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ISOSPIN ANALYSIS OF HADRONIC DIFFRACTIVE PRODUCTION

By Z. Koba

The Niels Bohr Institute, University of Copenhagen*

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Two important aspects of the isospin analysis of hadronic diffractive production processes are discussed. They are 1) cross-channel isospin representation, and 2) isospin analysis of "integrated cross-sections".

Examples are taken from single pion production in πN and NN collisions, and graphical illustration of the results are presented.

1. Introduction

Isospin symmetry is one of the best established properties of strong interactions of particles, and its usefulness in analyzing low-energy hadron interactions is very well known. Its application to exclusive experiments of high energy hadronic production processes is, however, rather limited in practice[1]. The reason is simple: for describing complicated processes one needs many isospin amplitudes, while the number of observable channels (of different charge combinations) remains small since in the usual hydrogen bubble chamber experiments it is difficult to identify final states involving two or more stable or semi-stable neutral particles[2].

Nevertheless, isospin analysis can be still utilized, as we shall see, in simpler cases of production processes and particularly for diffractive production, where the number of final particles is fairly small. In this lecture we should like to discuss two important points we have to face in practice when we make such an analysis. They are 1) cross-channel isospin representation and 2) isospin analysis of "integrated cross-sections". By the word "in-

* Address: The Niels Bohr Institute, University of Copenhagen, Copenhagen Ø, Blegdamsvej 17, Denmark.

[1] Application of isospin symmetry to the high energy inclusive cross-sections is more straightforward owing to the generalized optical theorem. See for instance, Chan H. M. *et al.*, *Phys. Rev. Letters*, **26**, 672 (1971); R. N. Cahn, M. B. Einhorn, *Phys. Rev.*, **D4**, 3337 (1971).

[2] One neutral particle can be identified by energy-momentum balance.

tegrated cross-sections" we mean cross-sections which are summed and/or averaged over helicities and integrated over a certain region of phase space.

We shall illustrate these points taking specific examples of single pion production in pion-nucleon and nucleon-nucleon collisions, because in these cases experimental data are already available. But the arguments are, of course, of more general validity. (General formulas can be found in Ref. [9].)

2. *Cross-channel isospin analysis*

One of the main interests in analyzing production processes is to find out the nature of exchange mechanism. In the diffractive production, in particular, we should like to verify that the "exchanged object" carries the vacuum quantum numbers. Thus the isospin analysis should be designed so as to meet this requirement.

2.1. A preliminary example: single pion production in π^+p collision

Analysis of single pion production in π^+p collision at 8 and 16 GeV/c has been carried out by Aachen–Berlin–Bonn–CERN–Cracow collaboration [1].

In the final state there are one nucleon and two pions, and the latter can be distinguished as the fast pion π_f and the slow pion π_s by the longitudinal momenta, $p_{\|,f} > p_{\|,s}$, the direction of incident pion being taken to be positive. Thus one may regard the reaction primarily due to the peripheral production schematized in Fig. 2.1.

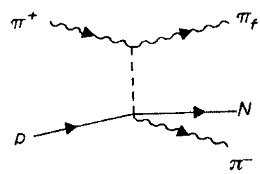

Fig. 2.1

Then we have three different charge combinations.

$$\pi^+p \to \pi_f^+(p\pi_s^0) \quad (2.1a)$$

$$\to \pi_f^+(n\pi_s^+) \quad (2.1b)$$

$$\to \pi_f^0(p\pi_s^+) \quad (2.1c)$$

The $(N\pi_s)$ system in the final state can have isospin $I = 1/2$ or $I = 3/2$. (In general it is a linear combination of $I = 1/2$ and $I = 3/2$.) Since the slow pion is more or less accompanying the nucleon, it is reasonable to specify two production amplitudes, $M_{1/2}$ and $M_{3/2}$ as those which give rise $I = 1/2$ and $I = 3/2$ states of $(N\pi_s)$ system, respectively[3]. Then

[3] Of course, $M_{1/2}$ and $M_{3/2}$ are functions of initial and final momenta and helicities.

it is straightforward to find the relevant Clebsch-Gordan coefficients and obtain the following amplitudes[4].

$$T_1 \equiv T(\pi^+ p \to \pi_f^+(p\pi_s^0)) = $$
$$= -\frac{1}{\sqrt{3}} M_{\frac{1}{2}} - \frac{2}{\sqrt{15}} M_{\frac{3}{2}} \tag{2.2a}$$
$$T_2 \equiv T(\pi^+ p \to \pi_f^+(n\pi_s^+)) = $$
$$= \frac{\sqrt{2}}{\sqrt{3}} M_{\frac{1}{2}} - \frac{\sqrt{2}}{\sqrt{15}} M_{\frac{3}{2}} \tag{2.2b}$$
$$T_3 \equiv T(\pi^+ p \to \pi_f^0(p\pi_s^+)) = $$
$$= \frac{\sqrt{3}}{\sqrt{5}} M_{\frac{3}{2}}. \tag{2.2c}$$

Here the normalization is to certain extent arbitrary, and we have chosen it so that

$$\sum_{j=1}^{3} |T_j|^2 = |M_{\frac{1}{2}}|^2 + |M_{\frac{3}{2}}|^2. \tag{2.3}$$

In practice one measures cross-sections averaged and summed over spin states and integrated over a certain region of phase space[5]. Thus one gets "integrated cross-sections"

$$\sigma_1 \equiv \sigma(\pi^+ p \to \pi_f^+(p\pi_s^0)) = $$
$$= \tfrac{1}{3} \int d\tau |M_{\frac{1}{2}}|^2 + \tfrac{4}{15} \int d\tau |M_{\frac{3}{2}}|^2 + $$
$$+ \frac{4}{3\sqrt{5}} \int d\tau \,\mathrm{Re}\, (M_{\frac{1}{2}}^* M_{\frac{3}{2}}) \tag{2.4a}$$
$$\sigma_2 \equiv \sigma(\pi^+ p \to \pi_f^+(n\pi_s^+)) = $$
$$= \tfrac{2}{3} \int d\tau |M_{\frac{1}{2}}|^2 + \tfrac{2}{15} \int d\tau |M_{\frac{3}{2}}|^2 - $$
$$- \frac{4}{3\sqrt{5}} \int d\tau \,\mathrm{Re}\, (M_{\frac{1}{2}}^* M_{\frac{3}{2}}) \tag{2.4b}$$
$$\sigma_3 \equiv \sigma(\pi^+ p \to \pi_f^0(p\pi_s^+)) = \tfrac{3}{5} \int d\tau |M_{\frac{3}{2}}|^2 \tag{2.4c}$$

where the symbol $\int d\tau$ represents sum and average over spin states and integration over phase space.

