Vol. 29 (1998) ACTA PHYSICA POLONICA B No 8

LOW-TEMPERATURE ASYMPTOTICS OF FREE ENERGY FOR 3D ISING MODEL IN AN EXTERNAL MAGNETIC FIELD

Martin S. Kochmański

Institute of Physics, Pedagogical University
T. Rejtana 16 A, 35–310 Rzeszów, Poland
e-mail: mkochma@atena.univ.rzeszow.pl

(Received January 28, 1998; revised version March 30, 1998)

The paper presents new method for calculating the low-temperature asymptotics of free energy of the 3D Ising model in external magnetic field ($H \neq 0$). For this purpose the method of transfer-matrix, and generalized Jordan–Wigner transformations are used. The results obtained are valid in the wide range of temperature and magnetic field values.

PACS numbers: 05.50. +q

1. Formulation of the problem

As it is well known, by till now an exact solution for the 2D Ising model in external magnetic field ($H \neq 0$) was not found. In the case of the 3D Ising model there does not exist an exact solution for vanishing magnetic field ($H = 0$), to say nothing of the case of non-zero magnetic field. Despite the great successes in the investigation of Ising models made by means of renormalization group method [1] and other approximate methods [2–5], the problem of calculation of various asymptotics for the 2D and 3D Ising models in the external magnetic field ($H \neq 0$) is still of great importance. In the paper [6] we calculated low-temperature asymptotics for the 2D Ising model in the external magnetic field ($H \neq 0$), as well as free energy for this model in the limit of asymptotically vanishing magnetic field. In this paper we discuss shortly the calculation of the low-temperature asymptotics for free energy in the 3D Ising model in external magnetic field ($H \neq 0$), following the approach and the ideas we have introduced in the paper [6].

Let us consider a cubic lattice built of N rows, M columns and K planes, to vertices of which are assigned the numbers σ_{nmk} from the two-entries set ± 1. These quantities here and everywhere below will be referred to as the

Ising "spins." The multiindex (nmk) numbers vertices of the lattice, with n numbering rows, m numbering columns, and k numbering planes. The Ising model with the nearest neighbors interactions in external magnetic field is described by the Hamiltonian of the form:

$$\mathcal{H} = -\sum_{(n,m,k)=1}^{NMK} \left(J_1\sigma_{nmk}\sigma_{n+1,mk} + J_2\sigma_{nmk}\sigma_{n,m+1,k} \right. \\ \left. + J_3\sigma_{nmk}\sigma_{nm,k+1} + H\sigma_{nmk}\right), \tag{1.1}$$

taking into account anisotropy of the interaction between the nearest neighbors ($J_{1,2,3} > 0$), and the interaction of the spins σ_{nmk} with external magnetic field H, directed "up" ($\sigma_{nmk} = +1$). The main problem consists of calculation of the statistical sum for the system:

$$\begin{aligned} Z_3(h) &= \sum_{\sigma_{111}=\pm1} \cdots \sum_{\sigma_{NMK}=\pm1} \mathrm{e}^{-\beta\mathcal{H}} \\ &= \sum_{\{\sigma_{nmk}=\pm1\}} \exp\Big[\sum_{nmk}(K_1\sigma_{nmk}\sigma_{n+1,mk} + K_2\sigma_{nmk}\sigma_{n,m+1,k} \\ &\qquad + K_3\sigma_{nmk}\sigma_{nm,k+1} + h\sigma_{nmk})\Big], \end{aligned} \tag{1.2}$$

where $K_{1,2,3} = \beta J_{1,2,3}$, $\quad h = \beta H$, $\quad \beta = 1/k_BT$. Typical boundary conditions for the variables σ_{nmk} are the periodic ones. We take this standard assumption everywhere below. Let us note here that the statistical sum (1.2) is symmetric with respect to the change $(h \to -h)$.

In this paper we consider a limited version of the problem, that is the calculation of the low-temperature asymptotics for free energy in the $3D$ Ising model in external magnetic field. More precisely, if the coupling constants ($J_{1,2,3}$ =const.) and external magnetic field (H =const.) are given, we consider the region of temperatures satisfying the condition: $h \sim \varepsilon^{-1}$, $\quad \varepsilon \ll 1$. To be more exact, we introduce a small parameter in the following way:

$$1 - \tanh(h/2) \sim \varepsilon, \qquad \varepsilon \ll 1\,. \tag{1.3}$$

Then we consider the problem of calculation of free energy per one Ising spin in the thermodynamic limit, with the accuracy up to the quantities of the order $\sim \varepsilon^2$ in expansions of the operators associated with the spin interaction with external magnetic field as well as among themselves (details of the approximation used will be presented below). To the author's knowledge, the problem formulated in this way, was not considered in existing literature and is of considerable importance.

