Vol. 29 (1998) *ACTA PHYSICA POLONICA B* No 6

TOOM PROBABILISTIC CELLULAR AUTOMATA STATIONARY STATES VIA SIMULATIONS* , **

Danuta Makowiec

Institute of Theoretical Physics and Astrophysics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk, Poland
e-mail: fizdm@univ.gda.pl

(Received October 6, 1997)

With computer simulations we investigate the basic thermodynamic features in the cellular automata governed by the stochastic three-spin majority vote, *i.e.* the Toom rule. The Gibbsianness of stationary states is tested by the relative entropy density. The critical exponents which characterize the ferro–paramagnetic phase transition are given.

PACS numbers: 05.50. +q, 05.70. Jk

1. Introduction

In general, there is rather a large discrepancy between physical systems present in the nature and models which can be investigated rigorously via statistical mechanics tools. Fortunately for the statistical mechanics modeling these both worlds seems to be very close to each other in the region of the critical properties. At criticality, the large scale properties are insensitive to the details of a lattice and microscale interactions. Therefore, equilibrium statistical mechanics has developed many powerful tools to study the phase transition and critical phenomena. There is little done in the theory of nonequilibrium systems. Hence, it is necessary and useful to study simple nonequilibrium systems. This paper concerns properties of stationary states which arise from the probabilistic cellular automata (PCA) with the evolution governed by the stochastic Toom rule [1]. The Toom PCA are the well known model which concerns the interacting particle system evolving stochastically in discrete time [2].

---

* Presented at the Marian Smoluchowski Symposium on Statistical Physics, Zakopane, Poland, September 1–10, 1997.

** This research was supported by grant KBN 129/P03/95/09. Simulations were performed partially in TASK — Academic Computer Center in Gdansk, Poland.

With our investigations we want to join the discussion on how close the stationary Toom system imitates an equilibrium system. Moreover, we want to characterize properties of the system that arises from the Toom interactions despite the fact whether the system possesses the Gibbsian property or not. Computer experiments are the source of our knowledge about stationary states. Using standard Monte Carlo methods we are given samples of configurations, which follow the distribution of the hidden Toom stationary measure. This procedure is well supported by the weak law of large numbers [2] and widely used (see *e.g.* [3, 4]).

Equilibrium statistical mechanics deals with Gibbs measures. These measures are known as the only measures which posses the "good" physical property, *i.e.*, isolating a part of a system from the rest will generally do not cause any drastic change in its behavior. The Gibbsian measures have been put in equilibrium statistical mechanics by hand [2, 3, 5], namely by assuming that the probability distribution for a finite lattice configuration σ to occur is expressed by the following famous formula:

$$\mathrm{d}\mu_{BG}(\sigma) = \frac{\mathrm{e}^{-\beta\mathcal{H}(\sigma)}}{\mathcal{Z}}\mathrm{d}\mu_0(\sigma)\,,$$

for $\beta = 1/kT$ the inverse of the temperature, μ_0 some *apriori* measure, and $\mathcal{H}$ is the energy carried by the configuration σ. In the case of infinite system this probability density is conditioned by a boundary configuration. Since one cannot expect all measures to be Gibbsian, the question arises if there are "weaker" conditions that capture "good" physical properties [2]. For example the stationary measure of the voter model (also called the Kawasaki model) competing with the Glauber dynamics although this stationary measure is non-Gibbsian, the Monte Carlo examinations provide for it the critical properties being the same as those for the universality class of Ising model [6, 7].

The theoretical investigations of the Toom PCA system are particularly difficult because there is not known the statistical description to the stationary measure at the area of the critical change [8]. In this area there is observed a battle between domains composed largely of -1's and domains composed largely of 1's in the stationary configurations. In the cellular automata theory such a behavior indicates the so-called complexity [9].

The results of the real experiments state that for some macroscopic observable $\mathcal{O}$ near the critical point depends on distance to the critical value along the power law, namely

$$\mathcal{O} \sim |T - T_{\mathrm{cr}}|^{\mu}\,.$$

According to the values of μ's thermodynamic systems are divided into different universality classes. It is tempting to suppose that the critical behavior of a system does not depend on whether the system is equilibrium or not [7, 10]. Therefore we also ask the question of what class of universality Toom PCA belong to.