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[4] These relations are always valid as one way of specifying isospin amplitudes, but the amplitudes $M_{1/2}$ and $M_{3/2}$ get direct physical significance only when the nucleon and the slow pion in the final state form a physical (*i.e.*, not just a kinematical) subsystem as in Fig. 2.1.

[5] The distinction between the fast and slow pions must be maintained throughout integration procedure.

At 16 GeV/c, for instance, the ABBCC collaboration [1] has got

$$\sigma_1 = 0.30 \pm 0.03 \text{ mb}$$
$$\sigma_2 = 0.31 \pm 0.03 \text{ mb} \tag{2.5}$$
$$\sigma_3 = 0.11 \pm 0.02 \text{ mb}$$

from which follow, solving (2.4)

$$\int d\tau |M_{\frac{1}{2}}|^2 = 0.54 \pm 0.05 \text{ mb}$$
$$\int d\tau |M_{\frac{3}{2}}|^2 = 0.18 \pm 0.03 \text{ mb}$$
$$\int d\tau \,\mathrm{Re}\,(M^*_{\frac{1}{2}} M_{\frac{3}{2}}) = 0.12 \pm 0.04 \text{ mb}. \tag{2.6}$$

What can we learn from these results concerning the exchange mechanism? If the reaction is taking place only through $I' = 0$ exchange[6], there will be no $I = 3/2$ state for the $(N\pi_s)$ system, and the ratio $\sigma_1 : \sigma_2 : \sigma_3$ will be $1:2:0$, as pointed out by Satz [2]. But the inverse is not always true. Even if we had verified that $\sigma_1 : \sigma_2 : \sigma_3 = 1:2:0$ and consequently $\int |M_{3/2}|^2 d\tau = \int \mathrm{Re}\,(M^*_{1/2} M_{3/2})\, d\tau = 0$, still we could not conclude that the non-vanishing $M_{1/2}$ is due to the $I' = 0$ exchange only, since $I' = 1$ exchange can as well lead to formation of $I = \frac{1}{2}$ state of $(N\pi_s)$ system. (Thus one has to make recourse to other considerations such as energy-dependence of these cross-sections in order to argue that $\int |M_{1/2}|^2 d\tau$ is mainly due to $I' = 0$ exchange [1].)

2.2. Combined analysis of π^+p and π^-p collisions

To proceed further in isospin analysis and pin down the contribution of $I' = 0$ exchange mechanism, we need obviously some more information. This is supplied by studying the reaction

$$\pi^- p \to \pi_f(N\pi_s) \tag{2.7}$$

at the same energy and at the same region of phase space. There are four observable channels of different charge combination, namely

$$\pi^- p \to \pi_f^-(p\pi_s^0) \tag{2.7a}$$
$$\to \pi_f^-(n\pi_s^+) \tag{2.7b}$$
$$\to \pi_f^0(p\pi_s^-) \tag{2.7c}$$
$$\to \pi_f^+(n\pi_s^-) \tag{2.7d}$$

[6] Hereafter we denote the isospin of the exchanged object by I' to make clear distinction from that of $(N\pi_s)$ system.

and one channel which is not detectable in the conventional hydrogen bubble chamber experiment

$$\pi^- p \rightarrow \pi_f^0(n\pi_s^0). \tag{2.7e}$$

Out of these, the reaction (2.4d) is a double charge exchange process and they are found to be very rare, so that we are justified to neglect contributions from the $I' = 2$ exchange mechanism. Then the remaining possibilities are $I' = 0$ and $I' = 1$ exchange, as in the case of π^+p collision.

Let us now compare, for example, (2.1a) and (2.7a). As is seen in Fig. 2.2, the only difference lies in the upper vertex, where the incident and the fast pion is

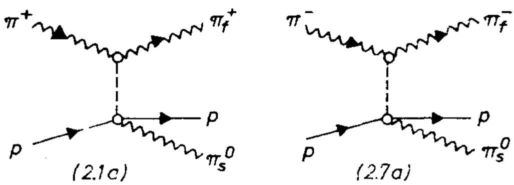

Fig. 2.2

either π^+ or π^-. If an "object"[7] of $I' = 0$ is being exchanged the upper vertices in the two cases will be exactly the same, but if it is an "object" of $I' = 1$ that is exchanged—one can imagine, if one likes, a ϱ^0-meson in the cross-channel — the couplings in the upper vertices have the same magnitude but the opposite sign because of the isovector property. Therefore the amplitudes for (2.1a) and (2.7a) should have the following form

$$T(\pi^+ p \rightarrow \pi_f^+(p\pi_s^0)) = aM^{I'=0} + bM^{I'=1}, \tag{2.8}$$

$$T(\pi^- p \rightarrow \pi_f^-(p\pi_s^0)) = aM^{I'=0} - bM^{I'=1} \tag{2.9}$$

where $M^{I'}$ denotes the amplitude due to I' exchange. Thus we shall be able to separate contributions from $I' = 0$ and $I' = 1$ exchange mechanism by comparing $\pi^{\pm}p$ collisions.