2. Partition function

Let us consider an auxillary $4D$ Ising model in external magnetic field H on simple $4D$ lattice $(N \times M \times K \times L)$. We write the Hamiltonian for the $4D$ Ising model with the nearest neighbor interaction in the form:

$$\mathcal{H} = -\sum_{n,m,k,l} (J_1\sigma_{nmkl}\sigma_{n+1,mkl} + J_2\sigma_{nmkl}\sigma_{n,m+1,kl} + J_3\sigma_{nmkl}\sigma_{nm,k+1,l} + J_4\sigma_{nmkl}\sigma_{nmk,l+1} + H\sigma_{nmkl}) \,, \tag{2.1}$$

taking into account anisotropy of the interaction between the nearest neighbors ($J_{1,2,3,4} > 0$), and interaction of the spins σ_{nmkl} with external magnetic field H, directed "up" ($\sigma_{nmkl} = +1$). In Eq. (2.1) the multiindex $(nmkl)$ numbers the vertices of the $4D$ lattice, and the indices (n, m, k, l) take the values from 1 to (N, M, K, L), respectively. As in the case of the $3D$ Ising model, we introduce periodic boundary conditions for the variables σ_{nmkl}. Then we write the partition function $Z_4(h)$ in the form:

$$Z_4(h) = \sum_{\sigma_{1111}=\pm 1} \cdots \sum_{\sigma_{NMKL}=\pm 1} \mathrm{e}^{-\beta\mathcal{H}} = \sum_{\{\sigma_{nmkl}=\pm 1\}} \exp\Big[\sum_{nmkl}(K_1\sigma_{nmkl}\sigma_{n+1,mkl} + K_2\sigma_{nmkl}\sigma_{n,m+1,kl} + K_3\sigma_{nmkl}\sigma_{nm,k+1,l} + K_4\sigma_{nmkl}\sigma_{nmk,l+1} + h\sigma_{nmkl})\Big] \,, \tag{2.2}$$

where the quantities K_i and h are defined as above (1.2) [7,8]. The expression (2.2) can be written using the well known method of transfer matrix, in the form of a trace from the L-th power of the operator $\hat{T}$:

$$Z_4(h) = Tr(\hat{T})^L \,, \qquad \hat{T} = T_4 T_h^{1/2} T_3 T_2 T_1 T_h^{1/2} \,, \tag{2.3}$$

where the operators $T_{1,2,3,4,h}$ are defined by the formulas:

$$T_1 = \exp\left(K_1 \sum_{nmk} \tau^z_{nmk}\tau^z_{n+1,mk}\right) \,, \quad T_2 = \exp\left(K_2 \sum_{nmk} \tau^z_{nmk}\tau^z_{n,m+1,k}\right) \,, \tag{2.4}$$

$$T_3 = \exp\left(K_3 \sum_{nmk} \tau^z_{nmk}\tau^z_{nm,k+1}\right) \,,$$

$$T_4 = (2\sinh 2K_4)^{NMK/2} \exp\left(K_4^* \sum_{nmk} \tau^x_{nmk}\right) \,, \tag{2.5}$$

$$T_h = \exp\left(h \sum_{nmk} \tau^z_{nmk}\right) \,, \tag{2.6}$$

and the quantities K_4 i K_4^* are coupled by the following relations:

$$\tanh(K_4) = \exp(-2K_4^*)\,, \quad \text{or} \quad \sinh 2K_4 \sinh 2K_4^* = 1\,. \tag{2.7}$$

The Pauli spin matrices $\tau_{nmk}^{x,y,z}$ commute for $(nmk) \neq (n'm'k')$, and for given (nmk) these matrices satisfy the usual relations [9]. It is easy to see that the matrices $T_{1,2,3,h}$ commute with each other, but do not commute with the matrix T_4. If the quantities $K_i = 0, \quad (i = 1,2,3)$, we immediately get the well known expressions describing the $3D$ Ising model on a simple cubic lattice. Namely, the transition to the $3D$ Ising model with respect to the coupling constants K_1, K_2, or K_3 is realized by taking $(K_1 = 0)$, or $(K_2 = 0)$, or $(K_3 = 0)$, and removing summation over n, $(N = 1)$, or over m, $(M = 1)$, or over k, $(K = 1)$, respectively. As a result we get the standard expressions [7] for the $3D$ Ising model in external magnetic field. In the process the operators $T_i, \quad (i = 1,2,3)$ in everyone of the cases are identically equal to the unit operator $(T_i \equiv \hat{1})$. A bit different situation appears in the case of the transition to the $3D$ Ising model with respect to the coupling constant K_4. In this case we take $(K_4 = 0, \quad L = 1)$, *i.e.* we remove summation over l. In consequence we get the following expression for the operator T_4, (2.5):

