

LOW MOMENTUM EFFECTIVE THEORY FOR
FEW-NUCLEON SYSTEMS*

ULF-G. MEISSNER

Forschungszentrum Jülich, Institut für Kernphysik (Theorie)
D-52425 Jülich, Germany*(Received June 5, 1998)*

Starting from a precise two-nucleon potential, I show how to use the method of unitary transformations to construct an effective potential that involves only momenta less than a given maximal value. I describe this method for an S-wave potential of the Malfliet–Tjon type. It is demonstrated that the bound and scattering state spectrum calculated within the effective theory agrees exactly with the one based on the original potential. Such a truly low momentum effective theory might pave the way for a consistent construction of effective chiral few-nucleon forces and for a systematic treatment of relativistic effects in few-body systems.

PACS numbers: 12.39.Pn, 12.39.Fe

1. Introduction and motivation

Chiral perturbation theory for two and more nucleons became a subject of a great research interest in the past few years, see *e.g.* the pioneering work in [1,2]. One hopes to be able to clarify the structure of nuclear forces in this way. However, only the low-momentum matrix elements of nuclear forces may be systematically treated in this approach since it is based on a consistent power counting of small momenta and pion masses compared to the typical hadronic scale of $\Lambda_{\text{had}} \simeq 1$ GeV (for some recent work along these lines see *e.g.* Refs. [3,4]). A natural problem arises due to the appearance of shallow nuclear bound states indicating a breakdown of perturbation theory. Furthermore, solving the Lippmann–Schwinger equation for constructing the deuteron necessarily involves momenta $|\vec{p}| > \Lambda_{\text{had}}$. For such momenta, the chiral effective potential constructed according to the conventional power counting rules is no longer applicable. This is witnessed by the fact that in

* Presented at the NATO Advanced Research Workshop, Cracow, Poland, May 26–30, 1998.

the calculations for the two-nucleon system in [2], an additional *ad hoc* cut-off to tame the high-momentum components had to be introduced. To be more precise, this cut-off function is not commensurate with the underlying chiral power counting since it introduces an infinite string of local operators with increasing dimension. One might therefore question the validity or usefulness of such an approach altogether. For recent discussions of this subject see [5] and a different power counting scheme has been presented in [6]. A similar problem arises in standard few- and many-body calculations based on realistic nucleon–nucleon (NN) forces. The potentials, if derived from meson exchange diagrams, are generally based on a non-relativistic expansion in powers of momenta over the nucleon mass and are then used in various types of bound state equations. These usually involve integrations over a much larger range of momenta as used in the construction of the NN potentials. The same is of course also true for the various phenomenological NN forces, which are chosen more or less *ad hoc* (with the exception of the pion tail). This affects in a non-trivial way the calculation of observables, such as masses, levels or electromagnetic response functions. For a recent discussion, see *e.g.* Ref. [7]. In this general context the question of the existence and the properties of a low-momentum effective theory for nucleons are thus of great importance. I show here that it is indeed possible to construct an effective two-nucleon potential from a given realistic potential which involves only low momenta, *i.e.* momenta below a chosen momentum cut-off, but which gives exactly the same results for bound and scattering states [8]. The cut-off scale introduced in our approach should be considered a physical quantity since it defines the Hilbert space in which the theory operates. This is different from the cut-off in a form factor or vertex function. It is important to stress that our approach differs from the treatment of the Schrödinger equation in an effective field theory framework proposed by Lepage [9]. In his approach, an effective field theory for nucleons only is constructed for very low momenta and eventually pions are added. While that is certainly a valid framework, we intend to stay closer to the already existing nuclear physics knowledge in that our approach will eventually allow to match the low-momentum theory in a well-defined way to the highly successful meson-exchange pictures of the nucleon–nucleon force. The results presented here should therefore be considered as a first step in a bigger program. Finally, I remark that while it seems to be known that such an exact momentum space projection can be done, to my knowledge this program has never been carried out before.