Once we have recognized this possibility, we can proceed more systematically [3]. First of all we may examine how many independent isospin amplitudes exist in the reaction

$$\pi N \rightarrow \pi N \pi. \tag{2.10}$$

As is well known, the initial state has $I = 1/2$, $I = 3/2$, while the final state has two kinds of $I = 1/2$, two kinds of $I = 3/2$, and one $I = 5/2$, since

$$\mathscr{D}_{1/2} \otimes \mathscr{D}_1 \otimes \mathscr{D}_1 = (\mathscr{D}_{1/2} \oplus \mathscr{D}_{3/2}) \otimes \mathscr{D}_1 = (\mathscr{D}_{1/2} \oplus \mathscr{D}_{3/2}) \oplus (\mathscr{D}_{1/2} \oplus \mathscr{D}_{3/2} \oplus \mathscr{D}_{5/2}).$$

Consequently we have 4 independent isospin amplitudes.

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[7] Here and in the following such an "object" need not be a definite particle or Regge pole.

As a convenient choice[8] for our problem we can specify these isospin amplitudes by the isospin I of the $(N\pi_s)$ system and the isospin I' of the exchanged "object" [4]. Then the four amplitudes are labelled as follows

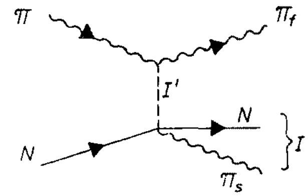

Fig. 2.3

$$M_{I=\frac{1}{2}}^{I'=0},\ M_{I=\frac{1}{2}}^{I'=1},\ M_{I=\frac{3}{2}}^{I'=1},\ M_{I=\frac{3}{2}}^{I'=2}$$

As we have mentioned before, the last one ($I' = 2$) can be neglected since in our problem its contribution is very small.

Now we see that we can measure altogether six-nonvanishing cross-sections (spin--averaged and integrated over a region of phase space) for (2.1a), (2.1b), (2.1c), (2.7a), (2.7b), (2.7c) and they can be expressed as linear functions of six quantities

$$\int |M_{I=\frac{1}{2}}^{I'=0}|^2 d\tau,\ \int |M_{I=\frac{1}{2}}^{I'=1}|^2 d\tau,\ \int |M_{I=\frac{3}{2}}^{I'=1}|^2 d\tau,$$

$$\int \mathrm{Re}\,(M_{I=\frac{1}{2}}^{I'=0} \cdot M_{I=\frac{1}{2}}^{I'=1*}) d\tau,\ \int \mathrm{Re}\,(M_{I=\frac{1}{2}}^{I'=0} \cdot M_{I=\frac{3}{2}}^{I'=1*}) d\tau$$

$$\int \mathrm{Re}\,(M_{I=\frac{1}{2}}^{I'=1} \cdot M_{I=\frac{3}{2}}^{I'=1*}) d\tau. \tag{2.11}$$

To be more explicit, we get

$$\sigma_1 \equiv \sigma(\pi^+ p \to \pi_f^+(p\pi_s^0)) = \frac{1}{9}\int \left|\sqrt{3}\, M_{\frac{1}{2}}^0 - \frac{1}{\sqrt{2}}\, M_{\frac{1}{2}}^1 - \sqrt{2}\, M_{\frac{3}{2}}^1\right|^2 d\tau, \tag{2.12a}$$

$$\sigma_2 \equiv \sigma(\pi^+ p \to \pi_f^+(n\pi_s^+)) = \tfrac{1}{9}\int |\sqrt{6}\, M_{\frac{1}{2}}^0 - M_{\frac{1}{2}}^1 + M_{\frac{3}{2}}^1|^2 d\tau, \tag{2.12b}$$

$$\sigma_3 \equiv \sigma(\pi^+ p \to \pi_f^0(p\pi_s^+)) = \tfrac{1}{2}\int |M_{\frac{3}{2}}^1|^2 d\tau, \tag{2.12c}$$

$$\sigma_4 \equiv \sigma(\pi^- p \to \pi_f^-(p\pi_s^0)) = \frac{1}{9}\int \left|\sqrt{3}\, M_{\frac{1}{2}}^0 + \frac{1}{\sqrt{2}}\, M_{\frac{1}{2}}^1 + \sqrt{2}\, M_{\frac{3}{2}}^1\right|^2 d\tau, \tag{2.13a}$$

$$\sigma_5 \equiv \sigma(\pi^- p \to \pi_f^-(n\pi_s^+)) = \tfrac{1}{9}\int |\sqrt{6}\, M_{\frac{1}{2}}^0 + M_{\frac{1}{2}}^1 - M_{\frac{3}{2}}^1|^2 d\tau, \tag{2.13b}$$

$$\sigma_6 \equiv \sigma(\pi^- p \to \pi_f^0(p\pi_s^-)) = \frac{1}{9}\int \left|\sqrt{2}\, M_{\frac{1}{2}}^1 + \frac{1}{\sqrt{2}}\, M_{\frac{3}{2}}^1\right|^2 d\tau. \tag{2.13c}$$

Further, the undetected cross-section for the channel (2.7e) is also expressed as

$$\sigma_7 \equiv \sigma(\pi^- p \to \pi_f^0(n\pi_s^0)) = \tfrac{1}{9}\int |M_{\frac{1}{2}}^1 - M_{\frac{3}{2}}^1|^2 d\tau. \tag{2.13d}$$

[8] There are of course other ways of specifying isospin amplitudes. All of them are related linearly to each other. A more conventional way is the direct channel representation, see Ref. [3]. A double peripheral representation is given in Ref. [8–9]. See also Section 4.

The numerical coefficients in (2.12), (2.13) are appropriate products of Clebsch-Gordan coefficients and can be obtained by the standard rules of composition and decomposition of angular momenta. The normalization has been chosen such that

$$\sum_{j=1}^{7} \sigma_j = \sum_{I,I'} \frac{2I+1}{2I'+1} \int |M_I^{I'}|^2 d\tau, \tag{2.14}$$

in distinction from the previous one, (2.3).

