ON THE CLASSIFICATION OF QUANTUM SYSTEMS WITH RESPECT TO THEIR INTEGRABILITY

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New methodology of classification of quantum systems with respect to their integrability basing on the analytical properties of statistical measures is proposed. Advantages of this methodology are discussed. Examples of possible applications of this approach are presented. Model of a paramagnetic atom moving in external magnetic field is discussed in detail. Predictions of different types of both quantum integrability and quantum chaos are given.

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1. Introduction

The aim of this paper is to introduce new methodology of classification of quantum systems with respect to their integrability. *A quantum system with N degrees of freedom is integrable if there exist N globally defined operators, $I_m(\hat{p}_1, ..., \hat{p}_N; \hat{q}_1, ..., \hat{q}_N)$, for $m = 1, ..., N$, whose mutual commutators vanish,*

$$[\hat{I}_m, I_n] = 0, \tag{1}$$

for all $m, n = 1, .., N$ [1–4]. In this paper we consider chaotic systems as systems where it is impossible to find such operators except the Hamiltonian in conservative systems. So far, there is a classification basing on the general properties of probability distribution of a quantity called spacing. Spacing is a difference between two adjacent levels in the energy spectrum of a quantum system. For chaotic systems the distribution of spacing may be approximated by the Wigner surmise [5–7]

$$P_W(s) = \begin{cases} \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right) & \text{for } s \geq 0\,, \\ 0 & \text{for } s < 0\,, \end{cases} \tag{2}$$

and for integrable systems is given by the Poisson distribution

$$P_{\mathrm{P}}(s) = \begin{cases} \exp(-s) & \text{for } s \geq 0, \\ 0 & \text{for } s < 0 . \end{cases} \tag{3}$$

These two distributions are distinctively different. Therefore it was possible to classify quantum systems with respect to their integrability basing on similarities between a histogram of spacing obtained for the considered system and functions given by (1), (2) or other formulas binding for different types of quantum chaotic systems. For more details see [7, 8].

The concept presented below is based on the observation that not the shape of the statistical measure but rather its class with respect to the differentiation is significant in the context of quantum integrability. Our main result is an indication that quantum chaotic system can be defined as a system with continuous probability distribution of spacing. It is shown that the observed shortage of small spacings usually explained as a consequence of the level repulsion, is a simple inference from our condition. Since integrability of the quantum system is connected with the number of independent globally defined invariants which are in involution with each other and our theory is only qualitative, we were not able to formulate a condition concerning integrable systems. However, using our methodology it seems possible to reveal the existence of a single invariant for a given, even very complicated, quantum system. And above all, it is very simple and does not require any additional significant computations since making a histogram of spacing, while dealing with such systems, is a standard procedure.

It must be stated that our classification is based on features that do not depend on the special kind of so called unfolding of the energy spectrum [9]. We have obtained our results using different unfolding procedures. Moreover, we have applied our classification to many systems considered by other authors, which may have applied very different unfolding techniques. But there are also other arguments against the objection that our classification might be based on the artificial effect caused by unfolding. The first one is that while constructing our methodologywe base on the analytical results, and the second is the fact that whether the function is continuous or discontinuous at some point is not a local feature but it is possible to conclude it on the basis of the function behaviour upon some finite extent.

The paper is organized in the following way: In the Section 2 we approach statistical measures we are talking about, in the Section 3 we test them by investigating the two-dimensional nonlinear quantum oscillator in both chaotic and integrable regions. The new methodology of classification of quantum systems with respect to their integrability is then introduced. In Sections 4 and 5 we apply our methodology to the periodically driven systems and to the one-dimensional system which is integrable by the definition, whereas its levels exhibit repulsion. In conclusion we summarize our results pointing out the most important achievements of our classification.

2. Statistical measures

Before presenting our methodology of classification a short description of statistical measures must be given. Systematic discussion of these measures is contained in [10, 11]. It was possible, as a result of an observation, that spacing is in general a two point approximation of the first derivative of the function $E(i) = E_i$:

$$s_i = \frac{1}{i+1-i}(E_{i+1} - E_i)\,. \tag{4}$$

The fact that probability distribution of the differential quotient is used to characterize quantum systems with respect to its integrability made it clear, that higher order differential quotients of the function E_i also might be helpful in this matter. Therefore the second order differential quotient was introduced as a supplementary characteristic of quantum chaos or integrability:

$$z_i = \Delta^2 E_i = \frac{1}{(i+1-i)^2}(E_i + E_{i+2} - 2E_{i+1})\,. \tag{5}$$

This statistical measure is based on three consecutive energy levels. This gives rise to a question whether it is the only measure of a clear physical meaning based on such number of levels. It follows from the theory of finite elements that this condition is fulfilled by three point approximations of the first derivative, which may be constructed with the use of three point asymmetrical and symmetrical elements in the following way:

$$x_i = \frac{1}{2(i+1-i)}(-3E_i + 4E_{i+1} - E_{i+2})\,, \tag{6}$$

$$y_{i+1} = \frac{1}{2(i+1-i)}(E_{i+2} - E_i)\,. \tag{7}$$

To find formulas for probability distributions of these quantities for chaotic systems the most general formula of the random matrix theory RMT [7, 8, 12], the Rozenzweig–Porter formula, may be used:

$$P(x_1, x_2, ..., x_N) = C_{N\beta}\left[\prod_{i>j=1}^{N}|x_i - x_j|^{\beta}\exp\left(-\frac{1}{2}\sum_{i=1}^{N}x_i^2\right)\right]\,, \tag{8}$$

where $\beta = 1, 2, 4$ for GOE, GUE, and GSE, respectively. The abbreviations GOE, GUE, GSE mean Gaussian Orthogonal Ensemble, Gaussian Unitary