2. Formalism

To be specific, consider a momentum space Hamiltonian for the two-nucleon system of the form

$$\mathcal{H}(\vec{p}, \vec{p}') = \mathcal{H}_0(\vec{p}) \delta(\vec{p} - \vec{p}') + \mathcal{V}(\vec{p}, \vec{p}'), \quad (1)$$

where \mathcal{H}_0 stands for the kinetic energy and the explicit form of the NN potential will be specified later. The aim is to decouple the low and high momentum components of this two-nucleon potential using the method of unitary transformation [10, 11]. For achieving that, one introduces the projection operators

$$\eta = \int d^3p |\vec{p}\rangle \langle \vec{p}|, \quad |\vec{p}| \leq \Lambda; \quad \lambda = \int d^3p |\vec{p}\rangle \langle \vec{p}|, \quad |\vec{p}| > \Lambda, \quad (2)$$

where Λ is a momentum cut-off whose value will be specified later and η (λ) is a projection operator onto low (high) momentum states with $\eta^2 = \eta$, $\lambda^2 = \lambda$, $\eta\lambda = \lambda\eta = 0$ and $\lambda + \eta = \mathbf{1}$. To be precise, the separation into low and high momentum components is to be understood in a limiting sense, we always consider $\lim_{\varepsilon \rightarrow 0} (\Lambda - \varepsilon)$. In this basis, the Schrödinger equation takes the form

$$\begin{pmatrix} \eta\mathcal{H}\eta & \eta\mathcal{H}\lambda \\ \lambda\mathcal{H}\eta & \lambda\mathcal{H}\lambda \end{pmatrix} \begin{pmatrix} \eta\Psi \\ \lambda\Psi \end{pmatrix} = E \begin{pmatrix} \eta\Psi \\ \lambda\Psi \end{pmatrix}. \quad (3)$$

We now perform a unitary transformation of the type

$$\mathcal{H} \rightarrow \mathcal{H}' = U^\dagger \mathcal{H} U, \quad (4)$$

so that $\eta\mathcal{H}'\lambda = \lambda\mathcal{H}'\eta = 0$. The corresponding unitary operator U is parametrized in terms of an operator A , following Okubo [10]:

$$U = \begin{pmatrix} (1 + A^\dagger A)^{-1/2} & -A^\dagger (1 + AA^\dagger)^{-1/2} \\ A(1 + A^\dagger A)^{-1/2} & (1 + AA^\dagger)^{-1/2} \end{pmatrix} \quad (5)$$

and A satisfies the condition $A = \lambda A \eta$. The requirement of decoupling the two spaces leads to the following nonlinear integral equation

$$\lambda (\mathcal{H} - [A, \mathcal{H}] - A\mathcal{H}A) \eta = 0 \quad (6)$$

for the operator A . In the context of the nuclear many-body theory one often introduces a mean field single particle basis, which defines a complete set of N -particle states. A low-energy subgroup of states form a model space and one is interested in effective interactions acting in that model space such that the same low energy spectrum results as for the underlying

N -body Hamiltonian. A way to arrive at that effective interaction is to decouple by a suitable transformation the two spaces (model space and the rest space), which leads to a decoupling equation of exactly the form Eq. (6). In that context it is often reformulated into a linear form on a two-body level using the exactly known interacting two-body states (some references are *e.g.* [12, 13]). This is indeed a feasible way to proceed also in our context. Here, this alternative way will not be discussed any further but rather we solve directly the nonlinear equation (6). If we denote by $\vec{q}(\vec{p})$ a momentum from the $\eta(\lambda)$ -space, Eq. (6) takes the form¹

$$\begin{aligned} \mathcal{V}(\vec{p}, \vec{q}) &= \int d^3 q' A(\vec{p}, \vec{q}') \mathcal{V}(\vec{q}', \vec{q}) + \int d^3 p' \mathcal{V}(\vec{p}, \vec{p}') A(\vec{p}', \vec{q}) \\ &- \int d^3 q' d^3 p' A(\vec{p}, \vec{q}') \mathcal{V}(\vec{q}', \vec{p}') A(\vec{p}', \vec{q}) = (E_{\vec{q}} - E_{\vec{p}}) A(\vec{p}, \vec{q}). \end{aligned} \quad (7)$$

The quantities $E_{\vec{q}, \vec{p}}$ are the kinetic energies related to the corresponding three-momenta. This equation can only be solved numerically. This is most easily done by iteration starting with

$$A = \frac{\mathcal{V}(\vec{p}, \vec{q})}{E_{\vec{q}} - E_{\vec{p}}}. \quad (8)$$

