

BOHMIAN TIME VERSUS PROBABILISTIC TIME

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One of the basic problems of quantum cosmology is the problem of time. Various solutions have been proposed for this problem. One approach is to use the Bohmian time. Another approach is to use the probabilistic time which was recently introduced by Castagnino. We consider both of these definitions as generalizations of a semi-classical time and compare them for a mini-super space.

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1. Introduction

In quantum cosmology, the Universe is described by a wave function ψ . This wave function can be obtained as a solution of the Wheeler–DeWitt equation (WDW) with appropriate boundary conditions. This equation is the relevant Schrödinger equation ($H\psi = i\hbar\frac{\partial\psi}{\partial t}$) which is obtained from the classical theory, using the Dirac quantization prescription. Since the general covariance of the classical theory gives the classical Hamiltonian as a subsidiary condition ($H = 0$), the solutions of the WDW equation are time-independent. But how can a time-independent wave function describe a dynamical Universe? Various solutions have been proposed for this so-called *time-problem*: internal time versus external time, semi-classical time, Bohmian time, probabilistic time, ... *etc.* Here we compare three of these times: the Bohmian time [1], the probabilistic time [2] and the semi-classical time [3]. First, we give two different definitions of the semi-classical time. Then, we introduce the Bohmian time and the probabilistic time as generalizations of the two definitions of the semi-classical time. Consequently, we calculate the rate of the expansion of the Universe in terms of these two times for a mini-super space. Finally, we compare these two times.

2. Semi-classical time

Consider a simple system having a Lagrangian [4] :

$$L = \frac{1}{2}m(\dot{Q}^2 - V(Q)) .$$

In this Lagrangian, there is no coupling between the kinetic energy and the potential energy terms that mimic gravity. The momentum P_Q , conjugate to Q , is equal to $m\dot{Q}$. Now consider a WKB solution of this problem associated with the energy E

$$\psi_{\text{WKB}}^{(E)}(Q) = \frac{N}{\sqrt{S'(Q)}} e^{\frac{i}{\hbar} S(Q)} ,$$

where $S'(Q) = \frac{dS}{dQ}$, $S(Q)$ being the classical Hamilton–Jacobi function obtained from

$$\frac{1}{2m}S' + mV = E .$$

To define a suitable time parameter, we take advantage of the classical equation

$$P_Q = m \frac{dQ}{dt} = S'(Q) . \quad (1)$$

Since in the Copenhagen interpretation of quantum mechanics (1) is merely applicable in the semi-classical limit, this time parameter is only definable in this limit. For complex systems, however, it is not necessary for all degrees of freedom to have semi-classical behaviour. Thus, *e.g.*, for our present Universe, gravitational degrees of freedom have semi-classical behaviour, whereas other fields have quantum behaviour. Here, one can use classical degrees of freedom to define the aforementioned time parameters. One can even obtain a time-dependent equation for the quantum fields from the time-independent WDW equation [3]. It seems clear that for the early Universe, where all fields had quantum behaviour, this particular time parameter is not well-defined in the Copenhagen interpretation. It is for this reason, that some people have considered time as a classical concept which is born in the semi-classical limit.

Now one can write

$$|\psi_{\text{WKB}}^{(E)}(Q)|^2 dQ = \text{const.} \frac{dQ}{S'(Q)} \propto \frac{dQ}{\dot{Q}} = dt . \quad (2)$$

This means that the probability $|\psi_{\text{WKB}}^{(E)}|^2$ is larger in the Q -interval where the classical system spends more time. Thus, we can use (2) to define time in the semi-classical limit too. These two different time parameters, defined for this simple system, coincide in the classical limit.

The question is whether we can extend these definitions to the region where quantum effects are not negligible and systems are not simple.

3. The Bohmian time

When we extend the relation (1), defined in its semi-classical limit, to the quantum realm, we get the Bohmian equation of motion [5]. In fact, one of the fundamental principles of the Bohmian mechanics is this relation which connects the canonical momentum with the derivative of the phase of the wave function. In this way, one can define a path for the particle, where t is the parameter of the path. One can use this time parameter to calculate the average tunneling time through a potential barrier [6], where there is no well-defined way for its calculation in the Copenhagen interpretation. In fact, one can design experiments by which one can test the validity of the Bohmian time in the quantum domain [7].