2. Toom PCA definition

The Toom PCA denote two-level spin systems $\Omega = \{-1, 1\}$ placed on the square lattice $L \times L$, and the evolution of any spin σ_x, $x \in L \times L$ is conditioned by the states of its north and east nearest neighbors, and by the spin itself. Thus the states of these three spins determine the probability for the next time spin state at x, *i.e.* $(\sigma^t_{N(x)}, \sigma^t_{E(x)}, \sigma^t_x) \longrightarrow \sigma^{t+1}_x$ according to the following formula

$$p(\sigma^{t+1}_x | (\sigma^t_{N(x)}, \sigma^t_{E(x)}, \sigma^t_x)) = \tfrac{1}{2}[1 + (1-2\varepsilon)\sigma^{t+1}_x \mathrm{sign}(\sigma^t_{N(x)} + \sigma^t_{E(x)} + \sigma^t_x)]\,. \tag{1}$$

The ε parameter imitates the temperature-like effects in the local interactions.

For the Toom PCA there exists the rigorous proof [11, 12] that in the *space* $\times$ *time* lattice the system is Gibbsian with the Hamiltonian given by the formula:

$$\mathcal{H}(\sigma(x,t)) = -\log p(\sigma^t_x | \sigma')\,. \tag{2}$$

However, there is little known about the projection of this Gibbs measure to the space layers [13]. The fundamental example of the non-Gibbsian measure arises as the projection to one-dimension of the very well known Gibbs measure of the Ising model in two-dimensions [14].

3. Verifying Gibbsianness

The good, although rather rough, indicator of the character of the probability distribution is the cumulant U of the fourth order of magnetization. If the cumulant $U = 0$ then the distribution is Gaussian. If $U = \frac{2}{3}$ then we deal with bimodal distribution what indicates that two phases are present in the configuration state. In Fig. 1 we show values of the cumulant U obtained for the Toom PCA with different lattice sizes. One can observe the ε -range vs. lattice size L on which the bimodal distribution which characterizes the system before the critical point transfers into the Gaussian distribution. Therefore, one can expect non-Gibbsian properties on the stationary Toom PCA evolving within this region. In the following we will consider the two Toom systems: $\{L = 200, \quad \varepsilon = 0.090\}$ and $\{L = 60, \quad \varepsilon = 0.092\}$.

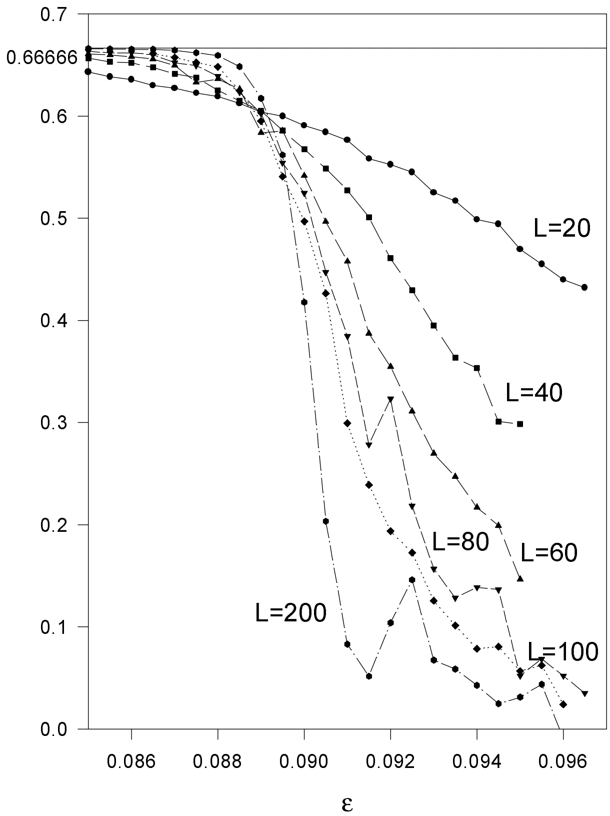


Fig. 1. The cumulant of the fourth order of the mean magnetization for different lattice size : $L = 20, 40, 60, 80, 100, 200$ in the critical region of the Toom PCA.