Ensemble and Gaussian Symplectic Ensemble. These ensembles have been introduced in RMT. In the following properties of GOE will be investigated, and for the purpose of this article the only important fact is that GOE consists of real random symmetrical matrices. Probability distributions for the considered finite elements were obtained with the use of an ensemble consisting of 3×3 matrices, so $N = 3$ was assumed in (8). The formulas for probability distribution of quantities s, x, y and z, see (4)–(7), for a quantum integrable system with three randomly distributed energy levels, which will be denoted as I(3) were also found in [10] and [11]. For the sake of completeness of this paper results both for GOE(3) and integrable system I(3) are summarized in the Appendix. These theoretical tools must be however tested before they are used in investigations concerning quantum chaos. Such tests should be done on a system with well known properties for which many energy levels can be easily computed. This will be done in the first part of the next section. In the second part of that section there will be given an extensive discussion answering the question how these tools might be helpful in the proper classification of quantum systems.

3. The classification of quantum systems

In order to tests measures mentioned in previous section we chose a system consisting of two coupled anharmonic oscillators analyzed in [13]:

$$H = \tfrac{1}{2}(p_1^2 + p_2^2) + V_1(x_1) + V_2(x_2) + V_{12}(x_1 - x_2)\,. \tag{9}$$

The potentials $V_i(x)$ are defined as

$$V_i(x) = v_i(\alpha_i x^2 + \beta_i x^4 + \gamma_i x^6), \quad (i = 1, 2, 12)\,. \tag{10}$$

The properties of this system were investigated for $\hbar^2 = 0.2$. Values of other parameters are gathered in the following table:

TABLE I

Values of parameters for potential defined by (9).

i	v_i	α_i	β_i	γ_i
1	100	1.56	−0.61	0.32
2	100	0.69	−0.12	0.03
12	v_{12}	−1.00	0.25	0.08

The most interesting parameter is v_{12} because for $v_{12} = 0$ this system is integrable and for $v_{12} = 100$ is almost chaotic. For the purposes of testing statistical measures reminded in the previous section it was necessary to compute approximately 10000 energy levels. Harmonic oscillator basis was used to construct Hamiltonian matrix. In the actual calculations two matrices 5538×5538 were diagonalized because sets of states with positive and negative parities do not couple with each other. For further calculations we took only 4000 middle levels from each spectrum. Data gathered from both spectra are showed on histograms. Figures 1(a), 1(b), 1(c), 1(d) present experimental histograms taken from the chaotic system, whereas figures 2(a), 2(b), 2(c), 2(d) show histograms calculated for the integrable case. Dashed and dotted curves plotted on these figures are theoretical probability distributions for I(3) and GOE(3), respectively (except Fig. 1(a),

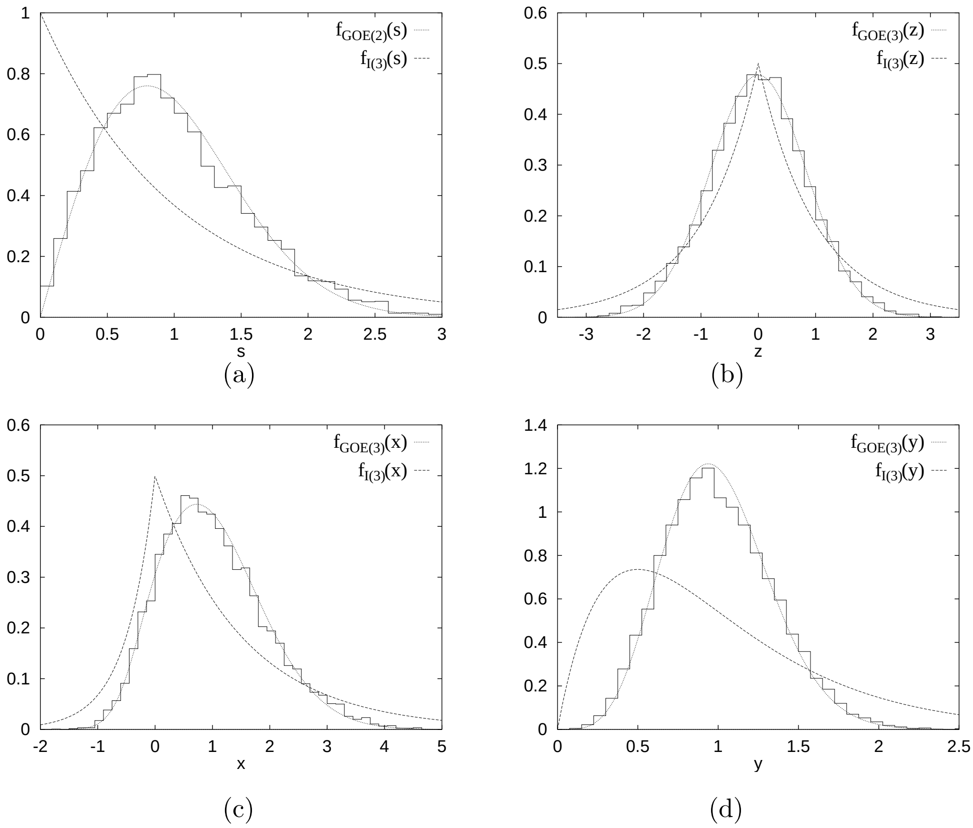

Fig. 1. The histograms of: 1(a) spacing, 1(b) the second order differential quotient, 1(c) asymmetrical three point element, 1(d) symmetrical three point element for two-dimensional nonlinear oscillator in the chaotic region (solid line). Analytically predicted probability distributions for I(3) model (dashed line) and GOE(3) (dotted line) except Fig. 1(a) where probability distribution for GOE(2) is presented.