In recent years, people have used the Bohmian time to take care of the time problem in quantum cosmology [1]. To show this, consider the following mini-super space

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 d\Omega_3^2,$$

where $a(t)$ is the radius of the Universe, $N(t)$ is an arbitrary function of t and $d\Omega_3^2$ is the metric of a unit three-sphere. The Hilbert–Einstein Lagrangian plus the Lagrangian of a homogeneous scalar field ϕ is given in this metric by

$$L = -a^{-3} \left\{ \frac{1}{2N} \left[\frac{\dot{a}}{a} \right]^2 + \frac{N}{2} (-a^{-2} + H^2) \right\} + a^3 \left\{ \frac{\dot{\phi}^2}{2N} - NV(\phi) \right\},$$

where $V(\phi)$ is an arbitrary potential and H^2 is related to the cosmological constant through the relation $H^2 = \frac{\Lambda}{3}$. The canonical momenta P_ϕ and P_a are, respectively, given by $a^3 \dot{\phi}$ and $-a\dot{a}$. If we obtain the classical Hamiltonian and make the substitutions $P_\phi \rightarrow -i\frac{\partial}{\partial\phi}$ and $P_a \rightarrow -i\frac{\partial}{\partial\phi}$, we obtain the WDW equation in the following form (in the gauge $N = 1$)

$$\left\{ \left[\frac{1}{2} a^{-3} \left(a \frac{\partial}{\partial a} \right)^2 + a^3 (-a^{-2} + H^2) \right] + \left[\frac{1}{2} a^{-3} \frac{\partial^2}{\partial \phi^2} + a^3 V(\phi) \right] \right\} \psi(a, \phi) = 0. \quad (3)$$

Writing the solutions of this equation in the form $R(a, \phi)e^{\frac{i}{\hbar}S(a, \phi)}$, we get the following Bohmian equations of motion

$$\dot{a} = -\frac{1}{a} \frac{\partial S}{\partial a}, \quad (4)$$

$$\dot{\phi} = \frac{1}{a^3} \frac{\partial S}{\partial \phi}. \quad (5)$$

Thus, knowing ψ , one can get the evolution of a and ϕ . The time independence of ψ for this problem (quantum cosmology) simplifies the integration of (4) and (5).

4. The probabilistic time

Recently, Castagnino has extended (2) to define time in the quantum cosmology. Consider the aforementioned mini-super space. The volume element is $\sqrt{-G(a)}da d\phi$, where $G(a) = \det(G_{ab})$, G_{ab} being the metric defined on the mini-super space. The probability of finding the metric in the interval $(a, a + da)$, independent of ϕ , is

$$dP = da \sqrt{-G(a)} \int |\psi(a, \phi)|^2 d\phi.$$

The idea of the probabilistic time is that the Universe stays in metric a for a period of time proportional to dP . Thus, we can define an element of the probabilistic time in the following way

$$d\theta = c da \sqrt{-G(a)} \int |\psi(a, \phi)|^2 d\phi,$$

where c is a constant. Castagnino has considered a non-relativistic particle described by the one-dimensional wave function $\psi(x, t)$. Suppose that we parametrize ψ by $\tau = \tau(t)$ instead of t , using an arbitrary measure $\mu(\tau)$. Then, we can write

$$|\psi(x, t)|^2 dx dt = |\psi_\mu(x, \tau)|^2 \mu(\tau) dx d\tau. \quad (6)$$

Now, the question is about how we can obtain the real time t from τ . Using (6), one can show that

$$dt = t_0 d\tau \mu(\tau) \int |\psi(x, \tau)|^2 dx,$$

where we have normalized $\psi(x, t)$ in the following way

$$\int_0^{t_0} \int |\psi(x, t)|^2 dx dt = 1.$$

For the problem under consideration we can do the same thing with a to obtain

$$d\theta = \theta_0 da \sqrt{-G(a)} \int |\psi(a, \phi)|^2 d\phi. \quad (7)$$

Using this relation, we can obtain the expansion rate of Universe in terms of the probabilistic time

$$\frac{da}{d\theta} = \left[\theta_0 \sqrt{-G(a)} \int |\psi(a, \phi)|^2 d\phi \right]^{-1}. \quad (8)$$