It is known that, if μ, ν are stationary measures for the same interactions and $i(\mu|\nu)$ the relative entropy density of μ with respect to ν is different from zero, *i.e.* $i(\mu|\nu) \neq 0$ then both measures are non-Gibbsian ones [2].

In order to find the density of the relative entropy it is particularly useful to explore the so-called properties of large deviations. It appears that the probability that a configuration σ — a sample, taken from the probability distribution ν, *looks in the volume Λ like a typical configuration of* μ decays exponentially in the volume of Λ with rate $i(\mu|\nu)$, *i.e.* [2]:

$$\mathrm{Prob}_\nu\{\sigma_\Lambda \text{ is typical for } \mu\} \sim \mathrm{e}^{-|\Lambda|\ i(\mu|\nu)}\,. \tag{3}$$

Hence

$$i(\mu|\nu) \sim \lim_{\Lambda\to\infty} -\frac{1}{|\Lambda|}\ln(\mathrm{Prob}_\nu\{\sigma_\Lambda \text{ is typical for } \mu\})\,. \tag{4}$$

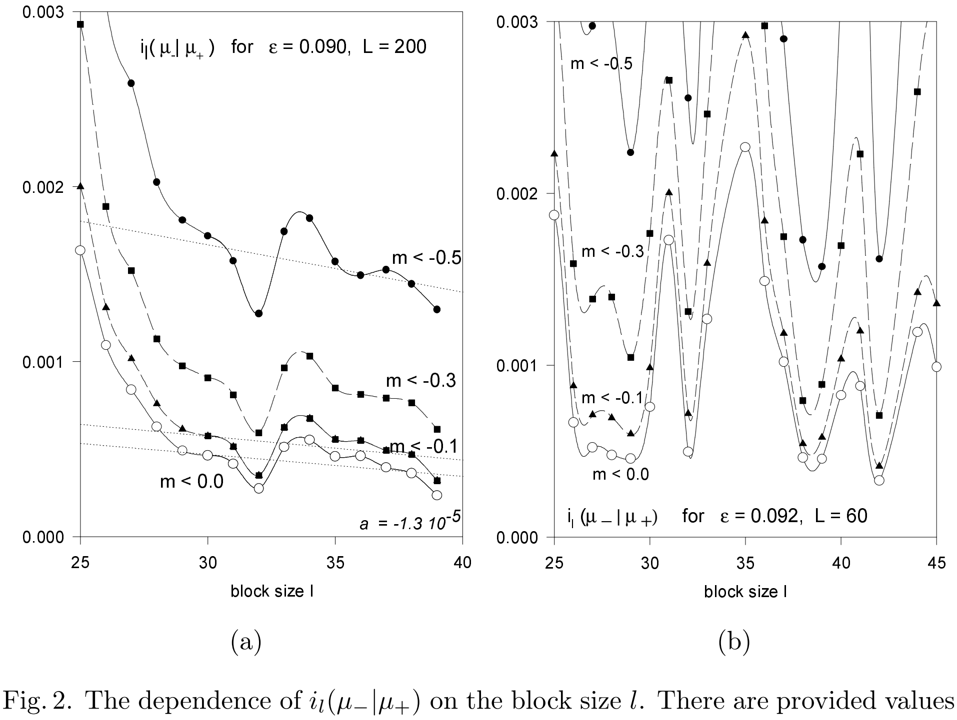
There are two basic measures in Toom PCA: the positively magnetized phase μ_+ and the negatively magnetized phase μ_-. Both of these phases have a large magnetization $\pm m$, respectively. In our experiments we search how rare is to see the $l \times l$-size square block with negative magnetization if the system is represented by μ_+ phase, globally. Namely, we investigate the decay of the following function

$$i_l(\mu_-|\mu_+) = -\frac{1}{l^2}\ln(\mathrm{Prob}_{\mu_+}\{m(\sigma_{l\times l}) < 0\}) \qquad (5)$$

to estimate the infinite block size limit

$$i_l(\mu_-|\mu_+) \longrightarrow_{l\to\infty} ? \qquad (6)$$

The data obtained for the large lattice ($L = 200$) and large blocks is presented in Fig. 2(a). The different curves in this figure represent the different conditions for the mean magnetization of a block. One can observe that when this condition goes down to 0 the corresponding functions of i_l