$$T_4^* \equiv T_4(K_4 = 0) = \prod_{nmk} (1 + \tau_{nmk}^x)\,, \tag{2.8}$$

where we used the relation (2.7). Then, after transition to the limit $(K_4 = 0, \quad L = 1)$ in (2.3), we can write the following expression for the partition function for the $3D$ Ising model:

$$Z_3(h) = Tr(T_4^* T_h^{1/2} T_3 T_2 T_1 T_h^{1/2})\,, \tag{2.9}$$

where the matrices T_i are defined as above (2.4)–(2.6), (2.8). Now we go over to the fermionic representation. For this purpose one should write the matrices T_i in terms of the Pauli operators $\tau_{nmk}^{\pm}$, [8]:

$$\tau_{nmk}^{\pm} = \tfrac{1}{2}(\tau_{nmk}^z \pm i\tau_{nmk}^y), \tag{2.10}$$

which satisfy anticommutation relations for one vertex, and which commute for different vertices.

As the next step one should proceed from the representation by Pauli operators (2.10) to the representation by Fermi creation and annihilation operators [10]. In the paper [10] an appropriate transformations (generalized transformations of the Jordan–Wigner type), with enable to go over to the fermionic representation were introduced:

$$\begin{aligned}
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{N}\sum_{p=1}^{M}\sum_{q=1}^{k-1}\alpha^\dagger_{spq}\alpha_{spq}+\sum_{s=1}^{N}\sum_{p=1}^{m-1}\alpha^\dagger_{spk}\alpha_{spk}+\sum_{s=1}^{n-1}\alpha^\dagger_{smk}\alpha_{smk}\right)\right]\alpha^\dagger_{nmk},\\
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{N}\sum_{p=1}^{M}\sum_{q=1}^{k-1}\beta^\dagger_{spq}\beta_{spq}+\sum_{s=1}^{n-1}\sum_{p=1}^{M}\beta^\dagger_{spk}\beta_{spk}+\sum_{p=1}^{m-1}\beta^\dagger_{npk}\beta_{npk}\right)\right]\beta^\dagger_{nmk},\\
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{N}\sum_{p=1}^{m-1}\sum_{q=1}^{K}\gamma^\dagger_{spq}\gamma_{spq}+\sum_{s=1}^{N}\sum_{q=1}^{k-1}\gamma^\dagger_{smq}\gamma_{smq}+\sum_{s=1}^{n-1}\gamma^\dagger_{smk}\gamma_{smk}\right)\right]\gamma^\dagger_{nmk},\\
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{N}\sum_{p=1}^{m-1}\sum_{q=1}^{K}\eta^\dagger_{spq}\eta_{spq}+\sum_{s=1}^{n-1}\sum_{q=1}^{K}\eta^\dagger_{smq}\eta_{smq}+\sum_{q=1}^{k-1}\eta^\dagger_{nmq}\eta_{nmq}\right)\right]\eta^\dagger_{nmk},\\
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{n-1}\sum_{p=1}^{M}\sum_{q=1}^{K}\omega^\dagger_{spq}\omega_{spq}+\sum_{p=1}^{M}\sum_{q=1}^{k-1}\omega^\dagger_{npq}\omega_{npq}+\sum_{p=1}^{m-1}\omega^\dagger_{npk}\omega_{npk}\right)\right]\omega^\dagger_{nmk},\\
\tau^+_{nmk} &= \exp\left[i\pi\left(\sum_{s=1}^{n-1}\sum_{p=1}^{M}\sum_{q=1}^{K}\theta^\dagger_{spq}\theta_{spq}+\sum_{p=1}^{m-1}\sum_{q=1}^{K}\theta^\dagger_{npq}\theta_{npq}+\sum_{q=1}^{k-1}\theta^\dagger_{nmq}\theta_{nmq}\right)\right]\theta^\dagger_{nmk},
\end{aligned} \tag{2.11}$$

