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ON THE ENERGY OF A STATIONARY SYSTEM IN A GENERAL FIELD THEORY

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Tolman's theorem concerning the expression for the energy of a stationary system in General Relativity, in terms of the Einstein canonical energy-momentum complex, is extended so as to apply to a general class of field theories. The general result is then criticised and a remedy is given which results quite naturally in an infinite number of different energy-momentum complexes. One of these complexes might reasonably be called "canonical" and for the special case of General Relativity, this "canonical" complex turns out to be the one proposed by Moller.

1. *Introduction*

The energy of a stationary system has been much discussed in the literature [1], [2], [3] and here we will reconsider the problem within the context of Tolman's theorem [2]. This theorem states that in General Relativity, given the asymptotic form of the metric, the expression for the energy of a stationary system in terms of the canonical energy-momentum complex can be expressed as a volume integral extending only over the part of the volume actually occupied by the sources of the field. It is proposed here to discuss a generalisation of this theorem to a general field theory supposed derivable from a Lagrangian. Tolman's theorem is easily generalized, but the physical content of his result (and it's generalization) is criticised. It turns out that it is possible to eliminate such criticisms by a simple procedure, and that this procedure leads quite naturally to an infinite number of new energy-momentum complexes. One of these complexes might reasonably be called "canonical" and this particular "canonical" complex, for the case of General Relativity, turns out to be the complex discovered by Moller. By construction, the other energy-momentum-complexes have properties similar to the Moller complex.

We will begin by introducing the general formalism to be used, and essentially we shall follow Bergmann [4].

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2. The general formalism

We will consider a field theory with field variables Ψ_A (where the "A" denotes any number of indices of any type). For our purposes we will take Ψ_A to be a tensor. The field equations for Ψ_A are derivable from an action principle with Lagrangian $\mathscr{L}$:[1]

$$\delta A = 0 \qquad A = \int_\Omega \mathscr{L} d_x^4 \qquad \mathscr{L} = \mathscr{L}(\Psi_A \Psi_{A,\mu}) \tag{2.1}$$

for arbitrary variations $\delta\Psi_A$ which vanish on the surface S of the arbitrary four-dimensional region Ω. We shall not suppose $\mathscr{L}$ an invariant density, but the existence of an invariant density (which in general will contain Ψ_A and its first and second derivatives) differing from $\mathscr{L}$ by only an ordinary divergence will be assumed. It will be supposed further that $\mathscr{L}$ behaves as an invariant density under linear transformations. The field equations from (2.1) are:

$$L^A \overset{\text{deff}^n}{=} \frac{\delta\mathscr{L}}{\delta\Psi_A} = \frac{\partial\mathscr{L}}{\partial\Psi_A} - \left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,\mu}}\right)_{,\mu} = 0 \; [= KP^A \text{ in the presence of sources}]. \tag{2.2}$$

The identities and conservation laws can be obtained by considering the effect of an infinitesimal transformation:

$$\bar{x}^\mu = x^\mu + \mathscr{E}^\mu (\mathscr{E}^\mu \text{ arbitrary infinitesimal functions of } x^\lambda). \tag{2.3}$$

In fact from (2.3) we have:

$$\bar{\delta}\Psi_A = \bar{\Psi}_A(x^\lambda) - \Psi_A(x^\lambda) = F^{B\nu}_{A\mu}\Psi_B \mathscr{E}^\mu_{,\nu} - \Psi_{A,\sigma}\mathscr{E}^\sigma \tag{2.4}$$

where $F^{B\nu}_{A\mu}$ are constants dependant on the transformation properties of Ψ_A. A simple calculation based on Noether's theorem then yields the usual identities and conservation laws [5]. In particular we have the strong conservation law [4] which we will take in the form:[2]

$$\theta^\nu_{\mu,\nu} \equiv 0 \qquad \theta^\nu_\mu = \frac{-1}{Kn}\left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,\nu}}\Psi_{A,\mu} - \delta^\nu_\mu\mathscr{L}\right) + \frac{1}{n}F^{B\nu}_{A\mu}P^A\Psi_B. \tag{2.5}$$

The first equation in (2.5) then implies the existence of a quantity $h^{\nu\lambda}_\mu$ such that:

$$h^{\nu\lambda}_\mu = -h^{\lambda\nu}_\mu \qquad \theta^\nu_\mu = h^{\nu\lambda}_{\mu,\lambda}. \tag{2.6}$$

Now since $\mathscr{L}$ is an invariant density under linear transformations a simple application of Noether's theorem, in which we take $\mathscr{E}^\mu$ in (2.3) as a linear function of x^λ, yields:

$$\theta^\nu_\mu = S^{\nu\lambda}_{\mu,\lambda} \qquad S^{\nu\lambda}_\mu = \frac{-1}{nK}\left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,\lambda}}F^{B\nu}_{A\mu}\Psi_B\right). \tag{2.7}$$

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[1] A comma denotes a partial derivative. Greek indices run 0, 1, 2, 3, whilst small Latin indices run 1, 2, 3.