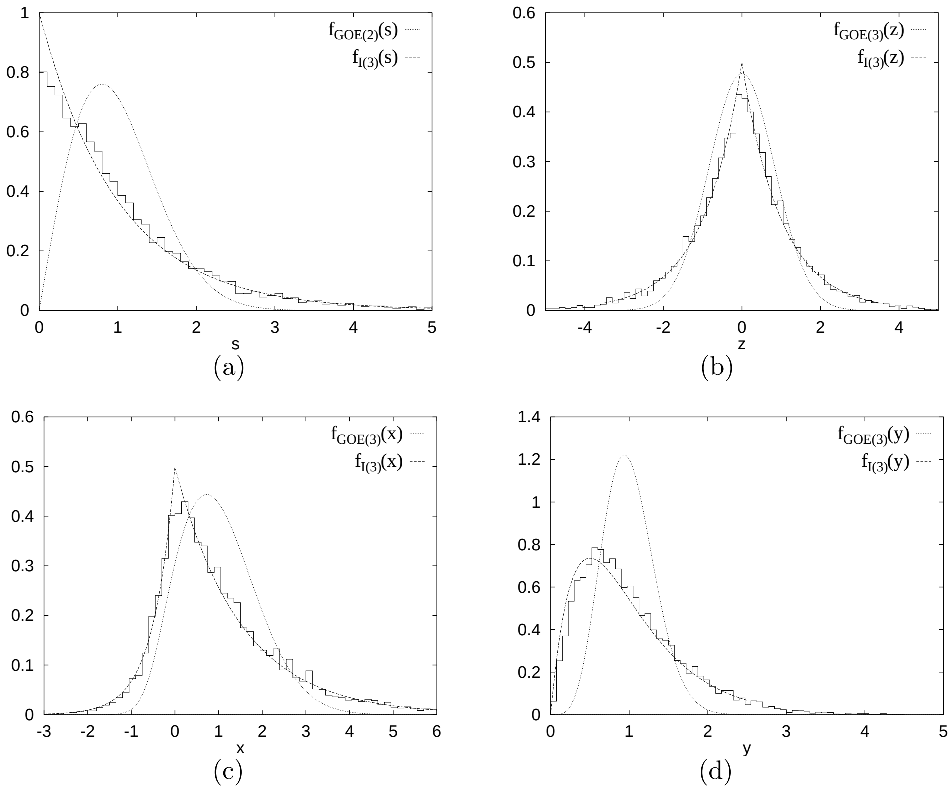

Fig. 2. The histograms of: 2(a) spacing, 2(b) the second order differential quotient, 2(c) asymmetrical three point element, 2(d) symmetrical three point element for two-dimensional nonlinear oscillator in the integrable region (solid line). Analytically predicted probability distributions for I(3) model (dashed line) and GOE(3) (dotted line) except Fig. 2(a) where probability distribution for GOE(2) is presented.

where the function from the Wigner surmise is plotted). The presented figures were taken as an experimental evidence, that theoretical predictions for GOE(3) and I(3) are good approximations of the statistical measures calculated for systems with large N.

Careful survey of both theoretical and numerical data presented here shows that analytical properties of the considered probability distributions are not accidental. There seem to exist some general rules from which one may predict general behaviour of a given statistical measure for any quantum system knowing only whether it is integrable or not. We found these rules and give them in three observations.

Observation 1 *Statistical measures calculated for quantum systems belonging to different classes with respect to their integrability are functions of different classes with respect to their differentiability.*

To account for this observation on the basis of presented data we will discuss our plots in more detail. We start from the probability distributions of spacing presented in Fig. 1(a). These well known functions are smooth everywhere except the origin. At this point probability distribution of spacing of the integrable system is a discontinuous function, whereas the probability distribution for the chaotic system is continuous but nodifferentiable one at this point. Therefore, using more formal language it may be said that the first one belongs to the class of discontinuous functions, whereas the second one to the class of continuous functions C^0. Fig. 1(b) presents distributions of the second differential quotient of the energy spectrum and like in previous case the only interesting point is zero. For this value of an independent variable both functions take their maximum values. In the case of integrable system the derivative of the probability distribution does not exist at this point, whereas for the chaotic one it does. Therefore, the probability distribution for the integrable system belongs to the class C^0 while the probability distribution for chaotic one is a smooth function (class C^∞). Analogous situation is for the probability distributions of the asymmetrical three point element plotted in Fig. 1(c). Also here the derivative of the probability distribution calculated for integrable system does not exist at the point for which the function takes its maximum value. The only difference is that it takes place at the point different from zero. This case is the most important one for our discussion because it clearly shows that the property noticed of different classes of functions is not connected with a certain distinguished point. The last case is presented in Fig. 1(d), where the probability distribution of three point symmetrical element is presented. It is the first one of the considered cases in which one may find nondifferentiability of the probability distribution of the integrable system not connected with a maximum of this function. Above we discussed only probability distributions of the three point asymmetrical and symmetrical elements for integrable system because functions corresponding to chaotic system are for these cases everywhere smooth *i.e.* belong to the class C^∞. Therefore, all four cases confirm our observation and the last two ones indicates that we do not generalize observations related to some specific features of probability distributions presented here.

Observation 2 *Probability distributions of the n-point finite elements calculated for quantum systems belonging to one class with respect to their integrability are functions of the same class with respect to their differentiability.*

We formulate this as a generalization of the observations related to the probability distribution of the three-point finite elements presented in Figs 1(b), 1(c), 1(d). They are independent statistical measures with different shapes. But, nevertheless, all functions presented here for the integrable system belong to the class C^0 whereas all functions for chaotic systems belong to the class C^∞. Although this observation is made on the basis of only two quantum models: GOE(N) and I(N) we think that it is very unlikely that six considered functions belong to the proper classes accidentally.