After four iterations, we then perform an average over the values of the operator A with different weight factors. This allows to speed up the convergence considerably (details on this procedure will be published elsewhere). We also provide a regularization scheme for the singularities of the operator A , which arise by solving this equation, *i.e.* at the cut-off momentum (as becomes obvious from Eq. (8)). To be specific, we redefine the original potential by multiplying it with some smooth functions $f(\vec{p})$ and $f(\vec{q})$ which are zero in some neighborhood of the point $|\vec{p}| = A$ and one elsewhere,

$$V(\vec{p}, \vec{q}) \rightarrow f(\vec{p}) V(\vec{p}, \vec{q}) f(\vec{q}), \quad (9)$$

symbolically shown in Fig. 1. The precise form of this procedure is of no interest for the following and will thus not be discussed in detail here. We only add that the regularization is chosen mild enough that it has no effect on the observables, as will be illustrated later on for a specific example. More formally, for a typical S-matrix element this equivalence amounts to

$$\langle \vec{p}_1 | S | \vec{p}_2 \rangle \longrightarrow \langle \vec{p}_1 | S(\alpha) | \vec{p}_2 \rangle \xrightarrow{\alpha \rightarrow 0} \langle \vec{p}_1 | S | \vec{p}_2 \rangle, \quad (10)$$

with α some typical range in momentum space, *cf.* Fig. 1.

¹ Our notation is such that $Q(\vec{p}, \vec{q})$ stands for the corresponding matrix element $\langle \vec{p} | Q | \vec{q} \rangle$ for any operator Q .

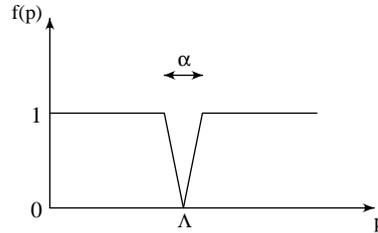


Fig. 1. Symbolic form of the function regularizing the singularity at $|\vec{p}| = |\vec{q}| = \Lambda$. In the real case, the edges are not sharp.

3. Results for a semi-realistic potential

We now restrict ourselves to the NN S-waves. To be specific, consider a momentum space Malfliet–Tjon [14] potential with an attractive and a repulsive part

$$\mathcal{V}_{\text{MT}}(\vec{q}_1, \vec{q}_2) = \frac{1}{2\pi^2} \left(\frac{V_R}{t + \mu_R^2} - \frac{V_A}{t + \mu_A^2} \right), \quad (11)$$

with $t = (\vec{q}_1 - \vec{q}_2)^2$. We choose the parameters as given in [15], $V_R = 7.29$, $V_A = 3.18$, $\mu_R = 614$ MeV and $\mu_A = 306$ MeV. From here on, we only consider the S-wave part of this potential which can be obtained analytically. Although this potential is quite simple, it captures essential features of the NN interaction, in particular, it supports exactly one bound state at $E = -2.23$ MeV. Since we are interested in an effective theory with small momenta only, we set the cut-off $\Lambda = 400$ MeV (or smaller). One now compares the original potential with the effective one. The latter is defined via

$$\mathcal{V}_{\text{eff}} = \mathcal{H}' - \mathcal{H}_0. \quad (12)$$

In the range of the small momenta, the potentials are very similar. However, one finds significant differences between the effective and the original potential when the cut-off, above which the nucleonic momenta are integrated out, is chosen very small, $\Lambda \leq 200$ MeV. This is shown in Fig. 2.

We now consider observables. Phase shifts can be derived from the S-matrix, or equivalently, from a K-matrix approach. Symbolically, the relation between the S- and the K-matrix can be expressed as

$$S = \frac{1 - i\pi qK}{1 + i\pi q\bar{K}}, \quad (13)$$

with q the on-shell relative momentum of the two nucleons corresponding to the laboratory energy being considered. We work in the framework of the

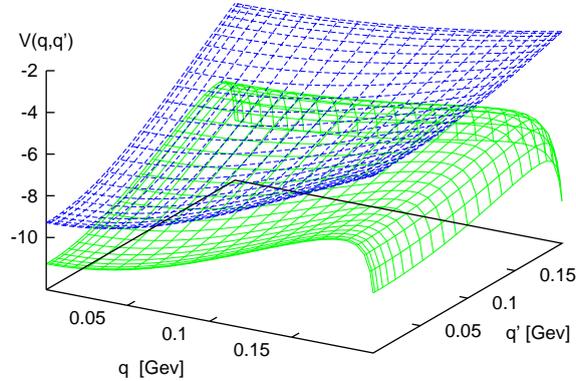


Fig. 2. Effective two-nucleon potential (green hatched area with solid lines) in comparison with the original potential, Eq. (11) (blue hatched area with dashed lines), for momenta less than 200 MeV.

latter because the K-matrix is purely real. This is, however, just a matter of convenience. The low-energy phase shifts and the bound-state energy are reproduced to a very high precision with the resulting effective potential acting only in the low momentum components. This is shown for the S-wave phase in Fig. 3 for two values of the cut-off $\Lambda = 200$ and 400 MeV, respectively.