[2] Here n is a normalization constant which is determined when θ^ν_μ is given explicitly for a particular theory.

In general of course we can find an infinite number of quantities $b_\mu^{\nu\lambda}$ such that:

$$\theta_\mu^\nu = b_{\mu,\lambda}^{\nu\lambda} \qquad \theta_{\mu,\nu}^\nu = b_{\mu,\lambda\nu}^{\nu\lambda} = 0. \tag{2.8}$$

Some of the "superpotentials" $b_\mu^{\nu\lambda}$ will possess anti-symmetry in ν and λ and some will not. We will call the "superpotential" $S_\mu^{\nu\lambda}$ the "canonical superpotential" and the complex θ_μ^ν the (usual) canonical energy-momentum complex. For our expression for the energy we will take:

$$E = -\int_V \theta_0^0 dV = \int_V \left\{ \frac{-F_{A0}^{B0} P^A \Psi_B}{n} + \frac{1}{Kn}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,0}} \Psi_{A,0} - \delta_0^0 \mathscr{L}\right)\right\} dV \tag{2.9}$$

where V is the hypersurface x^0 = constant.

3. The generalized Tolman theorem

We will now consider those field theories whose Lagrangians satisfy the "homogeneity" condition:

$$\frac{\partial \mathscr{L}}{\partial \Psi_A} \Psi_A + \frac{\partial \mathscr{L}}{\partial \Psi_{A,\mu}} \Psi_{A,\mu} = P\mathscr{L} \text{ (P a non-zero constant).} \tag{3.1}$$

Then (3.1) and (2.2) give

$$L^A \Psi_A - P\mathscr{L} = -\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,\mu}} \Psi_A\right)_{,\mu}. \tag{3.2}$$

If we now, use (3.2) (2.2) and (2.9) we find:

$$E = \int_{\bar{V}} \left\{ \frac{-F_{A0}^{B0} P^A \Psi_B}{n} - \frac{P^A \Psi_A}{Pn}\right\} dV + \int_V \left\{ \frac{1}{nK} \frac{\partial \mathscr{L}}{\partial \Psi_{A,0}} \Psi_{A,0} - \frac{1}{nKP}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,\mu}} \psi_A\right)_{,\mu}\right\} dV \tag{3.3}$$

where $\bar{V}$ is that part of the volume V in which $P^A \neq 0$. If we now assume that the system is stationary and that the asymptotic form of the field variables Ψ_A is known then we can calculate E from (2.7) by Gauss' law (using $\Psi_{A,0} = 0$);

$$E = \int_V \frac{1}{nK}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,a}} F_{A0}^{B0} \Psi_B\right)_{,a} dV = \int_S \frac{1}{Kn}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,a}} F_{A0}^{B0} \Psi_B\right) dS \tag{3.4}$$

where S is the 2-surface at infinity, surrounding V. Again since $\Psi_{A,0} = 0$ we can write the final term of (3.3) as:

$$\frac{1}{KPn}\int_V \left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,a}} \Psi_A\right)_{,a} dV = \frac{1}{KPn}\int_S \left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,a}} \Psi_A\right) dS \tag{3.5}$$

and in general evaluate it by using the asymptotic form of Ψ_A. We now chose a specific reference frame K in which $\Psi_{A,0} = 0$ and the integrals (3.4) and (3.5) converge, and in which $E \neq 0$. Then using the result of (3.4) we may evaluate (3.5) by comparison with (3.4) in the form:

$$\frac{1}{nKP}\int\left(\frac{\partial \mathcal{L}}{\partial \Psi_{A,a}}\Psi_A\right)dS = N_K E \tag{3.6}$$

where N_K is a constant depending only on the frame K.[3] Equation (3.3) now reads:

$$E(1+N_K) = \int_{\bar{V}}\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n}\frac{-P^A\Psi_A}{Pn}\right\}dV. \tag{3.7}$$

Then if $(1+N_K) \neq 0$, equation (3.7) is the generalization of Tolman's theorem giving the energy as an integral over $\bar{V}$ only.