and analogously for the operators τ^-_{nmk}. In the paper [10] we obtained formulas for the relations between various Fermi operators, and commutation relations for them. Further in this paper we will use the fact that the following equality of local occupation numbers is valid:

$$\begin{aligned}
\tau^+_{nmk}\tau^-_{nmk} &= \alpha^\dagger_{nmk}\alpha_{nmk}=\beta^\dagger_{nmk}\beta_{nmk}=\gamma^\dagger_{nmk}\gamma_{nmk}\\
&= \eta^\dagger_{nmk}\eta_{nmk}=\omega^\dagger_{nmk}\omega_{nmk}=\theta^\dagger_{nmk}\theta_{nmk}\,.
\end{aligned} \tag{2.12}$$

Then, applying the expressions (2.10)–(2.12) and considerations from the paper [6], we can write the partition function (2.9) in the form:

$$\begin{aligned}
Z_3(h) &= (2\cosh^2 h/2)^{NMK}\langle 0|T^*|0\rangle = A\langle 0|U+\mu^2 CUD|0\rangle\,,\\
U &\equiv T^l_h T_3 T_2 T_1 T^r_h\,,
\end{aligned} \tag{2.13}$$

where $A=(2\cosh^2 h/2)^{NMK}$ and $\mu=\tanh(h/2)$, and the operators $T_{1,2,3}$, $T^{l,r}_h$ and C,D are of the form:

$$T_1 = \exp\left[K_1\sum_{n,m,k=1}^{N,M,K}(\alpha^\dagger_{nmk}-\alpha_{nmk})(\alpha^\dagger_{n+1,mk}+\alpha_{n+1,mk})\right],$$

$$
\begin{aligned}
T_2 &= \exp\left[K_2 \sum_{n,m,k=1}^{N,M,K} (\beta^\dagger_{nmk} - \beta_{nmk})(\beta^\dagger_{n,m+1,k} + \beta_{n,m+1,k})\right], \\
T_3 &= \exp\left[K_3 \sum_{n,m,k=1}^{N,M,K} (\theta^\dagger_{nmk} - \theta_{nmk})(\theta^\dagger_{nm,k+1} + \theta_{nm,k+1})\right], \qquad (2.14)
\end{aligned}
$$

and

$$
\begin{aligned}
T_h^r &= \exp\Big\{\mu^2\Big[\sum_{nmk}\sum_{s=1}^{N-n} \alpha^\dagger_{nmk}\alpha^\dagger_{n+s,mk} \\
&\quad + \sum_{nn'mk}\sum_{t=1}^{M-m} \alpha^\dagger_{nmk}\alpha^\dagger_{n',m+t,k} + \sum_{nn'mm'k}\sum_{l=1}^{K-k} \alpha^\dagger_{nmk}\alpha^\dagger_{n'm',k+l}\Big]\Big\}, \\
T_h^l &= \exp\Big\{\mu^2\Big[\sum_{nmk}\sum_{l=1}^{K-k} \theta_{nm,k+l}\theta_{nmk} \\
&\quad + \sum_{nmkk'}\sum_{t=1}^{M-m} \theta_{n,m+t,k}\theta_{nmk'} + \sum_{nmm'kk'}\sum_{s=1}^{N-n} \theta_{n+s,mk}\theta_{nm'k'}\Big]\Big\},
\end{aligned}
$$

(2.15)

$$
C = \sum_{nmk} \theta_{nmk}\,, \qquad D = \sum_{nmk} \alpha^\dagger_{nmk}\,.
$$

Here and below $\sum_{n,m,\ldots}$ means summation over the complete set of indices ($n = 1, \ldots N$; $m = 1, \ldots M$; *etc.*). It is obvious that the operator $\hat{G}$:

$$
\hat{G} = (-1)^{\hat{S}}\,, \qquad \hat{S} = \sum_{nmk} \alpha^\dagger_{nmk}\alpha_{nmk}\,, \qquad (2.16)
$$

where $\hat{S}$ is the operator of the total number of particles, commutes with the operator T^*, (2.13). Therefore, we can divide all states of the operator T^* into states with even ($\lambda_{\hat{G}} = +1$) or odd number of particles ($\lambda_{\hat{G}} = -1$) with respect to the operator $\hat{G}$, (2.16). The form of the operators $T_{1,2,3}$ does not change during the course, only the boundary conditions for the operators ($\alpha_{nmk}, \ldots$) do. In the case of even states ($\lambda_{\hat{G}} = +1$) antiperiodic boundary conditions, and in the case of odd states periodic ones, are chosen [6].

