Vol. B 3 (1972) ACTA PHYSICA POLONICA No 2

SEMICLASSICAL MODEL OF HIGH ENERGY SCATTERING

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(*Received March 15, 1971*)

The problem of high energy p-p scattering is investigated in the approximation in which the colliding particles are treated classically. It is assumed that the interactions of these particles can be described in terms of some, (unspecified in detail), short range interactions and in terms of massive vector mesons exchanges. It is shown that for a reasonable choice of the classical world lines of the protons the high energy elastic differential cross-section is consistent with the Regge model predictions and with the experimental data. The amplitude of the inelastic π-p process with n vector mesons in the final state is also calculated. It is shown that, in general, the number of the produced vector particles is not governed by the Poisson distribution.

1. *Introduction*

Particles moving with very high momenta can be described in quantum mechanics in the semiclassical approximation. The wave function of such particles can be written in the eikonal form known in geometrical optics [1] and their motion can then be treated as the classical motion along certain trajectories. Highly energetic particles can be described by semiclassical methods also in the relativistic quantum field theory [3]. Classical world lines of these particles determine the current J_μ which acts as the external source of quantized vector field.

The subject of the present paper is an application of the semiclassical approximation in quantum field theory and a qualitative comparison of the results obtained with the experimental data on elastic p–p scattering. The interaction Hamiltonian is divided into two parts: the part corresponding to the unknown short distance interaction (hard part) and the part corresponding to the vector mesons exchanges (soft part). The second part of the interaction is assumed to have the semiclassical eikonal form mentioned above. It is shown that one can choose the current in such a way that the final form of the scattering cross-section is consistent with the predictions of the Regge model and with the p–p elastic scattering experimental data in the large s and small t region. In particular our model describes the fall-off of the p–p cross-section at small t and predicts the shrinking of the diffraction peak.

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In Section 2 we give the general description of the model. In Section 3 we evaluate the contribution to the scattering amplitude from the emission and absorption of the vector mesons by the same particle. In Section 4 we obtain the contribution from vector meson exchanges between both particles. In Section 5 we evaluate the amplitude of the inelastic process with n vector mesons in the final state. Finally, in Section 6, we compare our results with the Regge model and with the experimental data. We discuss also some possibilities of a further development of the model.

2. *General description of the model*

The elastic p–p scattering amplitude can be expressed in terms of the matrix elements of the S operator:

$$S(p_2, q_2; p_1, q_1) = \langle p_2 q_2 | S | p_1 q_1 \rangle \tag{2.1}$$

where p_1, q_1 and p_2, q_2 are the four-momenta of the colliding particles before and after the scattering.

In the present approach to high-energy collisions we will assume that protons interact strongly *via* some short-range forces which will not be specified in detail. In addition, protons will be assumed to interact *via* massive vector-meson exchanges. The coupling of vector mesons to protons will be treated in the approximation in which the recoil of the proton is neglected and the protons will be assumed to move in a fully prescribed manner as determined by their (unknown) short range interactions. The motion of the protons considered as the source of the vector field φ_μ will be treated classically. The short range interaction and the interaction with vector field will be called respectively the hard and the soft part.

The separation of the interaction into two parts will be expressed by the following factorization property of the S operator:

$$S = Te^{-i\int d^4x(H_{\text{hard}}+H_{\text{soft}})} = S_{\text{hard}} S_{\text{soft}}, \tag{2.2}$$

where:

$$S_{\text{hard}} = Te^{-i\int d^4x H_{\text{hard}}}, \tag{2.3a}$$

$$S_{\text{soft}} = Te^{-i\int d^4x H_{\text{soft}}} = Te^{-i\int d^4x J_\mu \varphi^\mu}. \tag{2.3b}$$

We have assumed that H_{hard} does not depend on the vector field and that the current J_μ is a c-number function.

The factorization of the S operator leads to the factorization of the scattering amplitude:

$$S(p_2, q_2;\ p_1, q_1) = \langle p_2 q_2 | S_{\text{hard}} | p_1 q_1 \rangle \langle 0 | S_{\text{soft}} | 0 \rangle, \tag{2.4}$$

where we have taken into account that the initial and final two proton states are the vacuum states with respect to the vector field.

The hard part of the interaction is written in the conventional form:

$$\langle p_2 q_2 | S_{\text{hard}} | p_1 q_1 \rangle = \langle p_2 q_2 | p_1 q_1 \rangle + \\ + i(2\pi)^4 \delta^{(4)}(p_2+q_2-p_1-q_1) T_{\text{hard}}(p_2, q_2; p_1, q_1) \tag{2.5}$$

and it can be split into two parts according to whether the nucleon lines connect p_1 with p_2 and q_1 with q_2 or p_1 with q_2 and q_1 with p_2:

$$T_{\text{hard}}(p_2, q_2; p_1, q_1) = T(p_2, q_2; p_1, q_1) - T(q_2, p_2; p_1, q_1). \tag{2.6}$$

Examples of diagrams contributiong to the first and the second part are given in Fig. 1. Finally, the full scattering amplitude can be expressed in the following form:

$$\begin{aligned} S(p_2, q_2; p_1, q_1) &= \langle p_2 q_2 | p_1 q_1 \rangle + i(2\pi)^4 \delta^{(4)}(p_2 + q_2 - p_1 - q_1) \times \\ &\times [T(p_2, q_2; p_1, q_1) \langle 0 | S_{\text{soft}} | 0 \rangle - (p_2 \leftrightarrow q_2)]. \end{aligned} \tag{2.7}$$