In discussing the content of this theorem one can only say that any significance it has seems to be mathematical rather than physical. Although convenient for calculations, (3.7) says nothing about the energy distribution. It does not of course follow from (3.7) that all the energy E of the stationary system under discussion lies inside $\bar{V}$ [6]. What the theorem does is to show that any contribution to E not directly expressible as an integral over $\bar{V}$ can be reduced by Gauss' law to an integral over the 2-surface at infinity and then evaluated, using the asymptotic form of Ψ_A, in terms of E. We are however restricted to coordinate systems in which not only $\Psi_{A,0} = 0$ but also ones in which (3.4) and (3.5) converge, with $E \neq 0$. It would be thus of value to find an energy-momentum complex such that the corresponding energy could be written immediately as an integral over $\bar{V}$. That is we seek a complex M^ν_μ such that:

$$M^{\nu}_{\mu,} = 0 \qquad -\int_V M^0_0 dV = -\int_{\bar{V}} M^0_0 dV = E \tag{3.8}$$

(where E is the value of $-\int \theta^0_0 dV$ in the frame K) for all frames in which $\Psi_{A,0} = 0$. Then if $\bar{V}$ is finite, E will in general converge for all such frames. Further with such an M^ν_μ more definite statements concerning energy distribution might be made. Fortunately such an expression is easy to construct — wemerely use (2.2), (3.2), (3.3) and (3.6) to write

$$E = \frac{1}{(1+N_K)}\int_{\bar{V}}\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n} + \frac{1}{nK}\left(\frac{\partial \mathcal{L}}{\partial \Psi_{A,0}}\Psi_{A,0} - \mathcal{L}\right) + \frac{1}{nPK}\left(\frac{\partial \mathcal{L}}{\partial \Psi_{A,\mu}}\psi_A\right)_{,\mu}\right\}dV \tag{3.9}$$

and then consider the integrand as the (0, 0) component of the complex

$$-N^\nu_\mu = \frac{-1}{(1+N_K)}\left[\frac{F^{B\nu}_{A\mu}P^A\Psi_B}{n} - \frac{1}{nK}\left(\frac{\partial \mathcal{L}}{\partial \Psi_{A,\nu}}\Psi_{A,\mu} - \delta^\nu_\mu \mathcal{L}\right) - \frac{1}{nP}\delta^\nu_\mu\left(\frac{1}{K}\frac{\partial \mathcal{L}}{\partial \Psi_{A,\lambda}}\psi_A\right)_{,\lambda}\right]. \tag{3.10}$$

[3] Since N_K depends on surface integrals at infinity, only the asymptotic form of the field and hence reference system is important and so one need not, in general, specify a particular frame but rather a class of frames such that the transformation of coordinates from one member of this class to another is asymptotically the identity transformation and that in each member of the class $\psi_{A,0} = 0$.

We then find that $N^{\nu}_{\mu,\nu} = \dfrac{-1}{nP(1+N_K)}\left(\dfrac{1}{K}\dfrac{\partial \mathscr{L}}{\partial \Psi_{A,\lambda}}\Psi_A\right)_{,\lambda\mu}$. Hence if we define M^{ν}_{μ} by:

$$M^{\nu}_{\mu} = \frac{1}{(1+N_K)}\left[\frac{F^{B\nu}_{A\mu}P^A\Psi_B}{n} - \frac{1}{nK}\left(\frac{\partial \mathscr{L}}{\delta \Psi_{A,\nu}}\delta\Psi_{A,\mu} - \delta^{\nu}_{\mu}\mathscr{L}\right) - \right.$$

$$\left. - \frac{1}{nP}\left\{\delta^{\nu}_{\mu}\left(\frac{1}{K}\frac{\partial \mathscr{L}}{\delta \Psi_{A,\lambda}}\Psi_A\right)_{,\lambda} - \left(\frac{1}{K}\frac{\partial \mathscr{L}}{\partial \Psi_{A,\nu}}\Psi_A\right)_{,\mu}\right\}\right] \tag{3.11}$$

then we are assured that $M^0_0 = N^0_0$ and that (3.8) is satisfied for a stationary system. Even if N_K is unknown, (3.11) determines M^{ν}_{μ} to within a constant factor, the quantity N_K being determined when the frame satisfying the condition in (3.6) is selected.