The phase shifts from the original potential are exactly reproduced in the approach based on the effective potential, as long as one stays below the chosen cut-off. This is the reason why the solid and dashed lines in the figure fall on top of each other. The Lippmann–Schwinger equation in the effective approach involves by construction only momenta below the cut-off and thus the bound state can be calculated completely consistently. We find that the bound state energy is also exactly reproduced. Furthermore, the deuteron state evaluated in the effective theory is of course unitarily transformed. Matrix elements of an arbitrary operator \mathcal{O} remain unchanged under this unitary transformation,

$$\langle \Phi_D | \mathcal{O} | \Phi_D \rangle = \langle \Phi'_D | \mathcal{O}' | \Phi'_D \rangle, \quad (14)$$

with $\mathcal{O}' = U^\dagger \mathcal{O} U$, $|\Phi'_D\rangle = U^\dagger |\Phi_D\rangle$ and $\Phi_D(p) = \langle p | \Phi_D \rangle$ denotes the deuteron wave function in momentum space. For illustration, we compare in Fig. 4 the original and the unitarily transformed deuteron momentum space wave functions for $\Lambda = 200$ MeV. Note that due to the regularization, the “spike” close to the cut-off is in fact a smooth function and does not introduce any singular derivatives. For a larger cut-off value, say $\Lambda = 400$ MeV, the two curves fall onto of each other apart from a small interval in the vicinity of

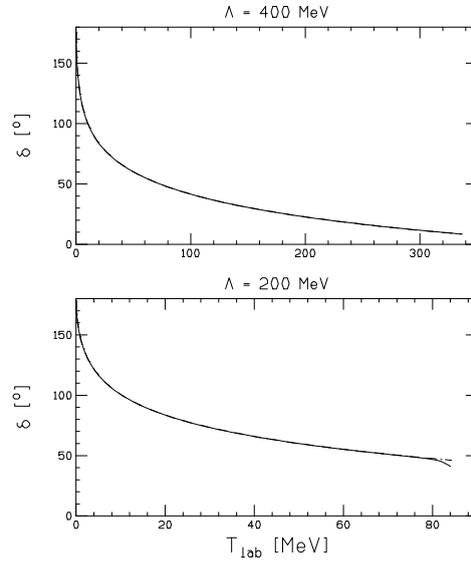


Fig. 3. Phase shifts from the effective potential (solid lines) and the original potential (dashed lines) as a function of the kinetic energy in the lab frame. Upper (lower) panel: $\Lambda = 400$ (200) MeV.

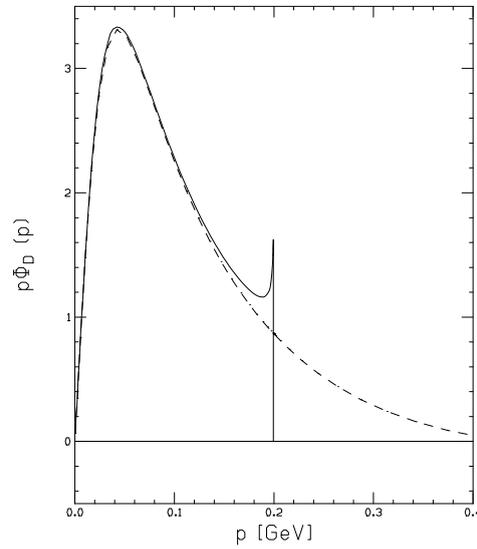


Fig. 4. Deuteron wave function $p\Phi_D(p)$ versus the momentum p from the effective potential (solid line) and the original potential (dashed line) for $\Lambda = 200$ MeV.