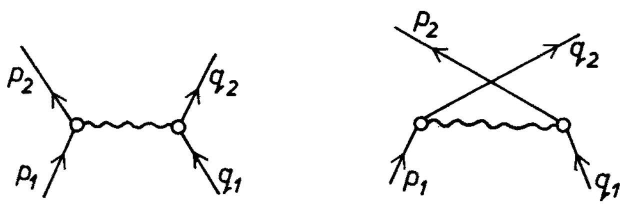

Fig. 1

The current J_μ will be written as the classical current of point particles:

$$J_\mu(x) = g \int_{-\infty}^{+\infty} d\xi_\mu \delta^{(4)}(x - \xi) + g \int_{-\infty}^{+\infty} d\eta_\mu \delta^{(4)}(x - \eta), \tag{2.8}$$

where ξ_μ and η_μ represent the world lines of the two particles and g is the proton-meson coupling constant. The world lines will be chosen in the following simple form:

$$\xi_\mu(\lambda) = \frac{(p_2 + p_1)_\mu}{2m} \lambda + \frac{(p_2 - p_1)_\mu}{2m} \sqrt{\varrho^2 + \lambda^2} + b_\mu/2, \tag{2.9a}$$

$$\eta_\mu(\lambda) = \frac{(q_2 + q_1)_\mu}{2m} \lambda + \frac{(q_2 - q_1)_\mu}{2m} \sqrt{\varrho^2 + \lambda^2} - b_\mu/2. \tag{2.9b}$$

This family of world lines is parametrized by the asymptotic momenta in the remote past and remote future:

$$\frac{d\xi_\mu}{d\lambda} = \begin{cases} p_{2\mu}/m & \lambda \to +\infty \\ p_{1\mu}/m & \lambda \to -\infty \end{cases} \tag{2.10}$$

and similarly for η_μ.

The four-vector b_μ is connected with the relative coordinates of both particles at $\lambda = 0$ and is related to the classical impact parameter. Finally, ϱ measures the curvature of the world line. One can easily show that the maximum acceleration of the particle is inversely proportional to ϱ:

$$\left. \frac{d^2 \xi_\mu}{d\tau^2} \right|_{\lambda=0} = a_\mu|_{\lambda=0} = \frac{2m(p_2 - p_1)_\mu}{(p_1 + p_2)^2} \frac{1}{\varrho}, \tag{2.11}$$

where τ is the proper time.

The S_{soft} operator can be easily evaluated with the help of the continuous Baker-Hausdorff formula [4] or by the functional method of Hori [2], [5]. The result is:

$$S_{\text{soft}} = :e^{-i\int d^4x J^\mu \varphi_\mu} : e^{\frac{i}{2}\int d^4x d^4y J^\mu(x) \Delta^F_{\mu\nu}(x-y) J^\nu(y)}, \tag{2.12}$$

where $\Delta^F_{\mu\nu}$ is the Feynman propagator of the vector field:

$$\Delta^F_{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \cdot \frac{-g_{\mu\nu}+k_\mu k_\nu/\mu^2}{\mu^2-k^2-i\varepsilon}. \tag{2.13}$$

Due to the current conservation $\partial^\mu J_\mu = 0$, the $k_\mu k_\nu$ term does not contribute.

If we write the Feynman function in the form:

$$\Delta^F(x) = \bar{\Delta}(x) + \frac{i}{2}\Delta^{(1)}(x) \tag{2.14}$$

we can easily see that the real part $\bar{\Delta}(x)$ contributes a phase factor to S_{soft} and the imaginary part $\Delta^{(1)}(x)$ contributes a real exponential function. Therefore the vacuum matrix element of S_{soft} can be written in the following form:

$$\langle 0|S_{\text{soft}}|0\rangle = e^{i\varphi(s,t)} e^{\frac{1}{2}\int d\Gamma J^*_\mu(k) J^\mu(k)}, \tag{2.15}$$

where $d\Gamma = d^3k/(2\pi)^3 2\sqrt{\boldsymbol{k}^2+\mu^2}$ and $J_\mu(k)$ is the Fourier transform of the current $J_\mu(x)$. In the derivation of (2.15) we have used the following relations:

$$\frac{1}{4}\int d^4x d^4y J_\mu(x) \Delta^{(1)}(x-y) J^\mu(y) = \frac{\pi}{2}\int \frac{d^4k}{(2\pi)^4} d^4x d^4y J_\mu(x) e^{-ikx} \times$$

$$\times \delta(k^2-\mu^2)^{iky} J^\mu(y) = \frac{\pi}{2}\int \frac{d^4k}{(2\pi)^4} J^*_\mu(k)\delta(k^2-\mu^2) J^\mu(k) =$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{J^*_\mu(k) J^\mu(k)}{\sqrt{\boldsymbol{k}^2+\mu^2}}\Bigg|_{k_0=\sqrt{\boldsymbol{k}^2+\mu^2}}. \tag{2.16}$$

Using (2.8) and (2.9) we can evaluate the Fourier transform of the current:

$$J_\mu(k) = g\int_{-\infty}^{+\infty} d\xi_\mu e^{ik\xi} + g\int_{-\infty}^{+\infty} d\eta_\mu e^{ik\eta} =$$

$$= \frac{ig\varrho}{m}\left[e^{\frac{i}{2}bk}(p_{2\mu}(p_1k)-p_{1\mu}(p_2k))\frac{K_1\left(\frac{\varrho}{m}\sqrt{(p_1k)(p_2k)}\right)}{\sqrt{(p_1k)(p_2k)}}+\right.$$

$$\left.+e^{-\frac{i}{2}bk}(q_{2\mu}(q_1k)-q_{1\mu}(q_2k))\frac{K_1\left(\frac{\varrho}{m}\sqrt{(q_1k)(q_2k)}\right)}{\sqrt{(q_1k)(q_2k)}}\right] = J_{1\mu}(k)+J_{2\mu}(k), \tag{2.17}$$

where the two parts J_1 and J_2 depend respectively on p_1, p_2, and q_1, q_2. The details of these calculations are given in the Appendix A.