The ABBC-collaboration [5] has obtained the following values from the data of $\pi^{\pm} p \to \pi N \pi$ at 16 GeV/c.

$$\int |M^0_{\frac{1}{2}}|^2 d\tau = 448 \pm 29 \ \mu\text{b}$$
$$\int |M^1_{\frac{1}{2}}|^2 d\tau = 50 \pm 59 \ \mu\text{b}$$ [9]
$$\int |M^1_{\frac{3}{2}}|^2 d\tau = 108 \pm 14 \ \mu\text{b}$$
$$\int \text{Re}\,(M^0_{\frac{1}{2}} M^{1*}_{\frac{1}{2}}) d\tau = 70 \pm 33 \ \mu\text{b}$$
$$\int \text{Re}\,(M^0_{\frac{1}{2}} M^{1*}_{\frac{3}{2}}) d\tau = 8 \pm 23 \ \mu\text{b}$$
$$\int \text{Re}\,(M^1_{\frac{1}{2}} M^{1*}_{\frac{3}{2}}) d\tau = 66 \pm 56 \ \mu\text{b}. \tag{2.15}$$ [9]

Thus our expectation is borne out: The $I' = 0$ exchange amplitude dominates in the single pion production — *i. e.* the latter has mainly a diffractive character.

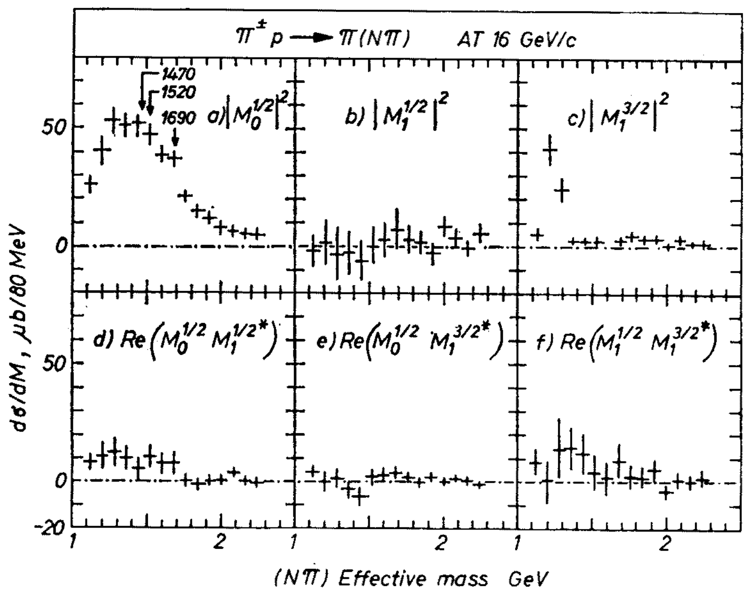

Fig. 2.4

[9] These numbers have been slightly changed, well within the errors, from the original text. See a comment on this point in Section 4.

A more detailed plot of these quantities as functions of $(N\pi_s)$ effective mass has been given by these authors. See Fig. 2.4. An interesting feature is the presence of a broad bump around 1300 MeV in the $I' = 0$ exchange contribution.

The same type of experiment is in progress by the same experimental group [6] on the $\pi^{\pm}p \rightarrow \pi N\pi$ process at 8 GeV/c. Then we shall be able to see clearly how the contributions from different isospin exchange mechanism depend on energy.

3. *Isospin analysis of "integrated cross-sections"*

3.1. An example: $NN \rightarrow NN\pi$. Isospin structure underdetermined

Let us now discuss pion production in NN collision[10]

$$N_1 N_2 \rightarrow N_1'(\pi N_2'). \tag{3.1}$$

The argument goes very much in the same way, but a new type of problem appears in this case, which will be studied in this section.

Firstly, from isospin analysis of single pion production in p--p collision alone one can conclude that $(\pi N_2')$ system is produced predominantly in $I = \frac{1}{2}$ state [7], but one cannot single out the contribution of $I' = 0$ exchange from data at a definite energy. (The situation is analogous to the case of $\pi^{\pm}p$ collision mentioned in 2.1.) Thus one has to compare pion production from $p-p$ and $p-n$ collisions at the same energy and at the same region of phase space in order to carry out a more detailed isospin analysis and clarify the nature of the exchange mechanism.

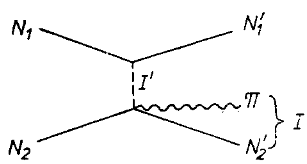

Fig. 3.1

In contrast to the case of πN collisions, there can be no $I' = 2$ exchange processes (*cf.* Fig. 3.1), so that we need not assume its absence. With the similar argument as in Section 2, we can easily verify that for the reaction (3.1) three isospin amplitudes are necessary and sufficient. For our purpose it is convenient to choose $M_{\frac{1}{2}}^0$, $M_{\frac{1}{2}}^1$, $M_{\frac{3}{2}}^1$ just in the same way as in πp collisions. On the other hand, the number of channels with identifiable final states is only 5, *i. e.*

$$p_1 p_2 \rightarrow p_1'(\pi^0 p_2') \tag{3.1a}$$

$$p_1 p_2 \rightarrow p_1'(\pi^+ n_2') \tag{3.1b}$$

$$p_1 p_2 \rightarrow n_1(\pi^+ p_2') \tag{3.1c}$$

[10] The pion in the final state can be associated with either of the nucleons. We denote the nucleon accompanying the pion by N'_2.

$$p_1 n_2 \to p_1'(\pi^- p_2') \tag{3.1d}$$

$$n_1 p_2 \to p_1'(\pi^- p_2'). \tag{3.1e}$$

Suppose, for a moment, that we could make a precisely differential measurement of cross-section for an infinitesimal region of phase space and for a definite set of helicity states. Then we should get relations of the following form

$$\frac{d\sigma_j}{d\vec{k}} = |a_j M_{I=\frac{1}{2}}^{I'=0} + b_j M_{I=\frac{1}{2}}^{I'=1} + c_j M_{I=\frac{3}{2}}^{I'=1}|^2. \tag{3.2}$$

$$j = 1, 2, 3, 4, 5$$

where a_j, b_j, c_j are known real coefficients. (See Table 3.1 below.) Thus we should have to determine 3 complex numbers ($M_{\frac{1}{2}}^0$, $M_{\frac{1}{2}}^1$ and $M_{\frac{3}{2}}^1$) in terms of 5 observed differential cross-sections. Since the overall phase has no significance, we should have in fact only 5 unknown real variables (3 moduli and 2 relative phases) and definite solutions would be obtained[11]. (In general there would be two sets of solutions.)