4. *Alternative procedure*

In some cases an alternative procedure is available. Suppose Ψ_A is a tensor with U upper indices and L lower indices $\Psi_A \to \Psi^{\alpha\ldots\beta}_{\mu\ldots\nu}$. Then we have from (2.4):

$$\bar{\delta}\Psi^{\alpha\ldots\beta}_{\mu\ldots\nu} = \bar{\Psi}^{\alpha\ldots\beta}_{\mu\ldots\nu}(x^{\sigma}) - \bar{\Psi}^{\alpha\ldots\beta}_{\mu\ldots\nu}(\bar{x}^{\sigma}) + \bar{\Psi}^{\alpha\ldots\beta}_{\mu\ldots\nu}(\bar{x}^{\sigma}) - \Psi^{\alpha\ldots\beta}_{\mu\ldots\nu}(x^{\sigma})$$

$$= -\Psi^{\alpha\ldots\beta}_{\mu\ldots\nu,\varrho}\mathscr{E}^{\varrho} - \Psi^{\alpha\ldots\beta}_{\mu\ldots\nu}(x^{\sigma}) + \Psi^{\sigma\ldots\varrho}_{\lambda\ldots\tau}\frac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}}\cdots\frac{\partial \bar{x}^{\beta}}{\partial x^{\varrho}}\frac{\partial x^{\lambda}}{\partial \bar{x}^{\mu}}\cdots\frac{\partial x^{\tau}}{\partial \bar{x}^{\nu}}. \tag{4.1}$$

Now from (2.3) we have $\dfrac{\partial \bar{x}^{\alpha}}{\partial x^{\sigma}} = (\delta^{\alpha}_{\sigma} + \mathscr{E}^{\alpha}_{,\sigma}) \ldots \dfrac{\partial x^{\lambda}}{\partial \bar{x}^{\mu}} = (\delta^{\lambda}_{\mu} - \mathscr{E}^{\lambda}_{,\mu}) \ldots$ *etc.* we can express (4.1) in the form (2.4) namely:

$$\bar{\delta}\Psi^{\alpha\ldots\beta}_{\mu\ldots\nu} = -\Psi^{\alpha\ldots\beta}_{\mu\ldots\nu,\varrho}\mathscr{E}^{\varrho} + F^{\tau\ldots\lambda\alpha\ldots\beta\pi}_{\varrho\ldots\sigma\mu\ldots\nu\omega}\Psi^{\varrho\ldots\sigma}_{\tau\ldots\lambda}\mathscr{E}^{\omega}_{,\pi}$$

$$F^{\tau\ldots\lambda\alpha\ldots\beta\pi}_{\varrho\ldots\sigma\mu\ldots\nu\omega} = \delta^{\tau}_{\mu}\ldots\delta^{\lambda}_{\nu}\ldots\delta^{\gamma}_{\varrho}\ldots\delta^{\alpha}_{\omega}\delta^{\beta}_{\sigma}\delta^{\pi}_{\gamma} + \ldots + \delta^{\alpha}_{\varrho}\ldots\delta^{\gamma}_{\sigma}\ldots\delta^{\tau}_{\mu}\ldots\delta^{\lambda}_{\nu}\delta^{\beta}_{\omega}\delta^{\pi}_{\gamma} -$$

$$-\delta^{\alpha}_{\varrho}\ldots\delta^{\beta}_{\sigma}\ldots\delta^{\tau}_{\gamma}\ldots\delta^{\lambda}_{\nu}\delta^{\tau}_{\omega}\delta^{\pi}_{\mu} - \ldots - \delta^{\alpha}_{\varrho}\ldots\delta^{\beta}_{\sigma}\ldots\delta^{\tau}_{\mu}\ldots\delta^{\lambda}_{\gamma}\delta^{\gamma}_{\omega}\delta^{\pi}_{\nu}.$$

Then

$$F^{\tau\ldots\lambda\alpha\ldots\beta\omega}_{\varrho\ldots\sigma\mu\ldots\nu\omega} = (U-L)(\delta^{\tau}_{\mu}\ldots\delta^{\lambda}_{\nu}\ldots\delta^{\alpha}_{\varrho}\ldots\delta^{\beta}_{\sigma}),$$

and then we easily find:

$$F^{\tau\ldots\lambda\alpha\ldots\beta\omega}_{\varrho\ldots\sigma\mu\ldots\nu\omega}\Psi^{\varrho\ldots\sigma}_{\tau\ldots\lambda} = (U-L)\Psi^{\alpha\ldots\beta}_{\mu\ldots\nu}. \tag{4.2}$$