The integral $\int d\Gamma J_\mu^* J^\mu$ can be therefore divided into three parts:

$$\int d\Gamma (J_1^* J_1 + J_2^* J_2 + 2 \operatorname{Re} J_1^* J_2). \tag{2.18}$$

The first two parts which we shall denote respectively by I_p and I_q will depend only on $t = (p_2 - p_1)^2 = (q_1 - q_2)^2$ so that:

$$I_p = I_q = I_1(t, \varrho). \tag{2.19}$$

The interference term $I_{12} = -2\int d\,\Gamma \operatorname{Re} J_1^* J_2$ has the oscillating factor $\cos bk$ and we expect that in view of these oscillations the contribution from $b \neq 0$ can be neglected in the first approximation. This statement will be verified later.

3. *Asymptotic large s and small t behaviour of I_1*

In this Section we will evaluate the I_1 part of (2.18). This quantity describes the interaction of the one particle current with itself and corresponds to the emission and absorption of virtual vector mesons by the particle.

I_1 can be written as the following integral:

$$I_1 = -\int d\Gamma J_1^*(k) J_1(k) = -\frac{g^2\varrho^2}{m^2} \int d\Gamma [(p_1k)p_2 - (p_2k)p_1]^2 \times$$
$$\times \frac{K_1^2\left(\frac{\varrho}{m}\sqrt{(p_1k)\,(p_2k)}\right)}{(p_1k)\,(p_2k)} = -\frac{g^2\varrho^2}{2m^2(2\pi)^3}\int \frac{d_3k}{\omega}\left[m_2\left(\frac{p_1k}{p_2k} + \frac{p_2k}{p_1k}\right) - \right.$$
$$\left. -2p_1p_2\right] K_1^2\left(\frac{\varrho}{m}\sqrt{(p_1k)\,(p_2k)}\right), \tag{3.1}$$

where

$$\omega = \sqrt{\boldsymbol{k}^2 + \mu^2}.$$

We will use the rest system of $p_1 + p_2$,

$$p_1 = (E, \boldsymbol{p}),\; p_2 = (E, -\boldsymbol{p}) \tag{3.2}$$

to evaluate (3.1).

After integration over the azimuthal angle we obtain:

$$I_1 = -\frac{g^2\varrho^2}{2\pi^2 m^2}\int_0^\infty \frac{\varkappa^2 d\varkappa}{\omega}\int_0^1 d\xi K_1^2\left(\frac{\varrho}{m}\sqrt{E^2\omega^2 - \varkappa^2 p^2 \varrho^2}\right) \times$$
$$\times \left[m^2 \frac{E^2\omega^2 + \varkappa^2 p^2 \xi^2}{E^2\omega^2 - \varkappa^2 p^2 \varrho^3} - p_1 p_2\right], \tag{3.3}$$

where $\varkappa$ denotes the length of the space part of the four vector k and $p = |\boldsymbol{p}|$.

We introduce new integration variables:

$$x = \frac{\varrho}{m}\sqrt{E^2(\varkappa^2+\mu^2)-\varkappa^2 p^2\xi^2}, \qquad z = \frac{p}{E}\,\xi.$$

This transformation brings (3.3) to the following form:

$$I_1 = -\frac{g^2m^2}{2\pi^2 Ep}\int\limits_{\frac{\varrho E\mu}{m}}^{\infty}\int\limits_{0}^{\frac{p}{E}} dz\frac{dx}{x}K_1^2(x)\sqrt{\frac{x^2-(\varrho^2\mu^2E^2/m^2)}{x^2-(\varrho^2\mu^2E^2z^2/m^2)}}\times$$

$$\times\left[\left(\frac{\varrho E\mu}{m}\right)^2 - x^2\frac{p_1p_2}{m_2} + \frac{1+z^2}{1-z^2}\left(x^2-\frac{\varrho^2E^2\mu^2}{m^2}\right)\right]. \tag{3.4}$$

After integration over z, this integral can be expressed in the from of a single integral over x:

$$I_1(s,t) = -\frac{g^2m^2}{2\pi^2E^2}\cdot\frac{1}{p/E}\int\limits_{\frac{\varrho E\mu}{m}}^{\infty}\frac{dx}{x}K_1^2(x)\left[\frac{p}{E}\,\frac{\sqrt{(v^2-1)(v^2-p^2/E^2)}}{1-(p^2/E^2)} -\right.$$

$$\left. - v^2\frac{p_1p_2}{m^2}\,\frac{1}{2}\left(\ln\frac{1+(p/E)}{1-(p/E)} - \ln\frac{v^2+(p/E)+\sqrt{(v^2-1)(v^2-p^2/E^2)}}{v^2-(p/E)+\sqrt{(v^2-1)(v^2-p^2/E^2)}}\right)\right], \tag{3.5}$$

where

$$v = \frac{m}{\varrho E\mu}\,x.$$

In terms of the Mandelstam variable t, the variables p and E become:

$$p = \sqrt{-\frac{t}{4}}, \qquad E = \sqrt{m^2-t/4} \tag{3.6}$$

so that for small $|t|$ we have:

$$p = \sqrt{-\frac{t}{4}}, \quad E\approx m, \quad \frac{p}{E}\approx\sqrt{-\frac{t}{4m^2}}. \tag{3.7}$$

For small $|t|$ the expression (3.5) reduces to the following simple form:

$$I_1(s,t)|_{t\to 0} = \frac{g^2|t|}{4\pi^2m^2}\int\limits_{\varrho\mu}^{\infty} dx K_1^2(x)\left(x-\frac{\varrho^2\mu^2}{x+\sqrt{x^2-\varrho^2\mu^2}}\right) =$$

$$= \frac{g^2|t|}{4\pi^2m^2}\int\limits_{\varrho\mu}^{\infty} dx\sqrt{x^2-\varrho^2\mu^2}\,K_1^2(x). \tag{3.8}$$