In reality, however, it is very difficult to make such differential measurements. Usually one has to average and sum with respect to helicity states and integrate over a region of phase space, and by this summation and integration the phase relations (which are non--linear ones) are spoiled. Consequently, we need 6 real integrated quantities listed in (2.11) in order to describe the 5 integrated cross-sections. Thus our problem is underdetermined in the case of $N-N$ collisions.

3.2. Inequalities as remains of phase relations

We shall show in the following that the situation is not so bad as it looks at first sight. The constraints due to phase relations (which are non-linear in differential cross-sections) are invalidated by integration but they are not completely lost; they remain, so to speak, in weaker forms of inequalities, as we shall see, and the latter turn out often to be quite effective in restricting values of various isospin contributions [8, 9].

For the purpose of finding out such inequalities, it is convenient to replace the phase space integration (and helicity summation) by a summation over a large number, W say, of points in the phase space. Then we can write, for any isospin amplitudes M an M',

$$\int |M|^2 d\tau = \sum_{j=1}^{W} \Delta\tau_j[\{\mathrm{Re}\, M(j)\}^2 + \{\mathrm{Im}\, M(j)\}^2] \tag{3.3}$$

$$\int \mathrm{Re}\,(M^* M') d\tau = \sum_{j=1}^{W} \Delta\tau_j[\mathrm{Re}\, M(j) \cdot \mathrm{Re}\, M'(j) + \mathrm{Im}\, M(j) \cdot \mathrm{Im}\, M'(j)] \tag{3.4}$$

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[11] In the case of πN collisions, the non-realistic assumption of precisely differential measurements of 6 cross-sections would lead, to an overdetermination of 3 moduli and 2 relative phases, thus affording a check of isospin invariance.

where $M(j)$, $M'(j)$ are the values of M, M' at the point j and $\Delta\tau_j$ is the phase space element at that point.

It is thus natural to introduce three $2W$-dimensional real vectors $\boldsymbol{M}^0_{\frac{1}{2}}$, $\boldsymbol{M}^1_{\frac{1}{2}}$ and $\boldsymbol{M}^1_{\frac{3}{2}}$ with the following components,

$$\boldsymbol{M}^{I'}_I = (\sqrt{\Delta\tau_1}\ \mathrm{Re}\ M^{I'}_I(1),\ \sqrt{\Delta\tau_1}\ \mathrm{Im}\ M^{I'}_I(1),\ \sqrt{\Delta\tau_2}\ \mathrm{Re}\ M^{I'}_I(2),$$
$$\sqrt{\Delta\tau_2}\ \mathrm{Im}\ M^{I'}_I(2),\ \ldots,$$
$$\sqrt{\Delta\tau_W}\ \mathrm{Re}\ M^{I'}_I(W),\ \sqrt{\Delta\tau_W}\ \mathrm{Im}\ M^{I'}_I(W)), \tag{3.5}$$

etc., because we then get

$$\int |M^0_{\frac{1}{2}}|^2 d\tau = (\boldsymbol{M}^0_{\frac{1}{2}})^2,\ etc., \tag{3.6}$$

and

$$\int \mathrm{Re}\,(M^0_{\frac{1}{2}} M^{1*}_{\frac{1}{2}}) d\tau = (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{1}{2}}),\ etc. \tag{3.7}$$

Notice that, although the vectors $\boldsymbol{M}^{I'}_I$ are $2W$-dimensional, we have in our problem to do with only three vectors so that we can limit ourselves to the subspace spanned by them. The dimension of this subspace is, of course, at most three.

It is now straightforward to apply the well worked out mathematical scheme of linear vector space [10]. A set of Gram determinants constructed from these 6 scalar products must be non-negative. That is to say

$$(\boldsymbol{M}^0_{\frac{1}{2}})^2 \geqslant 0, \tag{3.8a}$$

$$\begin{vmatrix} (\boldsymbol{M}^0_{\frac{1}{2}})^2 & (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{1}{2}}) \\ (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{1}{2}}) & (\boldsymbol{M}^1_{\frac{1}{2}})^2 \end{vmatrix} \geqslant 0, \tag{3.8b}$$

$$\begin{vmatrix} (\boldsymbol{M}^0_{\frac{1}{2}})^2 & (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{1}{2}}) & (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{3}{2}}) \\ (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{1}{2}}) & (\boldsymbol{M}^1_{\frac{1}{2}})^2 & (\boldsymbol{M}^1_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{3}{2}}) \\ (\boldsymbol{M}^0_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{3}{2}}) & (\boldsymbol{M}^1_{\frac{1}{2}} \cdot \boldsymbol{M}^1_{\frac{3}{2}}) & (\boldsymbol{M}^1_{\frac{3}{2}})^2 \end{vmatrix} \geqslant 0 \tag{3.8c}$$

The first one is trivial; the second and the third relations are the well-known Schwarz inequality and its generalization. It can be proved [10] that the set (3.8) is necessary and sufficient[12] and includes no redundancy. (Other sets of inequalities obtained by exchanging the roles of $M^{I'}_I$ are equivalent to this set.)

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[12] That is to say, if a given set of 6 numbers satisfy the inequalities (3.8), one can construct three vectors in such a way that their 6 scalar products are equal to the given 6 numbers, respectively.

Geometrical meaning of these inequalities is clear: The length of the segment $\boldsymbol{M}^0_{\frac{1}{2}}$, the area of the triangle formed by $\boldsymbol{M}^0_{\frac{1}{2}}$ and $\boldsymbol{M}^1_{\frac{1}{2}}$, the volume of the tetrahedron spanned by $\boldsymbol{M}^0_{\frac{1}{2}}$, $\boldsymbol{M}^1_{\frac{1}{2}}$ and $\boldsymbol{M}^1_{\frac{3}{2}}$ must be real. It can be also verified that (3.8c) is reduced to the phase relation of $M^0_{\frac{1}{2}}$, $M^1_{\frac{1}{2}}$ and $M^1_{\frac{3}{2}}$ when the equality sign holds.

