some systems with SR [25], namely the maximization of the response of the system to periodic forcing with certain frequency. This effect may be easily understood. First, chaotic time series are smooth in a short time scale. Second, if a weak periodic signal $\delta \cos 2\pi f_s t$ is masked by the addition of any chaotic time series $y(t)$, it may be effectively (*i.e.* without distortion) retrieved by subtracting another time series $y'(t)$, where y and y' are synchronized: this is the idea of secure communication [22]. In the case considered here the amplitude of periodic signal is smaller than the threshold Δy_{thr} and thus the signal retrieved in such a way cannot be observed. However, if y and y' are just below the synchronization threshold, attractor bubbling appears and Δy may from time to time exceed Δy_{thr}. The periodic signal varies much faster than the intermittent bursts (smooth in a short time scale) and simply superimposes on them (almost without distortion), what leads to high values of SNR.

3.3. Multiplicative periodic forcing. The discrete-time model

The SNR vs. a curves obtained from (2) for $\delta = 0$ and $\varepsilon = 0.05$ are shown in Fig. 3(a) for decreasing frequencies f_s. If $f_s = 0.125$ only monotonic increase of SNR with a is observed within the borders allowed for a: $0 < a < 4 - 2\varepsilon$. The maxima in SNR vs. a curves can be seen only for low frequencies of the periodic forcing. Their values increase and their location shifts towards smaller values of a as f_s decreases. SNR does not saturate even for very small frequencies of the multiplicative periodic forcing.

As the overall scaling law $\langle\tau\rangle \propto (a - a_c)^{-1}$ for the mean laminar phase length in a system with OOI is known one can suppose that a theory based on the adiabatic approximation should predict correctly the values of SNR as a function of a for small f_s. The adiabatic approximation, valid for small periodic forcing frequencies, is based on the assumption that the threshold crossing rate, *i.e.* the reverse of the mean laminar phase length, depends periodically on time [4,6] as under the influence of the periodic multiplicative forcing also the OOI threshold a_c becomes periodically time dependent [23] (*cf.* [13]). However, there are several difficulties in constructing such a theory in the present case. First, the intermittent bursts are not single spikes, as in the case of typical threshold-crossing systems, but have rather a certain distribution of durations. The analytic formula for this distribution, and for their mean length is not known, though it was shown numerically that bursts obey the same scaling laws as laminar phases, only with different proportionality constants [26]. Thus, as already mentioned, the system should be considered as a bistable rather than threshold-crossing one. But, second, to my knowledge there is no a general adiabatic theory of SR in asymmetric bistable systems. Thus in Ref. [23] the adiabatic theory based on the re-

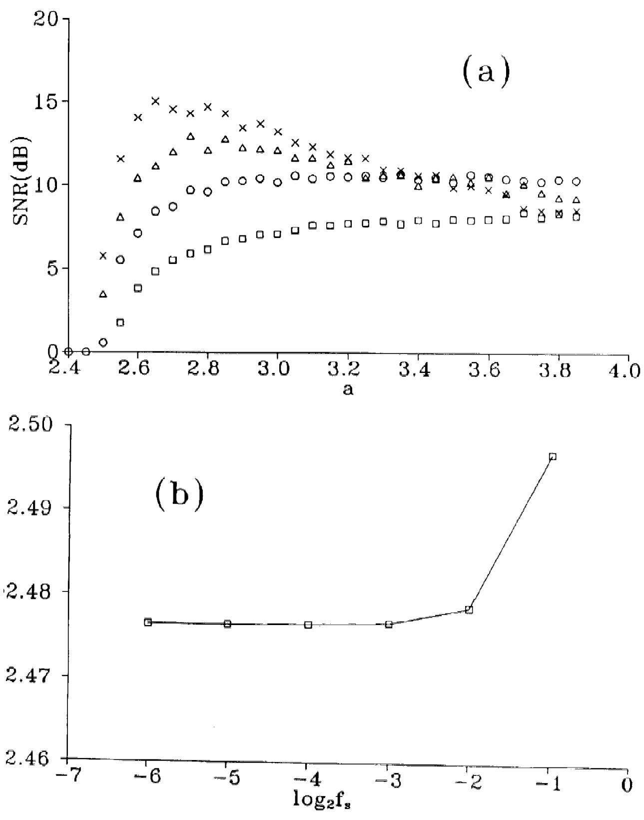

Fig. 3. SR in the map (2) with multiplicative periodic forcing. (a) SNR at the first harmonic vs. a, $\varepsilon = 0.05$, $f_s = 1/8$ Hz (□), $1/128$ Hz (○), $1/512$ Hz (△), $1/2048$ Hz (×), solid line — result of the simplified adiabatic approximation [23]; (b) Dependence of the OOI threshold $\tilde{a}_c$ on the frequency f_s of multiplicative periodic forcing with $\varepsilon = 0.05$ (results of (4)).

sults for threshold-crossing systems [6] was constructed which is valid only if the bursts occur seldom and have small lengths in comparison with laminar phases (*i.e.* just above the onset of OOI). The results are shown in Fig. 3(a) with a solid line. The numerical values of SNR are by some dB smaller than the ones evaluated on the basis of the above approximation, and, what is more important, SNR approaches zero for a considerably greater than it may be expected from the simple theory. It turns out that *e.g.* for $f_s = 1/8$ y_n falls to zero if $a < 2.47$ and no bursts appear for such a, so in the two-state approximation SNR $= 0$.