Observation 3 *There are several classes of both chaotic and nonchaotic systems.*

Indeed it is obvious that quantum harmonic oscillator constitutes a separate class of integrable systems, because all probability distributions of finite elements for this system are given by the Dirac delta function. Therefore, in the view of our classification different behaviour of quantum harmonic oscillator is observed not only for the distribution of three point finite elements but also for the distribution of spacing. It is worth noticing that it is not an extraordinary exception or a special feature of one dimensional systems as one could expect after considering the model discussed in Section 5. Instead, it seems to be an intrinsic feature of many dimensional harmonic oscillator as one can expect from [14]. Therefore, it seems that quantum mechanics distinguishes two different types of integrable systems. The fact that there are also different classes of quantum chaotic systems is more clear. For example it has been calculated in [10] that probability distributions of spacing for GUE(3) and GSE(3) belong to the class C^∞. This behaviour differs distinctively from that observed for GOE where probability distribution of spacing belongs to the class C^0. Therefore, from the point of view introduced here the kind of chaos in systems whose properties can be described by GOE should be different from that observed in systems described by GUE and GSE. Here we do not discuss properties of other statistical measures calculated for GUE(3) and GSE(3) because they have not been verified by the experimental data, and we should not consider them as properly describing the behavior of the real physical systems with large N. It must be stated that both integrable and nonchaotic systems belong to the same classes in our classification because of the quantitive character of integrability. Therefore it is impossible to introduce in this manner classes of integrable systems. We may call them nonchaotic instead. These three observations constitute our phenomenological methodology of classifying quantum systems with respect to their integrability on the base of statistical measures. To make it easier we formulate two inferences:

Inference 1 *The probability distributions of spacing for quantum chaotic systems are at least continuous functions.*

Inference 2 *The probability distributions of three point statistical measures for quantum chaotic systems belong to the class C^{∞}.*

We will show that the introduced methodology is better then the old one, and it also has greater capabilities for explaining the nature of real quantum systems. So why this approach is better then the old one? The answer is very simple. So far quantum system has been believed to be chaotic if the shortage of small spacings in its energy spectrum was observed. On the other hand, the only theoretical model of a spectrum of the integrable system was the model $\mathrm{I}(N)$ in which levels are randomly distributed and completely uncorrelated. These approaches do not seem appropriate. First, because there is at least one quantum integrable system with the shortage of small spacings: it is quantum harmonic oscillator. Secondly, because Poisson or Poisson-like distributions of spacing belonging to many integrable systems can be obtained from band matrices [13]. It is not well understood yet, but undoubtedly testifies that energy levels interact with each other. These facts are commonly known but they are considered as insignificant, in relation to profits from previous approach. Therefore we will indicate that the methodology presented here gives better results and does not have such unpleasant features. The trouble with harmonic oscillator vanishes. Probability distribution of spacing for this system is given by Dirac delta function which is by its definition discontinuous and integrable function. Our approach properly classifies systems with the significantly degenerate energy spectrum as nonchaotic ones. On the other hand, continuity of spacing distribution in connection with definition of spacing implies that there are relatively few small spacings in the energy spectrum of quantum chaotic systems. This fact agrees with the predictions of the old approach. It is also very important that using our approach we are able to properly classify systems which are wrongly classified by the old methodology. Here, we discuss two simple examples taken from [15]. In that paper, there are presented histograms of spacing for two quantum systems. Fig. 2 [15] shows a probability distribution of spacing for Rydberg states of a certain model of hydrogen atom in external magnetic field. As Rydberg atoms are known as exhibiting quantum chaos one may think that the histogram approximates Wigner probability distribution. Indeed, in this model there is a strong level repulsion but there is also a jump of this probability distribution for larger s. Histogram in Fig. 3 [15] is constructed for classically chaotic system and although its shape is slightly more complicated, it exhibits the same behaviour for small and big spacings as the previous one. Accordingly to the old classification both systems are chaotic. In our methodology they are nonchaotic, because this probability distributions seem to be discontinuous functions. Indeed, authors of the considered paper showed that this strange behaviour of probability distributions of spacing might be explained in terms of coupling between chaotic states obtained from RMT and randomly distributed states considered as belonging to integrable system. Thus in this case our methodology is perfectly confirmed.

4. Driven systems

The case of driven quantum systems seems to be slightly more complicated in view of our classification. The best known example of such systems is a quantum kicked rotator introduced by Casati *et al.* [16]. It is a quantized version of the standard map [17]. The Hamiltonian of the classical system reads:

$$H = \frac{p^2}{2I} - \frac{kI}{T}\cos\phi \sum_n \delta(t - nT)\,. \tag{11}$$

Below we use the dimensionless units, *i.e.*, we assume $I = 1$, $T = 2\pi$. Hamiltonian equations can be integrated over one period of the driving force to give the classical standard map

$$\phi_{n+1} = \phi_n + 2\pi p_n, \tag{12}$$

$$p_{n+1} = p_n + (K/2\pi)\sin\phi_{n+1}\,. \tag{13}$$

The kick strength $K = 2\pi k$ controls the stochasticity of the standard map. For $K \to 0$, the system is nearly integrable; for $K \approx 1$ the last Kolmogorov–Arnol'd–Moser torus disappears and diffusion in the p direction becomes possible. The classical system becomes fully chaotic. The quantum mechanical behavior generated by (12),(13) is found to be very different from the classical one [16, 18–22]. In particular the diffusion in phase space is suppressed and the motion is quasiperiodic [18, 19]. It was shown [20–22] that the mechanism of this suppression is similar to Anderson localization [23,24]. Consequently the quasienergy states are exponentially localized in the same way electronic states are localized in disordered solids. This phenomenon in analogy with Anderson localization is called dynamical localization.