3.3. Analysis of experimental data

We are now ready to analyse the experimental data given in Table 3.2. At 7.0 GeV/c Yekutieli *et al.* [11] have obtained 5 cross-sections $\sigma_1, \sigma_2, \ldots, \sigma_5$. If, for instance, we use σ_6 (which is not observed) in addition to them, we can solve the linear equation given

TABLE 3.1

Expressions for cross-sections $NN \to N\pi N$

Observable channels

$$\sigma_1 \equiv \sigma(p_1p_2 \to p_1'(\pi^0 p_2')) = \int d\tau \left| \frac{1}{\sqrt{3}} M^0_{1/2} - \frac{1}{3\sqrt{3}} M^1_{1/2} - \frac{2}{3\sqrt{3}} M^1_{3/2} \right|^2$$

$$\sigma_2 \equiv \sigma(p_1p_2 \to p_1'(\pi^+ n_2')) = \int d\tau \left| \frac{\sqrt{2}}{\sqrt{3}} M^0_{1/2} - \frac{\sqrt{2}}{3\sqrt{3}} M^1_{1/2} + \frac{\sqrt{2}}{3\sqrt{3}} M^1_{3/2} \right|^2$$

$$\sigma_3 \equiv \sigma(p_1p_2 \to n_1'(\pi^+ p_{0}',) = \int d\tau \left| \frac{\sqrt{2}}{\sqrt{3}} M^1_{3/2} \right|^2$$

$$\sigma_4 \equiv \sigma(p_1n_2 \to p_1'(\pi^- p_2')) = \int d\tau \left| \frac{\sqrt{2}}{\sqrt{3}} M^0_{1/2} + \frac{\sqrt{2}}{3\sqrt{3}} M^0_{1/2} - \frac{\sqrt{2}}{3\sqrt{3}} M^1_{3/2} \right|^2$$

$$\sigma_5 \equiv \sigma(n_1p_2 \to p_1'(\pi^- p_2')) = \int d\tau \left| \frac{2\sqrt{2}}{3\sqrt{3}} M^1_{1/2} + \frac{\sqrt{2}}{3\sqrt{3}} M^1_{3/2} \right|^2$$

Unobservable channels

$$\sigma_6 \equiv \sigma(p_1n_2 \to p_1'(\pi^0 n_1')) = \int d\tau \left| \frac{1}{\sqrt{3}} M^0_{1/2} + \frac{1}{3\sqrt{3}} M^1_{1/2} + \frac{2}{3\sqrt{3}} M^1_{3/2} \right|^2$$

$$\sigma_7 \equiv \sigma(n_1p_2 \to p_1'(\pi^0 n_2')) = \int d\tau \left| \frac{2}{3\sqrt{3}} M^1_{1/2} - \frac{2}{3\sqrt{3}} M^1_{3/2} \right|^2$$

TABLE 3.2

Experimental data: $NN \to N\pi N$

	7.0 GeV/c [11]	28.5 GeV/c [12–13]
σ_1	0.95 ± 0.16 mb	—[13]
σ_2	1.35 ± 0.30 mb	0.70 ± 0.05 mb
σ_3	1.20 ± 0.25 mb	0.05 ± 0.01 mb
σ_4	0.78 ± 0.08 mb	0.40 ± 0.02 mb
σ_5	0.33 ± 0.03 mb	0.05 ± 0.01 mb

[13] No experimental data available.

in Table 3.1 and express the six scalar products $(M^0_{\frac{1}{2}})^2$, ..., $(M^0_{\frac{1}{2}} \cdot M^1_{\frac{1}{2}})$. .... as linear combinations of σ_1, ..., σ_5 and σ_6. Since σ_1, ..., σ_5 are known, they are in fact functions of σ_6 only. Then we insert them into (3.8) and get inequalities for σ_6. In this way upper and lower bounds of σ_6 are determined, and consequently the upper and lower bounds of $(M^0_{\frac{1}{2}})^2$, ... $(M^1_{\frac{1}{2}} \cdot M^1_{\frac{3}{2}})$ are also found.

The authors of Ref. [11] have used $(M^0_{\frac{1}{2}} \cdot M^1_{\frac{1}{2}})$ as the parameter, and expressed the remaining five scalar products as functions of σ_1 ... σ_5 and $(M^0_{\frac{1}{2}} \cdot M^1_{\frac{1}{2}})$. Then inserting them into (3.8), the upper and lower bounds for $(M^0_{\frac{1}{2}} \cdot M^1_{\frac{1}{2}})$ is obtained, and consequently, the bounds of other scalar products are also determined. Of course this procedure is completely equivalent to the above mentioned one. The results are shown in Figs 3.2 and 3.3.

Further, at 28.5 GeV/c there are data [12–13] for σ_2, σ_3, σ_4 and σ_5. One can still