The latter discrepancy is rather not a result of the approximations applied but has its roots in the "averaging" properties of systems with OOI. The time-independent threshold for OOI with multiplicaltive periodic forcing $\tilde{a}_c$

may be evaluated as in the case $\varepsilon = 0$, by requiring that the time-averaged Lyapunov exponent in the direction perpendicular to the invariant surface $y_n = 0$ was equal to 1 [13]. This yields the condition [23]

$$\langle \ln [\tilde{a}_c x_n + \varepsilon (1 + \cos 2\pi f_s n)] \rangle = T_s^{-1} \sum_{n=0}^{T_s - 1} F(\tilde{a}_c, p(n)) = 0 \,, \tag{4}$$

where$\langle . \rangle$ denotes the average over one period $T_s = 1/f_s, p(n) = \varepsilon(1 + \cos 2\pi f_s n)$ and

$$F(a, p) = a^{-1} [(p + a) \ln (p + a) - a - p \ln p] \,. \tag{5}$$

The threshold $\tilde{a}_c$ evaluated in such a way is shown in Fig. 3(b) vs. the frequency f_s. *E.g.* if $f_s = 1/8$ this yields $a_c = 2.477$, in agreement with the numerical results. Therefore the system (2) with $\varepsilon > 0$ possesses a well-defined, time-independent OOI threshold; moreover, this threshold is frequency-dependent, a phenomenon not found in other chaotic systems exhibiting SR. It seems that even for small f_s this 'averaging' tendency prevails and only in the limit of extremely small f_s the OOI threshold becomes time dependent and closely follows the periodic forcing term; the characteristic time scale of the system is simply very long. Thus the theoretical description of SR in this case must go beyond the adiabatic approximation even for small f_s and it is still an open problem.

4. Summary and conclusions

In the present contribution, the effect of additive or multiplicative periodic forcing on the model chaotic threshold-crossing systems exhibiting OOI and attractor bubbling was investigated. In both cases SR was obtained in the two or three-state approximation.

In the case of additive forcing SNR depends on the forcing frequency, but for small f_s it is frequency independent. As it can be seen form the continuous-time model (3) for very high f_s, SNR increases and a minimum of SNR as a function of f_s is observed. The increase of SNR for high periodic forcing frequencies is connected with the smoothness of chaotic time series in a short time scale. Maybe, this is an important difference between stochastic and chaotic systems exhibiting SR. As far as I know, there is no systematic study of the influence of the correlation time of the noise on SNR in noise-driven systems with SR, in particular when this time is long in comparison with the forcing period. However, even a highly-coloured noise is not smooth in a short time scale and thus SNR need not increase with decreasing f_s as quickly as in chaotic systems. In my opinion, this problem deserves further investigation.

It is a good place here to emphasize that both in the case of additive and multiplicative periodic forcing SNR may be calculated also from the full output signal instead of its two or three-state approximation. In the case of additive forcing it was observed that SNR calculated from the full signal is for the whole range of a or k greater than SNR calculated from the two-state approximation and does not exhibit any maximum, but always decreases as the bursting behaviour becomes stronger. Thus the source of SR here is the addition of a threshold to the signal coming from attractor bubbling. In particular, SR does not improve the results of retrieval of periodic signal in secure communication using synchronized chaotic oscillators. In the case of multiplicative periodic forcing in (2) SR is observed also in the SNR vs. a curves obtained from the full signal y_n.

In the case of multiplicative periodic forcing SNR continues increasing even for very small f_s and the adiabatic approximation fails. This was shown to be connected with the properties of systems with OOI which tend to average the influence of the multiplicative forcing over time.

The results presented in this contribution are closely related to the observation of SR in a system with Pomeau-Maneville type-III intermittency [11]. However, the systems (2,3) have more in common with utilizing dynamical threshold-crossing devices in SR than the systems with "conventional" intermittency, in which a time sequence of chaotic and periodic phases with approximately the same amplitudes of oscillations is observed. On the other hand, *e.g.* in the case of additive periodic forcing, many properties of SR are similar to the ones obtained when this effect is investigated in bistable systems (*e.g.* SNR is strongly frequency-dependent for high f_s). This probably may be explained by the fact that intermmittent bursts are not single spikes, but their lengths have a certain distribution. Thus the two-state approximation produces a signal similar to what can be expected in asymmetric bistable systems. A close connection between such systems and threshold-crossing dynamics was pointed out also in the case of noise-driven systems in [27]; moreover, a dynamical system, but without OOI — a noise-driven semiconductor diode which produces time series consisting of short pulses of light of various length was proposed to look for SR in [28].

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