In the paper by M. Feingold *et al.* [26] the separation distribution of quasienergies for the kicked rotator was analyzed. The authors concluded that this distribution is the Poissonian one. It is similar to the corresponding distribution of the energies of localized electronic states in disordered solids [27] and supports the correspondence between these problems. This distribution is different from that obtained for time independent chaotic systems and indicates the existence of a brand new constant of motion.

As another example of a driven quantum system it will be considered one dimensional hydrogen atom in a microwave field. The following formula holds for the classical Hamiltonian of this system [25] (atomic units are used):

$$H = -\frac{1}{2n^2} + \varepsilon n^2 \cos\omega t \left(\frac{3}{2} - 2\sum_{s=1}^{\infty} s^{-1} J_s'(s) \cos s\lambda\right), \tag{14}$$

where J_s are Bessel functions. The classical map is then obtained in the form

$$\overline{N} = N + k\sin\phi, \tag{15}$$

$$\overline{\phi} = \phi + 2\pi\omega(-2\omega\overline{N})^{-\frac{3}{2}}, \tag{16}$$

where N is energy divided by ω, $\phi = \omega t - s\lambda$, $k = 0.822\pi\varepsilon\omega^{\frac{5}{3}}$. This "Kepler map" yields an approximate description of the motion of the classical electron. It is possible to locally approximate (15),(16) by a standard map [28]. Thus (15),(16) show that global diffusion is to be expected for $K > 1$. While quantizing the map (15),(16) which describes an unbounded motion in ϕ under a periodic perturbation,"*a new integral of motion will appear ("quasi-impulse"), besides quasienergy; for a given unperturbed level n_0, it will be just the fractional part of* $N_0 = -n_0/(2\omega_0) = -N_I$" [25]. One predicts thus for the quantum kicked rotator a new constant of motion, since the local correspondence between these systems holds. It was shown for the quantum rotator [29] that for $\hbar \to 0$ ($\hbar = 1/1944$) the spacing distribution resembles the predictions for chaotic systems given by GOE, while for $\hbar \to 1$ ($\hbar = 625/1944$) one gets almost perfect approximation of the Poissonian distribution. Some difficulty is however connected with the classifying of these systems for large values of K. According to the work by Izrailev [30] a Wigner type distribution of spacing of quantum kicked rotator is observed in this regime. It is in good agreement with that which should be predicted from the behaviour of quasieigenstates. In this range of parameters quasieigenstates are strongly overlapping and the localization vanishes. It seems to be in contradiction to what we have written about the correspondence between spacing distribution and integrals of motion since there seems that the value of K does not affect the constant of motion indicated before. However, if one divides both sides of the equation (13) by $\hbar$ it becomes clear that in fact this equation depends only on one parameter $K/\hbar$ and the case of small $\hbar$ is equivalent to the case of large K. Thus, for large values of K the additional integral of motion disappears and the system is chaotic as one should expect from the behavior of the spacing distribution.

5. A one dimensional system with strange spacing distribution

Here we will illustrate how our methodology works in the slightly more complicated case. The considered system was first proposed and initially investigated in [31]. Probability distribution of spacing in system with half integer angular momentum subjected to constant magnetic field and moving in one dimensional box showed a very strange shape. The system changes its properties (its linear dimension) as a result of coupling between magnetic field and angular momentum, in analogy to the paramagnetic atom. The Hamiltonian of such an atom may be written in the simplified form:

$$H = -\frac{d^2}{dx^2} + \kappa^2 \sigma_x \,, \tag{17}$$

where σ_x is the Pauli matrix:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

The quantity κ is a measure of the rate of transitions between two internal states of the atom. Different dimensions of the atom in different angular momentum states are reflected by boundary conditions. If it is in the state $|\frac{1}{2}, +\frac{1}{2}\rangle$, the box extends from $-a$ to $+a$ and when it is in the state $|\frac{1}{2}, -\frac{1}{2}\rangle$ the walls of this box are in $-b$ and $+b$, and it is assumed that $b - a = \Delta > 0$. Details of the analytic calculations for this model are given in [31]. Here we will discuss the case of the equation for eigenvalues belonging to symmetric eigenstates, which gives infinite number of levels:

$$2k \cot k\Delta = q_+ \tan q_+ a + q_- \tan q_- a \,,$$

$$E = +k^2, \quad q_\pm = \sqrt{k^2 \mp \kappa^2} \,. \tag{18}$$

The computations were performed with $a = 1$, $\kappa = 200$ and $\Delta = 0.01$. The spectrum of 10^5 energy levels was calculated. Then histograms of the introduced statistical measures were made. They are plotted in figures 3(a), 3(b), 3(c), 3(d) together with theoretical probability distributions for chaotic and integrable quantum systems. The histograms differ distinctively both from the predictions of RMT and the model of an integrable system with randomly distributed levels. The histogram of spacing is not convincing because of its strange shape. It is not clear if this function is discontinuous or rather only non-differentiable at its maximum. The shape of the histograms suggests rather the second possibility. Because we are not able to find out the answer to this question on the basis of numerical computations, one may propose that this system is chaotic because the second possibility is more probable and the system shows a classical example of the repulsion of levels. But using our methodology we must resort in this case to the procedure provided by the second inference and examine properties of probability distributions of three point elements. These probability distributions are presented in figures 3(b), 3(c), 3(d). As one can see, all probability distributions of these elements are discontinuous or belong to the class C^0. Therefore, on the basis of the inference 2 we can say that this system is not chaotic. But inference 2 does not entitle us to decide if these functions are continuous or not. We would like to stress that our methodology gives univocal results despite of the incomplete knowledge of the actual shape of the considered probability distributions and that they often contradict the old theory. Therefore, using our methodology we find out that this system is not chaotic.