The s dependence of I_1 is contained only in the s-dependence of the curvature radius ϱ which is an unknown function of s:

$$\varrho = \varrho(s). \tag{3.9}$$

Since we are dealing with high energy scattering it is reasonable to assume that ϱ is small, so that the curvature of the world line of the particle is large near the point $\lambda = 0$. Since $K_1(x)$ tends very rapidly to zero for $x \to \infty$ and ϱ is small, we can take into account only the asymptotic behaviour of $K_1(x)$ at small x, namely:

$$K_1(x)_{x\to 0} \approx x^{-1}. \tag{3.10}$$

If we neglect the contribution from the exponential tail of $K_1(x)$ at large x, we can approximate the right-hand side of (3.8) in the following way:

$$I_1(s, t)|_{t\to 0} \approx -\frac{g^2|t|}{4\pi^2 m^2} \ln \varrho\mu. \tag{3.11}$$

It is seen from (3.11) that I_1 is linear in t for small $|t|$ and that its only s dependence is through $\varrho(s)$. The explicit form of ϱ will be specified in Section 6 after calculating the contribution from the interference term I_{12}.

4. *Asymptotic behaviour of the interference term* I_{12}

The interference term can be written as:

$$I_{12} = -\frac{4g^2\varrho^2}{m^2} \int d\Gamma \cos(bk)\, K_1\left(\frac{\varrho}{m}\sqrt{(p_1k)(p_2k)}\right) K_1\left(\frac{\varrho}{m}\sqrt{(q_1k)(q_2k)}\right) \times$$
$$\times \left[(p_1q_1)\sqrt{\frac{(p_1k)(q_1k)}{(p_2k)(q_2k)}} - (p_1q_2)\sqrt{\frac{(p_1k)(q_2k)}{(p_2k)(q_1k)}}\right]. \tag{4.1}$$

Since the right-hand side of (4.1) contains two sets of momenta, namely p_1, p_2 and q_1, q_2, this integral is much more difficult to evaluate than (3.1). In order to simplify the calculations we will assume from the very beginning that $|t|$ is small and therefore that $p_1 \approx p_2$ and $q_1 \approx q_2$.

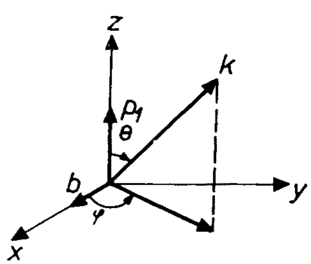

Fig. 2

In view of the space-like character of the impact parameter vector b_μ, we will assume that $b_0 = 0$. The integral I_{12} will be calculated in the CMS frame and the three dimensional cartesian coordinate system will be chosen in such a way, that (Fig. 2):

$$\boldsymbol{p}_1 = (0, 0, |\boldsymbol{p}_1|), \qquad \boldsymbol{b} = (|\boldsymbol{b}|, 0, 0).$$

After these simplifications the expression for I_{12} can be reduced to:

$$I_{12}(s,t)|_{t\to 0} \approx \frac{2g^2\varrho^2 t}{m^2} \int \frac{d_3 k}{2(2\pi)^3\omega} \cos(bk)\, K_1\left(\frac{\varrho}{m} p_1 k\right) K_1\left(\frac{\varrho}{m} q_1 k\right). \tag{4.2}$$

This integral will be calculated in spherical coordinates:

$$I_{12}(s,t) = \frac{g^2\varrho^2 t}{(2\pi)^3 m^2} \int_0^\infty \frac{\varkappa^2 d\varkappa}{\sqrt{\varkappa^2+\mu^2}} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi \cos(b\varkappa \sin\theta\cos\varphi)\times$$

$$\times K_1\left(\frac{\varrho}{m}(E\sqrt{\varkappa^2+\mu^2} - p\varkappa\cos\theta)\right) K_1\left(\frac{\varrho}{m}(E\sqrt{\varkappa^2+\mu^2} + p\varkappa\cos\theta)\right), \tag{4.3}$$

where $b = |\boldsymbol{b}|, p = |\boldsymbol{p}_1| = |\boldsymbol{q}_1|$ and E is the single particle CMS energy. The φ integration can be performed leading to the Bessel function J_0:

$$I_{12} = \frac{g^2\varrho^2 t}{4\pi^2 m^2} \int_0^\infty \frac{\varkappa^2 d\varkappa}{\sqrt{\pi^2+\mu^2}} \int_{-1}^{+1} d\xi J_0(b\varkappa\sqrt{1-\xi^2})\times$$

$$\times K_1\left(\frac{\varrho}{2m}(\sqrt{s}\sqrt{\varkappa^2+\mu^2} - \sqrt{s-4m^2}\,\varkappa\xi)\right) K_1\left(\frac{\varrho}{2m}(\sqrt{s}\sqrt{\varkappa^2+\mu^2} + \sqrt{s-4m^2}\,\varkappa\xi)\right), \tag{4.4}$$

where we have expressed CMS energy and momentum of the individual particle through the Mandelstam invariant variable s.