The next step is the transition to the momentum representation:

$$
\alpha^\dagger_{nmk} = \frac{\exp(i\pi/4)}{(NMK)^{1/2}} \sum_{qp\nu} \mathrm{e}^{-i(nq+mp+k\nu)}\xi^\dagger_{qp\nu}, \quad \beta^\dagger_{nmk} \to \eta^\dagger_{qp\nu}, \quad \theta^\dagger_{nmk} \to \zeta^\dagger_{qp\nu}.
$$

Here we have introduced, in terms of occupation numbers for fixed $(qp\nu)$, the corresponding ξ-, η- and ζ- Fermi creation and annihilation operators in the finite-dimensional Fock space of $2^8 = 256$ dimensions. Then, after a series of transformations and calculations we have at the following formula for the partition function (2.13):

$$Z^{+}_{3D}(h) = A\left(\prod_{0<q,p,\nu<\pi} A_1^4(q)\right)\left(\prod_{0<q,p,\nu<\pi} A_3^4(\nu)\right)\langle 0|T_3^*(h)T_2T_1^*(h)|0\rangle, \tag{2.17}$$

where the operators $T_1^*(h),\ T_2,\ T_3^*(h)$ are of the form

$$\begin{aligned}
T_1^*(h) &= \exp\Big[\sum_{0<q,p,\nu<\pi} B_1(q)(\xi^\dagger_{-q-p-\nu}\xi^\dagger_{qp\nu} + \xi^\dagger_{-q-p\nu}\xi^\dagger_{qp-\nu} \\
&\quad +\xi^\dagger_{-qp-\nu}\xi^\dagger_{q-p\nu} + \xi^\dagger_{-qp\nu}\xi^\dagger_{q-p-\nu})\Big], \\
T_2 &= \exp\Big\{2K_2 \sum_{0<q,p,\nu<\pi} [\cos p(\eta^\dagger_{qp\nu}\eta_{qp\nu} + \ldots) + \sin p(\eta^\dagger_{-q-p-\nu}\eta^\dagger_{qp\nu} \\
&\quad + \ldots + \eta_{qp\nu}\eta_{-q-p-\nu} + \ldots]\Big\}, \\
T_3^*(h) &= \exp\Big[\sum_{0<q,p,\nu<\pi} B_3(\nu)(\zeta_{qp\nu}\zeta_{-q-p-\nu} \\
&\quad + \zeta_{-qp\nu}\zeta_{q-p-\nu} + \zeta_{q-p\nu}\zeta_{-qp-\nu} + \zeta_{-q-p\nu}\zeta_{qp-\nu})\Big],
\end{aligned} \tag{2.18}$$

and $A_1(q,h),\ldots$ are defined by the expressions:

$$\begin{aligned}
A_1(q,h) &= \cosh 2K_1 - \sinh 2K_1 \cos q + \alpha(h,q)\sinh 2K_1 \sin q\,, \\
A_3(\nu,h) &= \cosh 2K_3 - \sinh 2K_3 \cos \nu + \alpha(h,\nu)\sinh 2K_3 \sin \nu\,, \\
B_1(q,h) &= \frac{\alpha(h,q)[\cosh 2K_1 + \sinh 2K_1 \cos q] + \sinh 2K_1 \sin q}{A_1(q,h)}\,, \\
B_3(\nu,h) &= \frac{\alpha(h,\nu)[\cosh 2K_3 + \sinh 2K_3 \cos \nu] + \sinh 2K_3 \sin \nu}{A_3(\nu,h)}\,, \\
\alpha(h,q) &= \tanh^2(h/2)\frac{1+\cos q}{\sin q}\,, \quad \alpha(h,\nu) = \tanh^2(h/2)\frac{1+\cos\nu}{\sin\nu}\,.
\end{aligned} \tag{2.19}$$

In the formula for $Z^{+}_{3D}(h)$ the sign $(+)$ means that we consider the case of even states $(\lambda_{\hat{G}} = +1)$ with respect to the operator $\hat{G}$, (2.16). It is obvious that for $h = 0$ we have the $3D$ Ising model for vanishing magnetic field. Then, for $K_1 = 0$ (or $K_2 = 0$, or $K_3 = 0$) the expression (2.17) for the statistical sum describes the $2D$ Ising model in external magnetic field [6].