(a) (b)

Fig. 2. The dependence of $i_l(\mu_-|\mu_+)$ on the block size l. There are provided values for linear regression for tails (last ten points). (a) the case $L = 200$, (b) the case $L = 60$.

go down less rapidly. However, the linear regression for the decay of the ten tail points shows although tiny (of the 10^{-5} order) but steady negative value. This result suggests for the limit density $i(\mu_-|\mu_+) \rightarrow 0$ (compare [15]). The Fig. 2(b) is to present the same estimation for the smaller lattice ($L = 60$). For this system the results are definitely undecidable, suggesting that $i(\mu_-|\mu_+) > 0.0005$. Hence the stationary measure in not Gibbsian..

4. Critical properties

The basic static critical exponents of spin systems are [3]:

— α : for free energy density $f_L(T) \sim |T_{\rm cr}(L) - T|^{2-\alpha_L}$

— β : for mean magnetization $m_L(T) \sim (T_{\rm cr}(L) - T)^{\beta_L}$ for $T < T_{\rm cr}$

— γ : for susceptibility $\chi_L(T) \sim (T - T_{\rm cr}(L))^{-\gamma_L}$ for $T > T_{\rm cr}$

Transferring temperature T into the temperature-like parameter ε we obtain the following results for Toom PCA:

L	$\varepsilon_{\rm cr}(L)$	β_L	γ_L	
20	0.0930 ± 0.005	0.97	0.23	
40	0.0920 ± 0.005	0.82	0.31	
60	0.0915 ± 0.005	0.67	0.56	(7)
80	0.0915 ± 0.005	0.65	0.65	
100	0.0910 ± 0.005	0.59	0.72	
200	0.0905 ± 0.005	0.52	0.80	

In finite size systems, the critical singularities are moved from each other depending on the lattice size. Here, $\varepsilon_{\rm cr}(L)$ are determined as the most rapid change with respect to ε of the three following characteristics: the cumulant of the magnetization U, and the two logarithm derivatives of $\langle |m| \rangle$ and $\langle m^2 \rangle$ [4].

Notice that according to the relation between critical exponents $\alpha = 2 - 2\beta - \gamma$, which comes out from the investigations of equilibrium thermodynamics, we are provided with the free energy density critical exponent: $\alpha_L = -0.17,\ 0.05, 0.10, 0.05, 0.10, 0.16$ for lattice sizes $L = 20,\ 40,\ 60,\ 80,$ 100, 200 respectively.

The finite-size scaling allows us to estimate the infinite lattice properties [4] by the following relations

$$m_L(\varepsilon) = L^{-\beta/\nu}\, m_0(L^{1/\nu} t)\,, \tag{8}$$

$$\chi_L(\varepsilon) = L^{(\gamma/\nu)-2}\, \chi_0(L^{1/\nu} t) \tag{9}$$

with

$$t = \left| \frac{\varepsilon - \varepsilon_{\rm cr}}{\varepsilon_{\rm cr}} \right|$$

and m_0, χ_0 the scaling, generally unknown, functions. Moreover,

$$\varepsilon_{\rm cr}(L) = \varepsilon_{\rm cr} + \lambda L^{-1/\nu}\,. \tag{10}$$

Hence, to characterize the Toom PCA in the thermodynamic limit, we need to know the critical exponent ν. It appears that some characteristics scales as $L^{1/\nu}$. It is obeyed by the cumulant U as well as logarithms of $\langle |m| \rangle$, $\langle m^2 \rangle$, $\langle m^4 \rangle$. The maximum slope for U, $\ln(\langle |m| \rangle)$ and $\ln(\langle m^2 \rangle)$ serves us estimates for ν, and then when applied to (10) also for $\varepsilon_{\rm cr}$. Finally we get [15]:

$$\nu = 0.92 \pm 0.03\,, \qquad \varepsilon_{\rm cr} = 0.0902 \pm 0.0005\,. \tag{11}$$

With the above listed values we can determine the other critical exponents (see Fig. 3(a)(b)) as

$$\beta = 0.31 \pm 0.05\,, \qquad \gamma = 0.76 \pm 0.15\,. \tag{12}$$

These values indicate $\alpha = 2 - 2\beta - \gamma = 0.62$.