the cut-off. We thus do not show that case here. Note, however, that the unprojected deuteron wave function is still not negligible at momenta of about 800 MeV. As an illustration, we consider the expectation value of the modulus of the momentum operator in the S-wave deuteron, $\langle \Phi_D(p) | \tilde{p} | \Phi_D(p) \rangle$, with $\tilde{p} = |\vec{p}|$. In the effective low momentum theory, the momentum operator becomes non-local. To be specific, I give the modulus of the effective momentum operator (dropping all vector arrows),

$$p_{\text{eff}}(p_1, p_2) = \int dp' p'^2 U^\dagger(p_1, p') p' U(p', p_2). \quad (15)$$

This non-locality can, of course, be handled without problems. The results for the modulus of \vec{p} are listed in Table I for the two cut-off values $\Lambda = 200, 400$ MeV. Of course, for the full MT-potential we do not need the regularization. However, to illustrate its influence, we have also performed a calculation with a MT-potential subject to the same regularization as used for the effective potential (labeled “regularized” in the table). For the lower cut-off, the few permille deviation between the exact result and the one based on the regularized potential is simply due to the fact that we did not optimize the numerical solution of the integral equation Eq. (7) to determine the operator A . If needed, one can improve these numbers to agree to arbitrary precision (which is not of relevance here). Any other operator can be handled in a similar fashion, in particular also two-body operators like they arise when one considers meson exchange currents.

TABLE I

Expectation value of $|\vec{p}|$ in the S-wave deuteron based on the exact and the effective MT-potential. For the exact case, we show the results with and without regularization at the singularity $|\vec{p}| = \Lambda$.

	$\Lambda = 200$ MeV	$\Lambda = 400$ MeV
Exact potential, not regularized	80.23 MeV	80.2308 MeV
Exact potential, regularized	79.90 MeV	80.2303 MeV
Effective potential, regularized	79.90 MeV	80.2304 MeV

4. Summary and outlook

I have shown how to construct an effective two-nucleon potential from a realistic potential of the Malfliet–Tjon type which involves only low momenta, $|\vec{p}| < \Lambda$. It was shown that this effective theory exactly reproduces

the features of the original one below the cut-off Λ . For the case at hand, the effective potential only has a shape that is significantly different from the original one when the cut-off is chosen small enough, say below 300 MeV. As already stressed in the introduction, this approach is now applied in few-body physics and in the chiral perturbation theory description of the NN interaction.

I am grateful to Evgenie Epelbaum and Walter Glöckle for a pleasant collaboration and for allowing me to present the material before publication.

REFERENCES

- [1] S. Weinberg, *Phys. Lett.* **B251**, 288 (1990); *Nucl. Phys.* **B363**, 3 (1991).
- [2] C. Ordóñez, L. Ray, U. van Kolck, *Phys. Rev. Lett.* **72**, 1982 (1994); *Phys. Rev.* **C53**, 2086 (1996); U. van Kolck, *Phys. Rev.* **C49**, 2932 (1994).
- [3] D.B. Kaplan, M.J. Savage, M.B. Wise, *Nucl. Phys.* **B478**, 629 (1996).
- [4] E. Epelbaum, W. Glöckle, Ulf-G. Meissner, nucl-th/9801064, *Nucl. Phys.* **A636** (1998), in print.
- [5] S.R. Beane, T.D. Cohen, D.R. Phillips, *Nucl. Phys.* **A632**, 445 (1998).
- [6] D.B. Kaplan, M.J. Savage, M.B. Wise, nucl-th/9801034.
- [7] H. Kamada, W. Glöckle, *Phys. Rev. Lett.* **80**, 2527 (1998).
- [8] E. Epelbaum, W. Glöckle, Ulf-G. Meissner, nucl-th/9804005 and work in preparation; E. Epelbaum, invited talk at Few-Body XVI, Autrans, France, June 1998.
- [9] P.G. Lepage, hep-ph/9706029.
- [10] S. Okubo, *Prog. Theor. Phys.* **12**, 603 (1954).
- [11] N. Fukuda, K. Sawada, M. Taketani, *Prog. Theor. Phys.* **12**, 156 (1954).
- [12] T.T.S. Kuo, *Lecture Notes in Physics*, Vol. 144, p. 248 (Springer Verlag, 1981).
- [13] K. Suzuki, R. Okamoto, *Prog. Theor. Phys.* **70**, 439 (1983).
- [14] R.A. Malfliet, J.A. Tjon, *Nucl. Phys.* **A127**, 161 (1969).
- [15] Ch. Elster, J.H. Thomas, W. Glöckle, nucl-th/9708017.