The change of the integration variables which simplifies (4.4) will be presented in three steps. The first step reads:

$$\varkappa = \sqrt{v^2-\mu^2}, \qquad \xi = \frac{u}{\sqrt{v^2-\mu^2}}. \tag{4.5}$$

After this transformation the integration region is contained inside of one branch of the hyperbola $v^2-u^2 = \mu^2$:

$$I_{12} = \frac{g^2\varrho^2 t}{4\pi^2 m^2} \int_\mu^\infty dv \int_{-\sqrt{v^2-\mu^2}}^{+\sqrt{v^2-\mu^2}} du J_0(b\sqrt{v^2-u^2-\mu^2})\times$$

$$\times K_1\left(\frac{\varrho}{2m}(\sqrt{s}\,v - \sqrt{s-4m^2}\,u)\right) K_1\left(\frac{\varrho}{2m}(\sqrt{s}\,v + \sqrt{s-4m^2}\,u)\right). \tag{4.6}$$

The second change of the variables contains the hyperbolic functions:

$$v = \frac{x}{\varrho}\cosh y, \qquad u = \frac{x}{\varrho}\sinh y \tag{4.7}$$

and brings (4.6) to the form:

$$I_{12}(s,t) = \frac{g^2 t}{2\pi^2 m^2} \int\limits_0^\infty dy \int\limits_{\varrho\mu}^\infty dx x J_0\left(\frac{b}{\varrho}\sqrt{x^2-\varrho^2\mu^2}\right) \times$$

$$\times K_1(x\cosh(y+\delta))\, K_1(x\cosh(y-\delta)), \tag{4.8}$$

where:

$$\operatorname{tgh}\delta = \sqrt{1-\frac{4m^2}{s}}. \tag{4.9}$$

The last step in our calculations contains the transformation:

$$\cosh(y+\delta) = \frac{\xi}{\eta}\cosh\delta, \qquad x = \eta. \tag{4.10}$$

Since $\eta/\xi \leqslant 1$ and $\cosh\delta$ is large, the argument of the second K_1 function in (4.8) can be approximated in the following way:

$$\xi(\cosh\delta\cosh 2\delta - \sqrt{\cosh^2\delta - \eta^2/\xi^2}\sinh 2\delta) \approx$$

$$\approx \xi\cosh\delta\,(\cosh 2\delta - \sinh 2\delta) = \xi e^{-2\delta}\cosh\delta.$$

The right-hand side of (4.8) reduces finally to:

$$I_{12} = \frac{g^2 t}{2\pi^2 m^2} f(\varrho\mu), \tag{4.11a}$$

$$f(\varrho\mu) = \int\limits_{\varrho\mu}^\infty \frac{d\xi}{\xi} \int\limits_{\varrho\mu}^\xi d\eta\,\eta J_0\left(\frac{b}{\varrho}\sqrt{\eta^2-\varrho^2\mu^2}\right) K_1(\xi\cosh\delta)\, K_1(\xi e^{-2\delta}\cosh\delta). \tag{4.11b}$$

The η integration can be easily performed:

$$\int\limits_{\varrho\mu}^\xi d\eta\,\eta J_0\left(\frac{b}{\varrho}\sqrt{\eta^2-\varrho^2\mu^2}\right) = \frac{\varrho}{b}\sqrt{\xi^2-\varrho^2\mu^2}\, J_1\left(\frac{b}{\varrho}\sqrt{\xi^2-\varrho^2\mu^2}\right) \tag{4.12}$$

so that:

$$f(\varrho\mu) = \frac{\varrho}{b}\int\limits_{\varrho\mu}^\infty d\xi\, \frac{\sqrt{\xi^2-\varrho^2\mu^2}}{\xi}\, J_1\left(\frac{b}{\varrho}\sqrt{\xi^2-\varrho^2\mu^2}\right) \times$$

$$\times K_1(\xi\cosh\delta)\, K_1(\xi e^{-2\delta}\cosh\delta). \tag{4.13}$$

Assuming that ϱ and $\tau = \varrho\mu\cosh\delta = \varrho\mu\sqrt{\frac{s}{4m^2}}$ are small, one can show (see Appendix B) that the function f tends to a constant when τ tends to zero. This constant is equal to:

$$\lim_{\tau\to 0} f(\varrho) = 2K_0(b\mu). \tag{4.14}$$

For small $|t|$ and large s we have therefore:

$$I_{12}(s,t) = \frac{g^2 t}{\pi^2 m^2} K_0(b\mu). \tag{4.15}$$

In contrast to I_1, I_{12} does not depend on the CMS energy in the asymptotic limit. Another possibility is that τ is large although ϱ is small. However, it is easy to show that $f(\varrho)$ tends to zero in this case, and the interference term gives no contribution to the scattering amplitude.

The whole exponent in (2.15) can now be written as:

$$I_1 + \frac{1}{2} I_{12} = \frac{g^2 t}{4\pi^2 m^2} \ln \varrho\mu + \frac{g^2 t}{2\pi^2 m^2} K_0(b\mu) = \frac{g^2 t}{4\pi^2 m^2} [\ln \varrho\mu + 2K_0(b\mu)] \tag{4.16}$$

and the scattering amplitude is:

$$\langle 0|S_{\text{soft}}|0\rangle = e^{I_1 + \frac{1}{2} I_{12}} = e^{-\frac{g^2 t}{4\pi^2 m^2}[\ln \varrho\mu + 2K_0(b\mu)]} \tag{4.17}$$

This formula will be discussed in Section 6.

5. *Production of vector mesons*

In this Section we will calculate the amplitude of the inelastic π—p collision for the production of n vector mesons. If the vector mesons were assumed to be ϱ^0-mesons we could further calculate the production amplitude with $2n$ pions in the final state.