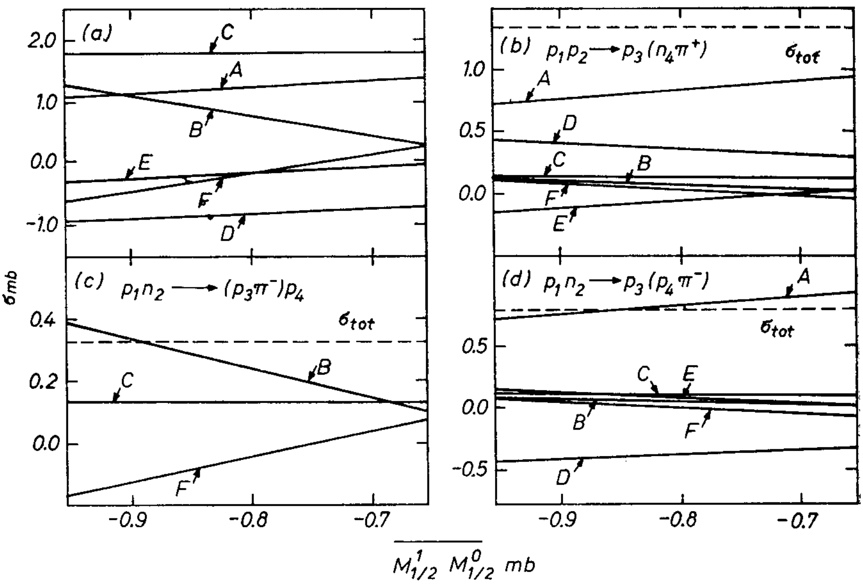

Fig. 3.2

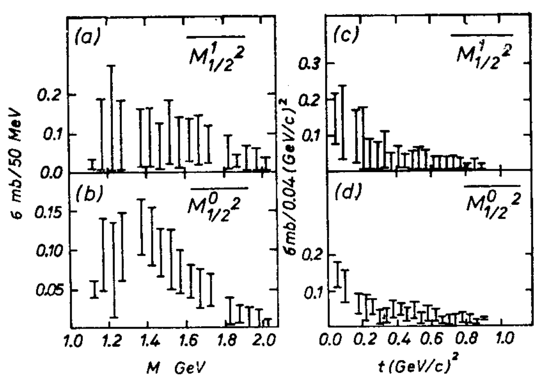

Fig. 3.3

use the inequalities (3.8) to estimate the upper and lower values of σ_1 and σ_6, and then evaluate the bounds of $(M_{\frac{1}{2}}^0)^2$, ..., $(M_{\frac{1}{2}}^1 \cdot M_{\frac{3}{2}}^1)$. This has been done by Yekutieli *et al.* and the comparison with the data of 7 GeV indicate very clearly that $(M_{\frac{1}{2}}^0)^2$ decreases very slowly, while $(M_{\frac{3}{2}}^1)^2$ which is very appreciable at 7.0 GeV/c, decreases very rapidly. See Table 3.3.

TABLE 3.3

Upper and lower bounds for unknown quantities at 28.5 GeV/c (Ref. [11])

	Min.	Max.
σ_1	0.38 mb	0.73 mb
$(M^0_{1/2})^2$	0.75 mb	0.8 mb
$(M^1_{1/2})^2$	0.125 mb	0.30 mb
$(M^1_{3/2})^2$	0.075 mb	0.075 mb
$(M^0_{1/2} \cdot M^1_{1/2})$	−0.35 mb	−0.10 mb
$(M^0_{1/2} \cdot M^1_{3/2})$	−0.02 mb	0.24 mb
$(M^1_{1/2} \cdot M^1_{3/2})$	−0.15 mb	0.03 mb

4. *Graphical illustration of three amplitude cases*

In the foregoing section we have seen that the integrated cross-sections for various charge channels of the reaction $NN \to N\pi N$ can be described in terms of six scalar products $(\boldsymbol{M} \cdot \boldsymbol{M}')$ of three vectors[14]. The same applies, of course, to the reaction $\pi N \to \pi N \pi$ as far as the $I' = 2$ exchange mechanism can be neglected. Since these vectors span a three--dimensional Euclidean space, it is possible to visualize the relations by means of a tetrahedron or pyramid [8–9].

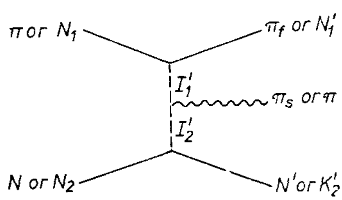

Fig. 4.1

For technical reasons (*i. e.*, simplicity of construction of the figures) we use this time a purely cross-channel representation (or a double peripheral representation), where isospin amplitudes are specified by isospin I_1', and I_2' of exchanged "objects". (See Fig. 4.1.) We denote three isospin amplitudes in this representation[15] by P, Q and R as defined by Table 4.1.

[14] Generally, when differential cross-sections of a certain reaction are described by n isospin amplitudes, the integrated cross-sections are expressed in terms of $\frac{1}{2}n(n+1)$ scalar products of n vectors.

[15] Our amplitudes in Sections 2 and 3 are related to the new ones by $M^0_{1/2} = \sqrt{3/2}\,P$, $M^1_{1/2} = Q+2R$, $M^1_{3/2} = Q-R$ in the case of πN, and $M^0_{1/2} = \sqrt{3/2}\,P$, $M^1_{1/2} = \sqrt{3/2}(Q+2R)$, $M^1_{3/2} = \sqrt{3/2}(Q-R)$ in the case of NN.

TABLE 4.1

	I'_1	I'_2
P	0	1
Q	1	0
R	1	1

Then we define three vectors $\boldsymbol{P}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ which are related to P, Q, and R just in the same way as $\boldsymbol{M}_{\frac{1}{2}}^{0}$, $\boldsymbol{M}_{\frac{1}{2}}^{1}$ and $\boldsymbol{M}_{\frac{3}{2}}^{1}$ are related to $M_{\frac{1}{2}}^{0}$, $M_{\frac{1}{2}}^{1}$ and $M_{\frac{3}{2}}^{1}$, explained in Section 3. As is summarized in Table 4.2, the integrated cross-sections for $\pi N \to \pi N \pi$ and $NN \to N\pi N$

TABLE 4.2

Double peripheral representation

	$NN \to N\pi N$	$\pi N \to \pi N \pi$
$A^2 = \lvert\boldsymbol{P}+\boldsymbol{Q}\rvert^2$	$2\sigma(p_1 n_2 \to p'_1 \pi^0 n'_2)$	$2\sigma(\pi^- p \to \pi^-_f p \pi^0_s)$
$B^2 = \lvert\boldsymbol{P}-\boldsymbol{Q}\rvert^2$	$2\sigma(p_1 p_2 \to p'_1 \pi^0 p'_2)$	$2\sigma(\pi^+ p \to \pi^+_f p \pi^0_s)$
$C_2 = \lvert\boldsymbol{Q}+\boldsymbol{R}\rvert^2$	$\sigma(n_1 p_2 \to p'_1 \pi^- p'_2)$	$2\sigma(\pi^- p \to \pi^0_f p \pi^-_s)$
$D^2 = \lvert\boldsymbol{Q}-\boldsymbol{R}\rvert^2$	$\sigma(p_1 p_2 \to n'_1 \pi^+ p'_2)$	$2\sigma(\pi^+ p \to \pi^0_f p \pi^+_s)$
$E^2 = \lvert\boldsymbol{P}+\boldsymbol{R}\rvert^2$	$\sigma(p_1 n_2 \to p'_1 \pi^- p'_2)$	$\sigma(\pi^- p \to \pi^-_f n \pi^+_s)$
$F^2 = \lvert\boldsymbol{P}-\boldsymbol{R}\rvert^2$	$\sigma(p_1 p_2 \to p'_1 \pi^+ n'_2)$	$\sigma(\pi^+ p \to \pi^+_f n \pi^+_s)$

are expressed in terms of the square of sum and difference of three pairs from $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, and it is quite straightforward to construct a pyramid spanned by these vectors. See Fig. 4.2.