3. Solution

Let us consider calculation of free energy per one Ising spin in external magnetic field in the approximation described shortly in the introduction. For this aim let us consider the operators $T_1^*(h)$ and $T_3^*(h)$ in the "coordinate" representation:

$$
\begin{aligned}
T_1^*(h) &= \exp\left[\sum_{nmk}\sum_{s=1}^{N-n} a(s)\alpha^\dagger_{nmk}\alpha^\dagger_{n+s,mk}\right], \\
T_3^*(h) &= \exp\left[\sum_{nmk}\sum_{l=1}^{K-k} c(l)\theta_{nm,k+l}\theta_{nmk}\right], \qquad (3.1)
\end{aligned}
$$

where the "weights" $a(s)$ and $c(l)$ are defined by the formulas:

$$
\begin{aligned}
a(s) &= \frac{1}{N}\sum_{0<q<\pi} 2B_1(q)\sin(sq) = z_1^{*s} + \tanh^2 h_1^* \frac{1-z_1^{*s}}{(1-z_1^*)^2}, \quad s=1,2,3,\ldots \\
c(l) &= \frac{1}{K}\sum_{0<\nu<\pi} 2B_3(\nu)\sin(l\nu) = z_3^{*l} + \tanh^2 h_3^* \frac{1-z_3^{*l}}{(1-z_3^*)^2}, \quad l=1,2,3,\ldots .
\end{aligned}
\qquad (3.2)
$$

We introduced renormalized quantities ($K^*_{1,3}$, $h^*_{1,3}$) defined as follows :

$$
\begin{aligned}
\sinh 2K^*_{1,3} &= \beta_{1,3}[\sinh 2K_{1,3}(1-\tanh^2(h/2))], \\
\cosh(2K^*_{1,3}) &= \beta_{1,3}[\cosh 2K_{1,3} + \tanh^2(h/2)\sinh 2K_{1,3}], \\
\beta_{1,3} &= [1+2\tanh^2(h/2)\sinh 2K_{1,3}e^{2K_{1,3}}]^{-1/2}, \\
\tanh^2 h^*_{1,3} &= \tanh^2(h/2)\frac{\beta_{1,3}\exp(2K_{1,3})}{\cosh^2 K^*_{1,3}}. \qquad (3.3)
\end{aligned}
$$

These formulas are valid for ($K_{1,3} \geq 0$). As in the case of the $2D$ Ising model [6, 11], in this case one can introduce a diagrammatic representation for the vacuum matrix element $S \equiv \langle 0|T_3^*(h)T_2T_1^*(h)|0\rangle$ also. Computation of the vacuum matrix element S, which enters the formula (2.17) for $Z^+_{3D}(h)$ in general case, where the "weights" (3.2) are arbitrary is, at least at present, impossible. Nevertheless, there exists a special case in which we can calculate the quantity S in the $3D$ case. Namely, this is the case where the "weights" (3.2) are independent of l and s. In this case one should, as in the $2D$ case [12], put the parameters $K_{1,3}$ equal zero ($K_{1,3} = 0$) in the formula (2.13), and then express the operators $T_h^{l,r}$ in terms of the Fermi β-operators (2.11) of creation and annihilation, having in mind to calculate S. After transition to the momentum representation, one should calculate the vacuum matrix element $S^*(y_1, y_3, z_2)$:

$$S^*(y_1, y_3, z_2) \equiv \langle 0|T^l(y_3)T_2T^r(y_1)|0\rangle\,, \qquad y_{1,3} \equiv \tanh^2 h_{1,3}\,,$$

where $z_2 = \tanh K_2$, the calculation of $S^*(y_1, y_3, z_2)$ becomes trivial. (Here we introduced the following change of notation: $h/2 \to h_1$ — for the operator T^r_h, and $h/2 \to h_3$ — for the operator T^l_h). We can write the result for $S^*(y_1, y_3, z_2)$ in the following form:

$$\begin{aligned} S^*(y_1, y_3, z_2) &= \left(2\cosh^2 K_2\right)^{\frac{NMK}{2}} \\ &\times \prod_{0<qp\nu<\pi} \Big[(1 - 2z_2\cos p + z_2^2)(1-\cos p) + 2z_2(y_1+y_3)\sin^2 p \\ &+ \; y_1y_3(1 + 2z_2\cos p + z_2^2)(1+\cos p)\Big]^4 . \end{aligned} \tag{3.4}$$