We treat the proton classically and we assume that in the remote past it is moving along a straight line. The π—p interaction curves the proton world line so that in the remote future the proton is moving in a different direction than that in the remote past. We will assume that its world line is described again by (2.9a). The π—p interaction will show up only in the bending of this world line. Since the proton is accelerated during the interaction with the pion, it can emit vector particles (this process is very similar to that of the bremsstrahlung). We must therefore calculate the following matrix element:

$$\begin{aligned} S_n(k_1, \ldots, k_n) &= \langle k_1, \ldots, k_n|S_{\text{soft}}|0\rangle = \\ &= \langle k_1 \ldots, k_n|: e^{-i\int d^4x J^\mu \varphi_\mu} :|0\rangle \langle 0|S_{\text{soft}}|0\rangle, \end{aligned} \tag{5.1}$$

where J_μ is the single proton current:

$$J_\mu(x) = g \int_{-\infty}^{+\infty} d\xi_\mu \delta^{(4)}(x-\xi) \tag{5.2}$$

and $k_1, \ldots, k_n$ are on-shell momenta of the mesons. The first factor in the right-hand side of (5.1) can be easily evaluated:

$$\langle k_1, \ldots, k_n|: e^{-i\int d^4x J^\mu \varphi_\mu}:|0\rangle = \frac{(-i)^n}{\sqrt{n!}} \prod_{m=1}^{n} J^\mu(k_m) \varepsilon_\mu^{\lambda_m}(\overset{\triangledown}{k}_m), \tag{5.3}$$

where $\varepsilon_\mu^\lambda(k)$ are the polarization vectors of vector mesons. The inelastic scattering cross-section can now be written as:

$$\sigma_n = \frac{(2\pi)^4}{2\lambda(s, m^2, m_\pi^2)} \int d\Gamma_{p_2} d\Gamma_{q_2} d\Gamma_{k_1} \dots d\Gamma_{k_n} \delta^{(4)}(p_1+q_1-p_2-q_1-\sum_{r=1}^{n} k_r)\times$$

$$\times \sum_{\text{spin}} |S_n(k_1, \dots, k_n)|^2, \tag{5.4}$$

where $\sum_{\text{spin}}$ denotes the summation over the polarizations of the vector particles in the final state and q_1, q_2 are the momenta of the pions before and after scattering and λ is the following function:

$$\lambda(s, m^2, m_\pi^2) = \sqrt{(s-m^2-m_\pi^2)^2-4m_\pi^2 m^2}. \tag{5.5}$$

After performing the spin summation we obtain:

$$\sigma_n = \frac{(2\pi)^4}{2\lambda} \frac{1}{n!} \int d\Gamma_{p_2} d\Gamma_{q_2} e^{-I_1} \int d\Gamma_{k_1} \dots d\Gamma_{k_n}\times$$

$$\times \delta^{(4)}(p_1+q_1-p_2-q_2-\sum_{r=1}^{n} k_r) \prod_{m=1}^{n} F(k_m), \tag{5.6}$$

where:

$$F(k) = -J_\mu^*(k)J^\mu(k). \tag{5.7}$$

For very energetic particles and soft vector mesons we can neglect the meson momenta in the delta function. The differential cross-section (*i.e.* the cross-section $d\sigma_n/dt$, where $t = (p_2-p_1)^2$ is the proton momentum transfer) takes the form of the Poisson distribution:

$$\frac{d\sigma_n}{dt} \sim \frac{a^n}{n!} e^{-a}. \tag{5.8}$$

However, the omission of k_r in the delta function cannot be justified in our model since the production amplitude has significant values also for the three momenta which are of the order of magnitude of a few meson masses. Therefore, in general the Poisson distribution does not follow from our model.

The problem of the emission of secondary particles in the eikonal approximation in quantum field theory was also considered by Barbashov *et al.* [8], [9]. The authors have calculated the cross-section of the production of n soft vector mesons in the straight-line path approximation and obtained the Poisson distribution for the number of vector mesons in the final state. However, they had to introduce separately the condition of the softness of vector mesons in the CM system. This condition is not Lorentz-invariant and leads (for large s) to the production of hard mesons in the forward direction in the laboratory system.

6. *Discussion*

The differential cross-section for the elastic p—p scattering can be written as the product of the hard and the soft parts of the scattering amplitude. Moreover, for small $|t|$ we can neglect the second term in (2.7). After substituting (4.17) into (2.7) the differential cross--section can be finally expressed as:

$$\frac{d\sigma}{dt}=\left(\frac{d\sigma}{dt}\right)_{\text{hard}}|\langle 0|S_{\text{soft}}|0\rangle|^2=\left(\frac{d\sigma}{dt}\right)_{\text{hard}}e^{-\frac{g^2t}{2\pi^2m^2}[\ln\varrho\mu+2K_0(b\mu)]}. \tag{6.1}$$

Since ϱ is small, in the first approximation we can neglect the interference term $2K_0(b\mu)$ (*cf.* the argument given at the end of Section 2).

Similarly as in the pure classical case the impact parameter will be a function of the momentum transfer. Since we are interested only in the small angle scattering, the impact parameter must be considerably large. We can assume that b is of the order of the Compton wave length of the proton, so that both protons interact between themselves only on the boundary of their interaction region and the trajectories of the protons cannot change significantly. If we assume that the exchanged vector meson is ϱ^0 then $b\mu\sim\mu/m\approx 0.83$. Since the interaction range is finite, b must be finite as $t\to 0$. The function $b(t)$ should be continuous and therefore its derivative $b'(t)$ should be finite. We can in general assume that $b(t)$ is a smooth function of t (at least for small t) and that it can be expanded in the power series around $t=0$:

$$b(t)=A+Bt+\ldots\quad (A,B>0). \tag{6.2}$$

This type of t dependence can be verified in the case of classical relativistic potential scattering in the CMS frame. (However, in the pure classical case b is also a function of the energy and for small t and large s it has the form: $b\approx A(s+t/2s)$.)

If we expand $K_0(b\mu)$ in (5.1) into a power series in t around $t=0$ ($b\mu=\mu/m$), we see that the second term in (5.2) contributes a t^2 term to the exponent and can be therefore neglected.