Thermodynamics considerations provide the other estimation for α also, namely $\alpha = 2 - d\,\nu$. According to this equality we obtain $\alpha = 0.16$ (compare to [8]). Notice that this value for α together with the value for β obtained earlier leads to $\gamma = 1.22$.

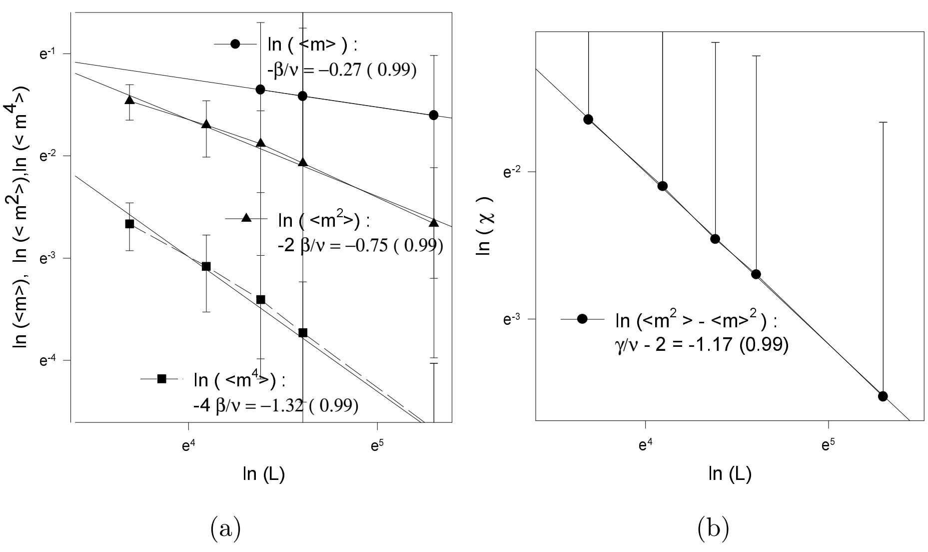


Fig. 3. The estimates for β (a) and γ (b) obtained from the finite size scaling (8) and (9).

There are PCA systems (*e.g.* mentioned in the Introduction the dynamics of competing Glauber and Kawasaki rules [6, 7]) for which the stationary measures appear to have critical properties the same as those known for the Ising model. On the other hand, there exist PCA (*e.g.* lattice diffusive system [16] or directed percolation [17]) which exhibit critical features to be a crossover between the rigorous Ising model and its mean-field approximation. Comparing our results to these two fundamental universality classes, *i.e.*, to the Ising model in 2-dim with exponents $\alpha = 0$, $\beta = 0.125$, $\gamma = 1.75$, $\nu = 1$ and to the mean-field model where $\alpha = 0$, $\beta = 0.5$, $\gamma = 1.0$, $\nu = 0.5$, we see that the Toom system also represents a crossover between the models. Although, concentrating on the β value only, one can notice that the Toom model is very close to other PCA systems like lattice gas ($\beta = 0.22$ [16]) or directed percolation ($\beta = 0.276$ [17]), however the values of remaining critical exponents are rather different from those obtained in the systems listed.

Closing our report on critical exponents estimations we must underline the fact, that data from the experiments was very chaotic in the sense that results obtained possess the high level of STD-errors. The phenomenologically based Harris criterion states that a system with $\alpha > 0$ is dynamically chaotic [18]. It means that any perturbation to the dynamics would effect in the rapid change of macroscopic properties. We can suspect that the significant difference in critical exponent values observed when the lattice size is changed, is caused by the similar effect. Therefore we had to reject data obtained for small lattices, *i.e.* $L = 20$ and $L = 40$ as not reliable for studying Toom interactions in the thermodynamic limit. On the other hand, in the Toom model there must be the noticeable effect caused by the sensitivity to the initial configuration [19]. Observed by us STD errors should be related to this notion of chaos.