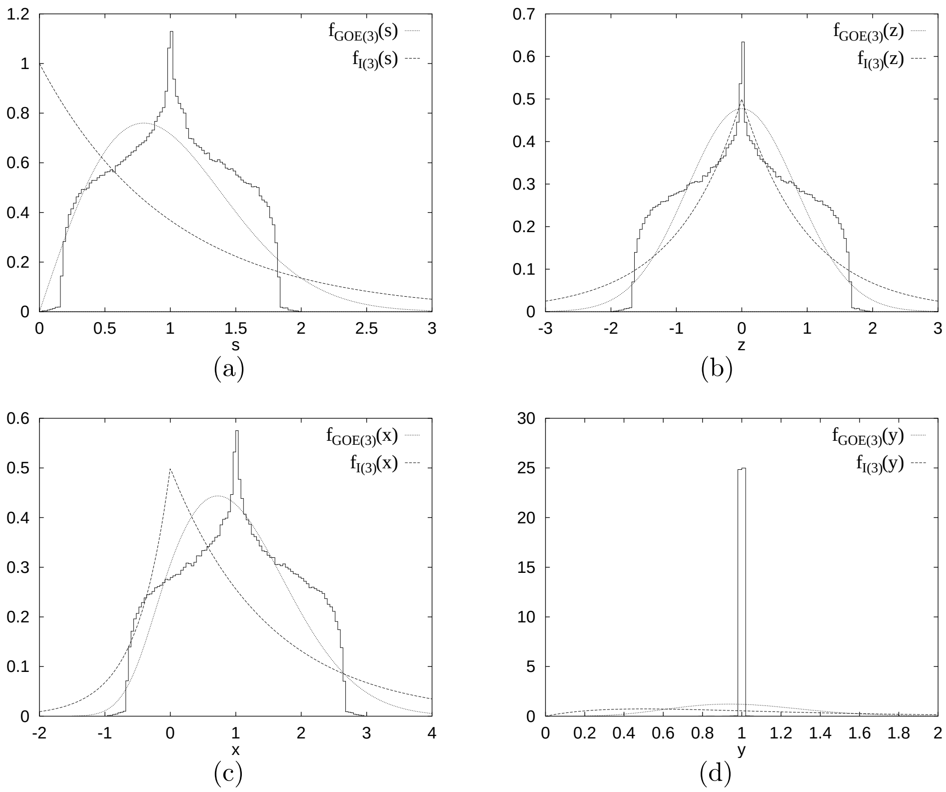

Fig. 3. The histograms of: 3(a) spacing, 3(b) the second order differential quotient, 3(c) asymmetrical three point element, 3(d) symmetrical three point element for one-dimensional model of paramagnetic atom, bin size 0.02 (solid line). Analytically predicted probability distributions for I(3) model (dashed line) and GOE(3) (dotted line) except Fig. 3(a) where probability distribution for GOE(2) is presented.

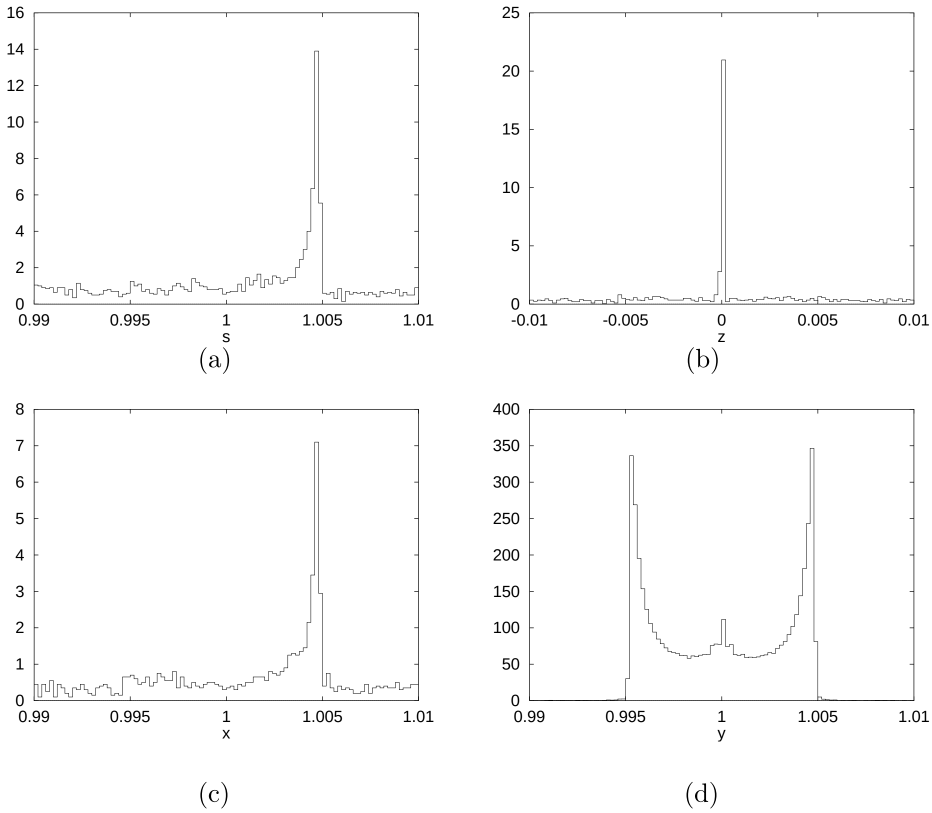

Fig. 4. The histograms of: 4(a) spacing, 4(b) the second order differential quotient, 4(c) asymmetrical three point element, 4(d) symmetrical three point element for one-dimensional model of paramagnetic atom with bin size 0.0002 (solid line). Analytically predicted probability distributions for I(3) model (dashed line) and GOE(3) (dotted line) except Fig. 4(a) where probability distribution for GOE(2) is presented.