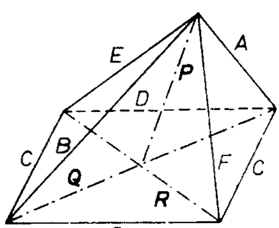

Fig. 4.2

In the case of πN collision, all the sides A, B, C, D, E, F are given, so that the pyramid is fixed (within the experimental errors, of course). See Fig. 4.3. The fact that a pyramid can actually be constructed shows that the inequalities (3.8) are satisfied and the requirement of isospin symmetry is fulfilled. Indeed it is for this purpose that we have slightly modified the values (well within the experimental errors) for $\int |M_{\frac{1}{2}}^{1}|^2 d\tau$ and $\int \mathrm{Re}\,(M_{\frac{1}{2}}^{1} M_{\frac{3}{2}}^{1*})\, d\tau$. In this case the third inequality (3.8c) is almost an equality and, correspondingly, the pyramid of Fig. 4.3 is nearly a planar figure ($\boldsymbol{R} \approx 0$), suggesting an approximate linear dependence of $M_{\frac{1}{2}}^{1}$ and $M_{\frac{3}{2}}^{1}$.

In the case of NN collision at 7.0 GeV/c, the values of B, C, D, E, F, are available, but not that of A. Thus the pyramid is not uniquely determined; it can take any shape

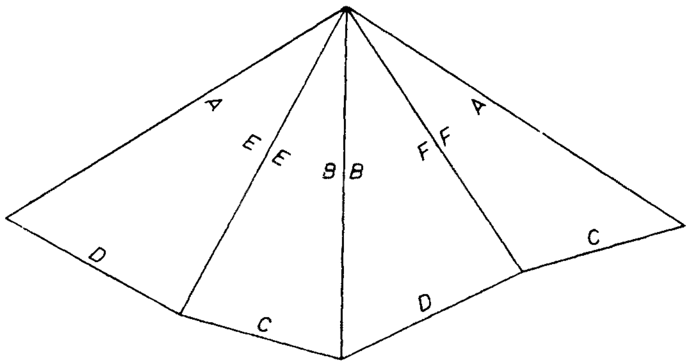

Fig. 4.3

between two extreme cases where two triangles BCE and BDF become coplanar. See Fig. 4.4.

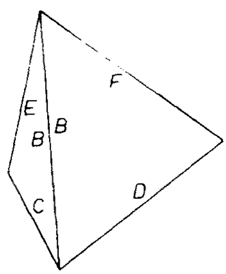

Fig. 4.4

Finally in the case of NN collision at 28.5 GeV, only C, D, E, F are known, and these connected 4 sides can be located in any shape between extreme cases. Nevertheless one can still make a fairly good determination of upper and lower bounds of B and then those of $|\boldsymbol{P}|^2$, $|\boldsymbol{Q}|^2$ *etc.*, owing to the smallness of C and D[16].

As is seen from these examples, the graphical method gives a very simple picture and interpretation of the isospin relations, in particular the inequalities, but its application is usually limited to the cases of 2 or 3 isospin amplitudes, because of our poor intuition.

This lecture is based on a series of works carried out in collaboration with R. Møllerud, L. Veje, J. Bjørneboe in Copenhagen and N. Törnqvist in Helsinki, to whom I am grateful. In particular I owe Jens Bjørneboe for continual discussions on this subject and comments on this lecture note. I thank also Dr W. Kittel and his colleagues in ABBC

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[16] This is very encouraging for further application of isospin analysis to very high energy reactions, where similar situations are expected.

collaboration and Professor G. Yekutieli and his colleagues in Rehovot for informing me of their results prior to publication.

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ADDENDUM

After these lecture notes were prepared, the following works on the isospin analysis of production experiments became available.

J. A. Charlesworth, N. Intizar, W. W. Neale and J. G. Rushbrooke, *Isospin Analysis of Exchange Mechanisms and* (ΣK)*-Systems in* $NN \to (\Sigma K)N$ *at 6 GeV/c*, (Cavendish report HEP-72-4). *Isospin Analysis of Exchange Mechanims in* $NN \to (\Lambda K)N$ *at 6 GeV/c*, (Cavendish report HEP-72-7).

In the above works the methods of analysis described in these lectures are applied and it is concluded that at $\sim$ 6 GeV/c:

1) In $NN \to (\Sigma K)N$ the amplitude for producing $I = 3/2$ (ΣK)-system by $I' = 1$ exchange is the largest, the mass distribution of (ΣK) showing evidence for $\Delta(1950)$ production, while the $I' = 0$ exchange is only $\sim 20\%$ of the total reaction amplitude.

2) In $NN \to (\Lambda K)N$, $I' = 0$ exchange accounts for $\sim 40\%$ of the total reaction amplitude.

3) A Deck model can explain the shape of $I' = 0$ exchange part of the combined $(\Sigma K)_{\frac{1}{2}}$ and (ΛK) mass spectrum.

Further, a work on the exchange isospin analysis of $K^-p \pm \bar{K}\pi N$ and $K^-p \to K^*(890)\,\pi N$ at 10 GeV/c by Aachen–Berlin–CERN–London (I. C.)–Vienna Collaboration is near completion. (W. Kittel, private communication.)

(*May 15, 1972*)