Let us assume that the s dependence of the curvature radius ϱ is of the following form:

$$\varrho(s)=\varrho_0s^{-1}. \tag{6.3}$$

This formula is consistent with our previous assumption that ϱ is small for very high energy. Using (6.3) and neglecting terms quadratic in t we see that (6.1) can be reduced to the Regge model form with straight line trajectories:

$$\frac{d\sigma}{dt}=\left(\frac{d\sigma}{dt}\right)_{\text{hard}}\left(\frac{s}{s_0}\right)^{\frac{g^2t}{2\pi^2m^2}}. \tag{6.4}$$

Since t is negative in the s channel, the resulting scattering cross-section (6.4) decreases exponentially with increasing $|t|$, Moreover, $d\sigma/dt$ decreases with increasing s at fixed t (shrinkage of the diffraction peak). We see therefore that this semiclassical model explains both characteristic features of the differential cross-section for the elastic p—p scattering at large energies and small angles.

For large values of the momentum transfer t the second term in (2.7) cannot be neglected. It follows that in this case not only the functions I_1 and I_{12} but also the phases $\varphi(s, t)$ and $\varphi(s, u)$ contribute to the amplitude. The integral representation of I_1 is known for arbitrary momentum transfer and therefore we can find the form of I_1 as a function of t. Since the lower integration limit in (3.5):

$$\frac{\varrho E\mu}{m} = \frac{\varrho_0\mu}{sm}\sqrt{m^2-t/4} \tag{6.5}$$

is small for every t, we claim that only the behaviour of $K_1(x)$ at small x is significant. After separating the leading term (*i.e.* the term containing large quantity $\ln \varrho\mu$ and that contributing to the large $|t|$ behaviour of I_1) from (3.5) we obtain:

$$I_1 = -\frac{g^2}{4\pi}\left[\left(1-\frac{2m^2-t}{\sqrt{t(t-4m^2)}}\ln\frac{\sqrt{4m^2-t}+\sqrt{-t}}{\sqrt{4m^2-t}-\sqrt{-t}}\right)\ln(\varrho^2\mu^2)-\right.$$
$$\left.-\frac{2m^2-t}{\sqrt{t(t-4m^2)}}\ln\frac{4m^2-t}{4m^2}\ln\frac{\sqrt{4m^2-t}+\sqrt{-t}}{\sqrt{4m^2-t}-\sqrt{-t}}\right]. \tag{6.6}$$

Let us notice that the expression in the round brackets is equal to the infrared divergent part of the vertex function with two external fermion lines in quantum electrodynamics. This result is not surprising since our method is closely related to the Bloch-Nordsieck method of treating infrared divergences. This for of I_1 is also consistent with the form of the eikonal scattering amplitude obtained by Fried and Gaisser [10].

However, we cannot say anything about the shape of I_1 and φ as functions of t for arbitrary values of this variable.

A model similar to ours has been considered by Torgersson [3] for straight-line particle trajectories:

$$\xi_\mu = (\lambda, 0, \tfrac{1}{2}b, |\boldsymbol{p}_1|\lambda/m), \quad \eta_\mu = (\lambda, 0, -\tfrac{1}{2}b, -|\boldsymbol{p}_2|\lambda/m). \tag{6.7}$$

The corresponding scattering amplitude is then equal to:

$$\langle 0|S_{\mathrm{soft}}|0\rangle = e^{-i\frac{g}{2\pi}\frac{1+v^2}{v^2}K_0(b_\mu)}, \tag{6.8}$$

where v is the particle velocity. This expression is consistent with the result of summing up the multiphoton exchange amplitudes in quantum electrodynamics (Fig. 3) considered by Wu and Cheng [6] and Ma and Chang [7].

In the Torgersson model the exponent of the formula (2.15) is purely imaginary and the contributions from the $\int d^4x d^4y J^1_\mu \Delta^{F\mu\nu} J^1_\nu$ terms are divergent. They are removed by a renormalization procedure. In our model these divergent terms are contained in the phase φ; their divergences can be removed by a similar method. Since the straight line interference terms describe the high energy contribution from the multiphoton exchange diagrams, it would be interesting to check whether one can choose the particle trajectory in such a way that the resulting amplitude will be consistent with the high energy behaviour of the radiative corrections to the diagrams in Fig. 3.

The problem of high energy inelastic hadron collisions in the bremsstrahlung approximation has been investigated by several authors [11], [12], [13]. However, all these papers deal with the multiple production of scalar or pseudoscalar mesons in the "sudden approximation". In this approximation the four-momentum of the nucleon is not continuous at

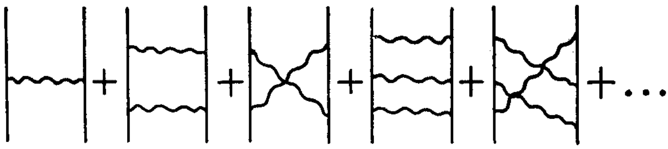

Fig. 3

$\lambda = 0$, where λ is the parameter of the world line. Białas and Ruijgrok [14] use the coherent state as the final state of the produced mesons. Their results show that the distribution of the number of produced mesons is narrowed in comparison with the Poisson distribution. Similarly as in the present model, the Poisson distribution can be obtained only after neglecting the conservation laws expressed by the delta function in the formula for the cross-section. In the "sudden approximation" the integral $I_1 = -\int d\Gamma J_\mu^* J^\mu$ is divergent and therefore this method cannot be applied in our model. One could, however, introduce the meson momentum cut-off but this procedure is not Lorentz invariant.

To summarize: the particle world line in the semiclassical approximation in quantum field theory can be chosen in such a way that the final form of the elastic cross-section is consistent with the Regge model and with the experimental data on elastic p—p scattering at large energy and small momentum transfer. In particular, such a semiclassical model accounts for the rapid fall-off of the differential cross-section at small t and large s and for the shrinkage of the diffraction peak. On the other hand, for a straight line particle trajectory (or for momentum transfer equal to zero) the final result is consistent with the sum of the infinite set of multiphoton exchange diagrams in quantum electrodynamics.