Our methodology was introduced not on the basis of the formal reasoning from well known theorems and axioms, but rather as a certain generalization of our observations so its predictions should be verified as often as possible. It appears that the discussed case provides such a possibility. Large amount of available energy levels enable us to change the size of the bin on histograms from 0.02 to 0.0002. These new histograms are shown in figures 4(a), 4(b), 4(c), 4(d). They are distinctively different from those with larger bin size. It is worth noticing that the probability distribution of spacing is not universal for all scales of variable s. Histograms for quantities s, x and z seem now to approximate Dirac delta function and the histogram for y consists of two symmetrical peaks. It is obvious that histograms consisting of only one narrow peak may be obtained from the spectrum of the quantum harmonic oscillator. Such an oscillator is the best known integrable system. These similarities of spectra properties suggest that considered system should be almost integrable. Therefore, presented histograms entitle us to come up with a hypothesis that the model of moving atom is almost integrable, although its true nature is not known yet. This conclusion stays in good agreement with our knowledge about integrability of quantum systems. It is a known fact that one dimensional systems are integrable. Besides, already in [31] it was found that rigidity function $\overline{\Delta}_3(L)$ [8,31] has the same character as for harmonic oscillator. In this way we have shown that the considered system is actually almost integrable so it belongs to the introduced class of nonchaotic systems as we deduce using our methodology. This fact is very important for the theory of quantum chaos because it is the first integrable system in which the repulsion of levels was observed (see [31]).

6. Conclusions

Now we would like to sum up the most important steps of our reasoning. There has been so far no deeper understanding of the importance of the statistical measures introduced in [10]. First, because for all systems for which it is possible to make histograms of these statistical measures it is also possible to make histogram of spacing, and this was considered to be decisive for the proper classification of quantum system with respect to its integrability. Secondly because theoretical probability distributions of these measures were calculated only for RMT models and integrable I(3) model. There seemed to be no connection between these probability distributions and histograms of introduced measures calculated for quantum systems whose nature was yet unknown. Our approach proved that these two statements are not correct. Moreover, it appears that new statistical measures are indeed useful and that they are simple complementary characteristics of quantum chaos, as was predicted in the paper [10]. Their significance for proper interpretation of the behaviour of the probability distributions of spacing seems now undoubtful.

The methodology introduced here is based on three simple observations proved to be clear, simple, and powerful in determination whether quantum system is chaotic or not. Its application to such different systems like: N-dimensional quantum harmonic oscillator, nonlinear 2-dimensional oscillator, several driven Hamiltonian systems, Rydberg atom, and a special model of the one dimensional paramagnetic atom subjected to the magnetic field allowed us to properly classify these systems. We think that these examples illustrate how general is our methodology and suggest potential possibilities of using it in the future.

One of the most interesting predictions seems to be univocal classification of second system described in [15] as a nonchaotic one. It is very important because this system is completely chaotic in the classical limit and there is so far no idea how to explain this phenomenon.

The most important is, in our opinion, the following conclusion: It seems that after the pioneering work where the spacing distributions of different chaotic and generic integrable systems were developed the general problems of quantum chaos where abandoned contrary to more specific applications. Although our classification might proof to be false we believe that this approach goes in the right direction, because it indicates that there is a need for an explanation of new feature in the behaviour of both chaotic and integrable quantum systems. Our predictions which need to be verified are:

1. There are different classes of quantum integrable systems.
2. Systems described by GUE and GSE significantly differ from those described by GOE.

Appendix

Below we present analytical formulas for statistical measures described in Section 2:

$$f_{\mathrm{GOE}(2)}(s) = \begin{cases} \frac{\pi s}{2} \exp\left(-\frac{\pi s^2}{4}\right) & \text{for } s \geq 0\,, \\ 0 & \text{for } s < 0\,, \end{cases} \tag{19}$$

$$f_{\mathrm{I}(3)}(s) = \begin{cases} \exp(-s) & \text{for } s \geq 0\,, \\ 0 & \text{for } s < 0\,, \end{cases} \tag{20}$$

$$\begin{aligned} f_{\mathrm{GOE}(3)}(x) \ = \ & \frac{81}{2284488\pi^2} \Big(910\sqrt{13}\pi\, x - 315\sqrt{13}x^3 \\ & +(1638x^2 + 2704\pi) \exp\left(-\frac{25x^2}{52\pi}\right) \\ & + \sqrt{13}x(315x^2 - 910\pi) erf\left(\frac{5x}{2\sqrt{13\pi}}\right)\Big) \exp\left(-\frac{27x^2}{52\pi}\right) \\ & \text{for } x \geq 0\,, \end{aligned} \tag{21}$$

$$
\begin{aligned}
f_{\mathrm{GOE}(3)}(x) \;=\; & \frac{81}{2284488\pi^2}\Big(910\sqrt{13}\pi x - 315\sqrt{13}x^3 \\
& -(390x^2 - 2704\pi)\exp\left(-\frac{441x^2}{52\pi}\right) \\
& -\sqrt{13}x(315x^2 - 910\pi)\mathit{erf}\left(\frac{21x}{2\sqrt{13\pi}}\right)\Big)\exp\left(-\frac{27x^2}{52\pi}\right) \\
& \text{for } x < 0,
\end{aligned}
\tag{22}
$$

where:

$$
\mathit{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt\,\exp(-t^2)\,,
\tag{23}
$$