This result can be used further to calculate free energy in the approximation discussed above (1.3). For this aim let us note that the conditions $[\tanh^2 h^*_{1,3}/(1 - z^*_{1,3})^2] \to 1$ are equivalent, accordingly to (3.3), to the conditions $(\exp(-2K_{1,3})(1-\tanh^2 h/2) \to 0)$. It follows from this equation that for fixed ($J_{1,3} =$ const., $H =$ const.) these conditions are satisfied in the region of temperatures T, where $(h/2) \sim \varepsilon^{-1}$, $\varepsilon \ll 1$. In this case we can use the result (3.4). Namely, let us consider the formulas (2.19) for $B_{1,3}$, written in terms of the renormalized parameters ($h^*_{1,3}$, $K^*_{1,3}$):

$$B_{1,3} = \frac{\tanh^2 h^*_{1,3}\frac{\sin q(\nu)}{1-\cos q(\nu)} + 2z^*_{1,3}\sin q(\nu)}{1 - 2z^*_{1,3}\cos q(\nu) + {z^*_{1,3}}^2}\,, \tag{3.5}$$

where $z^*_{1,3} = \tanh K^*_{1,3}$. Next, since the following equalities are satisfied:

$$\frac{z^*_{1,3}}{1 + {z^*_{1,3}}^2} = \frac{z_{1,3}(1 - \tanh^2 h/2)}{1 + 2z_{1,3}\tanh^2 h/2 + z^2_{1,3}}\,,$$

then, if we introduce a small parameter $[1 - \tanh(h/2)] \sim \varepsilon$, $(\varepsilon \ll 1)$, and expand $B_{1,3}$ into a series in powers of ε ($z^*_{1,3} \sim \varepsilon$), we obtain

$$B_{1,3} = \frac{(\tanh^2 h^*_{1,3} + 2z^*_{1,3})\sin q(\nu)}{1 - \cos q(\nu)} + \sim \varepsilon^2 \,.$$

This formula gives the following expressions for the "weights" $a(s)$ and $c(l)$, (3.2) in this approximation:

$$a(s) = \tanh^2 h^*_1 + 2z^*_1\,, \qquad c(l) = \tanh^2 h^*_3 + 2z^*_3\,, \tag{3.6}$$

with exactness of the order of smallness $\sim \varepsilon^2$. As a result in this approximation the "weights" $a(s)$, $c(l)$ do not depend on (s,l). Finally, if we substitute to the expression (3.4) for $S^*(y_1, y_3, z_2)$, the parameters $y_1 \to a(s)$ and $y_3 \to c(l)$, (3.6), we have at the following formula for free energy per one Ising spin $F_{3D}(h)$ in the thermodynamic limit:

$$\begin{aligned}
&-\beta F_{3D}(h) \asymp \ln(2^{3/2}\cosh K_1^* \cosh K_2 \cosh K_3^* \cosh^2 h/2) \\
&+\frac{1}{2\pi}\int_0^\pi \ln\Big[(1-2z_2\cos p + z_2^2)(1-\cos p) \\
&+2z_2(\tanh^2 h_1^* + \tanh^2 h_3^* + 2z_1^* + 2z_3^*)\sin^2 p \\
&+(\tanh^2 h_1^* + 2z_1^*)(\tanh^2 h_3^* + 2z_3^*)(1+2z_2\cos p + z_2^2)(1+\cos p)\Big] dp \,, \quad (3.7)
\end{aligned}$$

where $\beta = 1/k_B T$, and $z_2 = \tanh K_2$, and $h_{1,3}^*$ and $K_{1,3}^*$ are coupled with h and $K_{1,3}$ by the relations (3.3). One can show that, as it was done for the $1D$ and $2D$ Ising models [6, 12], in the case of the states odd ($\lambda_{\hat{G}} = -1$) with respect to the operator $\hat{G}$, (2.16), the formula for $F_{3D}(h)$ is described in the thermodynamic limit by (3.7). Let us note that the asymptotics (3.7) obtained above can be applied also in the case of rather strong magnetic fields (H), as far as it satisfies the condition $(1-\tanh h) \sim \varepsilon$, $\varepsilon \ll 1$, $(T = \text{const.})$.

It is well-known, that there is a great number of publications (see, for example, the series in many volumes *Phase Transitions and Critical Phenomena*, Ed. by C. Domb and M.S. Green) related, somehow or other, to low-and high-temperature expansions for the Ising model in zero — as well as in non-zero external magnetic field (see, for example, [13–17]). It should be noted, that the form of these expansions is not practically suitable for the calculations and phase diagram constructions. This one is quite obvious from the continuous attempts to search for the new approaches to the problem solution (see, for example [18–21]). From this point of view, our result (3.7) is just another attempt of that kind. As it follows from the deriving procedure, the validity range in the field H and temperature T of expansion (3.7) is great enough. In other words, the boundaries of this validity range are as though "floating" and just that one makes our result differ from the others. The last one will be discussed elsewhere.