APPENDIX A

In this Section we will calculate the Fourier transform of the current $J_\mu(x)$. After substituting (2.9a) into

$$J_\mu^{(1)}(k) = g \int_{-\infty}^{+\infty} d\xi_\mu e^{ik\xi} \tag{A.1}$$

we obtain:

$$J_\mu^{(1)}(k) = g e^{i\frac{bk}{2}} \int_{-\infty}^{+\infty} d\lambda \left(\frac{p_+}{2m} + \frac{p_-}{2m}\frac{df}{d\lambda}\right)_\mu e^{ik\left(\frac{p_+}{2m}\lambda + \frac{p_-}{2m}f(\lambda)\right)}, \tag{A.2}$$

where:

$$p_\pm = p_1 \pm p_2, \quad f(\lambda) = \sqrt{\varrho^2 + \lambda^2}.$$

Expression (A.2) can be written in the following form:

$$J_\mu^{(1)}(k) = ge^{i\frac{bk}{2}} \int_{-\infty}^{+\infty} d\lambda \frac{p_{+\mu}}{2m} e^{i\pi_+\lambda + i\pi_- f(\lambda)} +$$

$$+ge^{i\frac{bk}{2}} \int_{-\infty}^{+\infty} d\lambda \frac{p_{-\mu}}{2m} \frac{1}{i\pi_-} \frac{d}{d\lambda} (i\pi_- f) e^{i\pi_+\lambda + i\pi_- f(\lambda)}, \tag{A.3}$$

where:

$$\pi_\pm = \frac{p_\pm k}{2m}.$$

After integration by parts we obtain:

$$J_\mu^{(1)}(k) = \frac{g}{2m} e^{i\frac{bk}{2}} \left(p_+ - \frac{\pi_+}{\pi_-} p_-\right)_\mu \int_{-\infty}^{+\infty} d\lambda e^{i\pi_+\lambda + i\pi_- f(\lambda)} =$$

$$= \frac{g\pi_+}{2m} e^{i\frac{bk}{2}} \left(\frac{p_+}{\pi_+} - \frac{p_-}{\pi_-}\right)_\mu \frac{1}{i} \frac{\partial}{\partial \pi_-} \int_{-\infty}^{+\infty} \frac{d\lambda}{f(\lambda)} e^{i\pi_+\lambda + i\pi_- f(\lambda)}. \tag{A.4}$$

The introduction of the new integration variable:

$$\lambda = \varrho \sinh \psi$$

brings (A.4) to the following form:

$$J_\mu^{(1)}(k) = \frac{g\pi_+}{2m} e^{i\frac{bk}{2}} \left(\frac{p_+}{\pi_+} - \frac{p_-}{\pi_-}\right)_\mu \frac{1}{i} \frac{\partial}{\partial \pi_-} \int_{-\infty}^{+\infty} d\psi e^{i\varrho\sqrt{\pi_+^2 - \pi_-^2} \sinh(\psi + \varphi)}, \tag{A.5}$$

where:

$$\sinh \varphi = \frac{\pi_-}{\sqrt{\pi_+^2 - \pi_-^2}}.$$

After shifting the integration variable we obtain:

$$J_\mu^{(1)}(k) = \frac{g\pi_+}{m} \left(\frac{p_+}{\pi_+} - \frac{p_-}{\pi_-}\right) e^{i\frac{bk}{2}} \frac{1}{i} \frac{\partial}{\partial \pi_-} K_0\left(\varrho\sqrt{\pi_+^2 - \pi_-^2}\right) =$$

$$= -2ig\varrho \left(\frac{p_+}{p_+ k} - \frac{p_-}{p_- k}\right)_\mu e^{i\frac{bk}{2}} \frac{\pi_+ \pi_-}{\sqrt{\pi_+^2 - \pi_-^2}} K_1\left(\frac{\varrho}{m}\sqrt{(p_1 k)(p_2 k)}\right). \tag{A.6}$$

Formula (A.6) gives exactly the first term on the right-hand side of (2.17). The second term, $J^{(2)}(k)$, can be obtained in a similar way.

APPENDIX B

In this Section we will prove the formula (4.14). Substituting into (4.13) the new integration variable:

$$z = \xi \cosh \delta$$

we obtain:

$$f(\varrho) = \frac{\varrho}{b \cosh \delta} \int\limits_{\tau}^{\infty} \frac{dz}{z} \sqrt{z^2 - \tau^2}\, J_1\left(\frac{b\mu}{\varrho} \sqrt{z^2 - \tau^2}\right) K_1(z) K_1(ze^{-2\delta}), \tag{B.1}$$

where:

$$\tau = \varrho\mu \cosh \delta.$$

Since τ is small for high s, the main contribution to $f(\varrho)$ comes from the small z region and we can replace $K_1(z)$ by z^{-1}. The replacement of $K_1(z)$ by z^{-1} does not change the oscillating character of the integrand and does not destroy the convergence of the integral. On the other hand one can show that the contribution to $f(\varrho)$ from large values of z is negligible in comparison with the contribution from small z. Therefore, $f(\varrho)$ can be approximated by:

$$f(\varrho) = \frac{\varrho e^{2\delta}}{b \cosh \delta} \int\limits_{\tau}^{\infty} \frac{dz}{z^3} \sqrt{z^2 - \tau^2}\, J_1\left(\frac{b\mu}{\tau} \sqrt{z^2 - \tau^2}\right). \tag{B.2}$$

After introducing the new integration variable:

$$\tau x = \sqrt{z^2 - \tau^2}$$

we obtain:

$$\lim_{\tau \to 0} f(\varrho) = \frac{4}{b\mu} \int\limits_{0}^{\infty} dx \frac{x^2}{(x^2+1)^2} J_1(b\mu x) = 2K_0(b\mu). \tag{B.3}$$

This is the formula (4.14).

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