$$
f_{\mathrm{I}(3)}(x) = \begin{cases} \frac{1}{2}\exp\left(-\frac{2\,x}{3}\right) & \text{for } x \ge 0\,, \\ \frac{1}{2}\exp\left(2\,x\right) & \text{for } x < 0\,, \end{cases}
\tag{24}
$$

$$
\begin{aligned}
f_{\mathrm{GOE}(3)}(y) \;=\; & \frac{81}{4\pi^2}\,y\,\left(6\,y\exp\left(-\frac{9\,y^2}{\pi}\right)\right. \\
& \left. + \left(9\,y^2 - 2\,\pi\right)\exp\left(-\frac{27\,y^2}{4\,\pi}\right)\mathit{erf}\left(\frac{3\,y}{2\,\sqrt{\pi}}\right)\right),
\end{aligned}
\tag{25}
$$

$$
f_{\mathrm{I}(3)}(y) = 4\,y\,\exp\left(-2\,y\right),
\tag{26}
$$

$$
f_{\mathrm{GOE}(3)}(z) = \frac{3}{2\,\pi}\exp\left(-\frac{9\,z^2}{4\,\pi}\right),
\tag{27}
$$

$$
f_{\mathrm{I}(3)}(z) = \frac{1}{2}\,\exp\left(-|z|\right).
\tag{28}
$$

We do not remind here formulas for probability distributions of spacing for GOE(3) and I(3) because for the first case it is known that Wigner function is a very good approximation of spacing distribution even for $N \to \infty$ and in the second case the Poisson distribution like for I(2) is obtained.

REFERENCES

[1] G.M. Zaslavsky, *Phys. Rep.* **80**, 157 (1981).

[2] J. Hietarinta, *Phys. Lett.* **A93**, 55 (1982).

[3] J. Hietarinta, *J. Math. Phys.* **25**, 1833 (1984).

[4] B. Eckhardt, *Phys. Rep.* **163**, 205 (1988).

[5] E.P. Wigner, Proceedings of the International Conference on Neutron Interactions with the Nucleus, 1957, Columbia University Report CU-17 (TID-7547) (unpublished), p.49.

[6] E.P. Wigner, Gatlinburg Conference on Neutron Physics by Time of Flight, 1959, Gatlinburg, Tennessee, November 1956, Oak Ridge National Laboratory Report ORNL-2309 (unpublished), p.59.

[7] C.E. Porter, *Statistical Theories of Spectra: Fluctuations*, Academic Press, New York 1965.

[8] M.L. Mehta, *Random Matrices and the Statistical Theory of Energy Levels*, Academic Press, New York 1967.

[9] L.E. Reichl, *The Transition to Chaos in Conservative Classical Systems: Quantum Manifestations*, Springer Verlag, New York 1992.

[10] M. Duras, K. Sokalski, *Phys. Rev.* **E54**, 3142 (1996).

[11] M. Duras, K. Sokalski, *Acta Phys. Pol.* **B** (submitted).

[12] N. Rosenzweig, C.E. Porter, *Phys. Rev.* **120**, 1698 (1960).

[13] T.H. Seligman, J.J.M. Verbaarschot, M.R. Zirnbauer, *Phys. Rev. Lett.* **53**, 215 (1984).

[14] C.B Whan, *Phys. Rev.* **E55**, R3813 (1997).

[15] J. Zakrzewski, K. Dupret, D. Delande, *Phys. Rev. Lett.* **74**, 522 (1996).

[16] G. Casati, B.V. Chirikov, F.M. Izrailev, J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, G. Casati and J. Ford, eds., *Lecture Notes in Physics,* vol. 93, Springer, Berlin 1979.

[17] B.V. Chirikov, *Phys. Rep.* **52**, 263 (1979).

[18] B.V. Chirikov, F.M. Izrailev, D.L. Shepelyansky, *Sov. Sci. Rev.* **C2**, 209 (1981).

[19] T. Hogg, B. Huberman, *Phys. Rev. Lett.* **48**, 711 (1982); *Phys. Rev.* **A28**, 22 (1983).

[20] S. Fishman, D.R. Grempel, R.E. Prange, *Phys. Rev. Lett.* **49**, 509 (1982).

[21] D.R. Grempel, R.E. Prange, S. Fishman, *Phys. Rev.* **A29**, 1639 (1984).

[22] R.E. Prange, D.R. Grempel, S. Fishman, in *Chaotic Behavior in Quantum Systems*, *Proc. Como Conference on Quantum Chaos*, G. Casati, ed., Plenum, New York 1984.

[23] D.J. Thouless, in *Ill Condensed Matter, Proc. Les Houches Summer School* vol. 31, R. Balian, R. Maynard and G. Toulouse, eds. North-Holland, Amsterdam 1979.

[24] P.A. Lee, T.V. Ramakrishanan, *Rev. Mod. Phys.* **57**, 287 (1985).

[25] G. Casati, I. Guarneri, D.L. Shepelyansky, *Phys. Rev.* **A36**, 3501 (1987).

[26] M. Feingold, S. Fishman, D.R. Grempel, R.E. Prange, *Phys. Rev.* **B31**, 6852 (1985).

[27] S.A. Molcanov, *Commun. Math. Phys.* **87**, 429 (1981).

[28] A.J. Lichtenberg, M.A. Lieberman, *Regular and Stochastic Motion* Springer-Verlag, New York 1983.

[29] H. Frahm, H.J. Mikeska, *Phys. Rev. Lett.* **60**, 3 (1988).

[30] F.M. Izrailev, *Phys. Rev. Lett.* **56**, 541 (1986).

[31] Th.W. Ruijgrok, K. Sokalski, *Physica* **A208**, 263 (1994).