5. Conclusions

Toom PCA are perfectly suited for investigating links between systems solved rigorously and real systems.

The deterministic Toom cellular automata (*i.e.* $\varepsilon = 0$) are the example of the so-called complex automata [9], for which there is not known the natural measure. However, by violating the rigidity in the execution of the deterministic rule by introducing the random error, we could hope to move the system into the object for which the natural measure can be provided. It occurs true for Toom PCA away from the critical region. Being in the critical regime the Toom PCA is extremely sensitive to all details of the individual system. Therefore, having in mind that critical properties are the motivation to our investigations, the conclusion on the Toom model usefulness in statistical physics is not obvious. Our observations support the opinion that there are nonequilibrium systems which might be applied to equilibrium statistical physics only under much care.

REFERENCES

[1] A. Toom, *Cellular Automata with Errors: Problems for Students of Probability*, in *Topics of Contemporary Probability and its Applications* ed. J. Kaurie Snell, CRC Press, 1995; Ch.H. Bennett, G. Grinstein, *Phys. Rev. Lett.* **55**, 657 (1985).

[2] A.C.D. van Enter, R. Fernández, A.D. Sokal, *J. Stat. Phys.* **72**, 879 (1993).

[3] D. Stauffer, H.E. Stanley, *From Newton to Mandelbrot. A Primer in Theoretical Physics with Fractals for the Personal Computer*, Polish edition by WNT, Warszawa, 1996.

[4] D.P. Landau, in *Exploring Phase Transitions with Modern Monte Carlo Methods*, Proceedings of the Eighth Joint EPS-APS International Conference on Physics Computing, eds P. Borcherds, M. Bubak and A. Maksymowicz, Academic Computer Center Cyfronet, Kraków, 1996.

[5] Ya.G. Sinai, *Theory of Phase Transitions: Rigorous Results*, Akademiai Kiadó, Budapest, 1982.

[6] J-S. Wang, J.L. Lebowitz, *J. Stat. Phys.* **51**, 893 (1988).

[7] B.C.S. Grandi, W. Figueiredo, *Phys. Rev.* **E53**, 5484 (1996).

[8] D. Makowiec, *Phys. Rev.* **E56**, 5195 (1997).

[9] H. Gutowitz, *Cellular Automata and the Sciences of Complexity (part I)*, *preprint*, 1995, available at *WWW* server alife.santafe.edu

[10] G. Grinstein, C. Jayaparash, Ye He, *Phys. Rev. Lett.* **55**, 2527 (1985); O. Martin, *Physica* **D45**, 345 (1990).

[11] S. Goldstein, R. Kuik, J.L. Lebowitz, Ch. Maes, *Commun. Math. Phys.* **125**, 71 (1989).

[12] J.L. Lebowitz, C. Maes, E. Speer, *J. Stat. Phys.* **59**, 117 (1990).

[13] Ch. Maes, K. Vande Velde, *Physica* **A206**, 587 (1994); Ch. Maes, K. Vande Velde, *Relative Energies for Non-Gibbsian States* preprint texas mp archive nr 96-56 (1996).

[14] R.H. Schonmann, *Commun. Math. Phys.* **124**, 1 (1989).

[15] D. Makowiec, *Phys. Rev.* **E55**, 6582 (1997).

[16] F.J. Alexander, I. Edrei, P.L. Garrido, J.L. Lebowitz, *J. Stat. Phys.* **68**, 497 (1992).

[17] R.J. Baxter, A.J. Guttmann, *J. Math.* **A21**, 3193 (1988).

[18] A.B. Harris, *J. Phys. C* **7**, 1671 (1974); W. Selke, L.N. Shchur, A.L. Talagov, *Monte Carlo Simulations of Dilute Ising Models* in *Annual Reviews of Computational Physics: I*, ed. D. Stauffer, World Scientific, Singapore 1994.

[19] D. Makowiec, *Physica* **A234**, 435 (1996).