4. Final remarks

The main result given by the formula (3.7) can be applied, in the setting of equilibrium thermodynamics, to the analysis of three-dimensional Ising magnetic, lattice gas and to the three-dimensional models of binary alloys [22, 23] under the conditions for the temperature and magnetic field given by (1.3). Such analysis, as well as construction of appropriate phase diagrams for the models mentioned above is, in our opinion, of great interest. They deserve to be considered in a separate publication. Therefore we deliberately do not compare here our result (3.7) with those which were obtained by others. The other important feature of the method presented here is the possibility to derive the expressions for the free energy of the $3D$ Ising model in the limiting case of the magnetic field tending to zero ($H \to 0\,,\ N,\ M,\ K \to \infty$), if we know exact solution for the $3D$ Ising model in the zero external magnetic field ($H = 0$). This possibility results from equations (3.2)–(3.3) describing renormalised interaction constants $K^*_{1,3}$, and corresponds, as it was shown in the paper [6], to the results obtained by Yang [24] for the $2D$ Ising model.

I am grateful to H. Makaruk, R. Owczarek and I. Tralle for their assistance in preparation of the final form of this paper.

REFERENCES

[1] K.G. Wilson, J. Kogut, *Phys. Rep.* **C12**, N 2 (1974).

[2] B. McCoy, T.T. Wu, *Two Dimensional Ising Models*, Harvard U. Press. Cambridge, Mass. 1973 .

[3] Sh. Ma, *Modern Theory of Critical Phenomena*, Univ. California, U.A. Ben., Inc., 1976.

[4] Ya.G. Sinay, *Theory of Phase Transitions*, Nauka, Moscow 1980 (in Russian).

[5] Current Problems in Statistical Mechanics, *Physica* **A177**, Nn 1–3, (1991).

[6] M.S. Kochmański, *Phys. Rev.* **E56**, 3831 (1997).

[7] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, Inc., 1982.

[8] Yu.A. Izyumov, Yu.N. Skryabin: *Statistical Mechanics of Magnetically Ordered Systems*, Nauka, Moscow 1987, (in Russian).

[9] K. Huang: *Statistical Mechanics*, J. Wiley and Sons, Inc., New York–London 1963.

[10] M.S. Kochmański, *J. Tech. Phys.* **36**, 485 (1995).

[11] M.S. Kochmański, *Zh. Eksp. Teor. Fiz.* **111**, 1717 (1997) (in Russian); *JETP* **84**, 940 (1997).

[12] M.S. Kochmański, *J. Tech. Phys.* **37**, 67 (1996).

[13] M. Wortis, *Linked Claster Expansion* in *Phase Transitions and Critical Phenomena* Vol. 3 *Series Expansions for Lattice Model*, ed. C. Domb, M.S. Green, Academic Press, London, New York 1974, pp. 114–180.

[14] G.A. Backer,*Quantitative Theory for Critical Phenomena*, Academic Press, Boston 1990.

[15] C. Bailline, R. Gupta, K. Hawick, S. Pawley, *Phys. Rev.* **B45**, 10438 (1992).

[16] Z. Onyszkiewicz, *J. Magn. Mater.* **139**, 3 (1995).

[17] R. Gupta, P. Tamayo, Critical Exponents for the 3D Ising Model, Technical Report of Los Alamos National Laboratory, Los Alamos (1996).

[18] S.O. Warnaar, B. Nienhuis, K.A. Seaton, *Phys. Rev. Lett.* **69**, 710 (1992).

[19] V.V. Bazhanov, B. Nienhuis, S.O. Warnaar, *Phys. Lett.* **B322**, 198 (1994).

[20] V. Matveev, R. Shrock, *Phys. Rev.* **E53**, 254 (1996).

[21] U. Grimm, B. Nienhuis, *Phys. Rev.* **E55**, 5011 (1997).

[22] J.M. Ziman: *Models of Disorder*, Cambridge Univ. Press, Cambridge 1979.

[23] C.J. Thompson, *Classical Equilibrium Statistical Mechanics*, Cl. Press, Oxford 1988.

[24] C.N. Yang, *Phys. Rev.* **85**, 809 (1952).