5. Comparison of the Bohmian time and the probabilistic time

In the first section we showed that for a simple system, the Bohmian and the probabilistic times coincide in the semi-classical limit. To compare these two times in the quantum domain for the more complicated systems, we consider the aforementioned mini-super space. In our discussion, we obtained the expansion rate of the Universe in terms of both the Bohmian time (4) and the probabilistic time (8). The comparison of these two expansion rates provides a good way of comparing these two time parameters. The relation (4) relates the expansion rate to the phase of the wave function, where as the relation (8) relates the expansion rate to the amplitude of the wave function. If we write $\psi(a, \phi)$ in the form $R(a, \phi)e^{\frac{i}{\hbar}S(a, \phi)}$ and substitute it in the WDW, we get two equations, one of which is the following:

$$-a \frac{\partial}{\partial a} \left(R^2 a \frac{\partial S}{\partial a} \right) + \frac{\partial}{\partial \phi} \left(R^2 \frac{\partial S}{\partial \phi} \right) = 0. \quad (9)$$

If ψ is independent of ϕ , this equation leads to:

$$R^2 a \frac{\partial S}{\partial a} = \text{const.}$$

or

$$|\psi|^2 = R^2 = \frac{\text{const.}}{a} \left(\frac{\partial S}{\partial a} \right)^{-1}.$$

If we insert this into (8), we get

$$\frac{da}{d\theta} = \left[\theta_0 \sqrt{-G(a)} \int \frac{\text{const.}}{a} \left(\frac{\partial S}{\partial a} \right)^{-1} d\phi \right]^{-1}.$$

Since ψ was assumed to be independent of ϕ , so is S . Thus, considering the fact that for the mini-super space under consideration $\sqrt{-G(a)} = a^2$, we get

$$\frac{da}{dt} = -\frac{1}{a} \frac{\partial S}{\partial a},$$

where we have defined t as $\theta[-\theta_0(\text{const.}) \int d\phi]^{-1}$.

Notice that only in the non-realistic case of a Universe free of matter, the Bohmian time coincides with the probabilistic time. Here, we have not referred to the semi-classical limit. In fact, for a system with one degree of freedom, the Bohmian time and the probabilistic time coincide, both in semi-classical regime and in the quantum domain as we have already shown. But, for a complicated system, the form of (9) does not allow these two times to coincide – either in the semi-classical regime or in the quantum domain. Now, the important question is about the relative merit of these two times. Equation (8) indicates that the probabilistic time gives the expansion rate of the Universe independent of the amount of matter –something quite unnatural– where as the expansions rate in terms of the Bohmian time depends on the amount of matter in the Universe. On the other hand, while the Bohmian time reduces to the classical time in the semi-classical limit no matter what the degrees of freedom of system is, the probabilistic time coincides with the Bohmian one in the semi-classical limit only when the degrees of freedom of the system is one. So, only in this case, it reduces to the classical time. Therefore, the probabilistic time is not a suitable parameter for the description of a dynamical Universe.

6. Conclusion

If we do not accept the philosophy that time is a semi-classical concept, then both the Bohmian time and the probabilistic time are more suitable definitions for the time parameter than the semi-classical time. Here, we have shown that the probabilistic time has some problems that are not present for the Bohmian time. For example, the rate of the expansion of the Universe depends on the amount of matter present in it, if it is expressed in terms of the Bohmian time. But the same rate, is independent of amount of matter, if it is expressed in terms of the probabilistic time.

Furthermore, the Bohmian time reduces naturally to the semi-classical time, where as the probabilistic time has this property only for simple systems.

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