On returning to (2.7) we now find using (4.2);

$$\theta^{\mu}_{\mu} = S^{\mu\lambda}_{\mu,\lambda} = \frac{-1}{nK}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,\lambda}}F^{B\mu}_{A\mu}\Psi_B\right)_{,\lambda} = \frac{-1}{K}\left(\frac{\partial \mathscr{L}}{\partial \Psi_{A,\lambda}}\varepsilon\Psi_A\right)_{,\lambda} \tag{4.3}$$

where we have put $\varepsilon = \left(\dfrac{U-L}{n}\right)$. In particular, if $U = L$, $\theta^{\mu}_{\mu} = 0$. If we now consider the cases where $U = L$ ($\varepsilon = 0$) and return to the equation (2.9), making use of (3.2), (2.2) and (4.3), we can rewrite it in the form:

$$E\int_{\bar{V}}\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n}\frac{-P^A\Psi_A}{nP}\right\}dV+\int_V\left\{\frac{1}{nK}\frac{\partial\mathscr{L}}{\partial\Psi_{A,0}}\Psi_{A,0}+\frac{\theta^\mu_\mu}{\varepsilon nP}\right\}dV. \tag{4.4}$$

Then we can use the expression $E = -\int\theta^0_0 dV$ to rewrite (4.4) as:

$$E\left(1+\frac{1}{\varepsilon nP}\right)=\int_{\bar{V}}\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n}-\frac{P^A\Psi_A}{nP}\right\}dV+$$

$$+\int_V\left\{\frac{1}{nK}\frac{\partial\mathscr{L}}{\partial\Psi_{A,0}}\Psi_{A,0}+\theta^a_a\right\}dV. \tag{4.5}$$

We now consider the integrals $\int\theta^a_a dV$ and (3.4). We chose a frame K in which these integrals converge and in which $E \neq 0, \Psi_{A,0} = 0$, and then in this frame we define a constant M_K by:

$$\frac{1}{\varepsilon nP}\int_V\theta^a_a dV = M_K E. \tag{4.6}$$

In this frame then (4.5) and (4.6) yield:

$$E\left(1+\frac{1}{\varepsilon nP}-M_K\right)=\int_V\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n}-\frac{P^A\Psi_A}{nP}\right\}dV \tag{4.7}$$

which if $\left(1+\frac{1}{\varepsilon nP}-M_K\right)\neq 0$ recovers Tolman's theorem. We now continue as before, using firstly (2.2), (2.6), (3.2) and (4.3) to write;

$$\frac{P^A\Psi_A}{nP}=\frac{\mathscr{L}}{nK}+\frac{b^{\mu\lambda}_{\mu,\lambda}}{nP\varepsilon} \tag{4.8}$$

and then (4.5), (4.6) and (4.8) to give:

$$E=\frac{1}{\left(1+\dfrac{1}{\varepsilon nP}-M_K\right)}\int_{\bar{V}}\left\{\frac{-F^{B_0}_{A_0}P^A\Psi_B}{n}+\frac{1}{nK}\left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,0}}\Psi_{A,0}-\mathscr{L}\right)\frac{-1}{\varepsilon nP}b^{\mu\lambda}_{\mu,\lambda}\right\}dV. \tag{4.9}$$

We can here consider the integrand of (4.9) as the (0, 0) component of the complex:

$$-N^\nu_\mu=-\frac{-1}{\left(1+\dfrac{1}{\varepsilon nP}-M_K\right)}\left[\frac{F^{B\nu}_{A\mu}P^A\Psi_B}{n}-\frac{1}{nK}\left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,\nu}}\Psi_{A,\mu}-\delta^\nu_\mu\mathscr{L}\right)+\right.$$

$$\left.+\frac{1}{\varepsilon nP}\delta^\nu_\mu b^{\alpha\beta}_{\alpha,\beta}\right].$$

We then find $N^{\nu}_{\mu,\nu} = \dfrac{b^{\alpha\beta}_{\alpha,\beta\mu}}{\varepsilon nP\left(1+\dfrac{1}{\varepsilon nP}-M_K\right)}$ and if we define M^{ν}_{μ} by:

$$M^{\nu}_{\mu} = \frac{1}{\left(1+\dfrac{1}{\varepsilon nP}-M_K\right)}\left[\frac{F^{B}_{A\mu}P^{A}\Psi_{B}}{n} - \frac{1}{nK}\left(\frac{\partial\mathscr{L}}{\partial\Psi_{A,\nu}}\Psi_{A,\mu}-\delta^{\nu}_{\mu}\mathscr{L}\right) + \right.$$

$$\left. + \frac{1}{\varepsilon nP}(\delta^{\nu}_{\mu}b^{\alpha\beta}_{\alpha,\beta}-b^{\alpha\nu}_{\alpha,\mu})\right] \tag{4.10}$$

then M^{ν}_{μ} satisfies $M^{0}_{0} = N^{0}_{0}$ and (3.8) for a stationary system. Further having selected a suitable frame K in which the conditions of (3.6) hold, then with the M_K evaluated in the frame K have in K:

$$E = -\int_V \theta^{0}_{0}dV = -\int_V M^{0}_{0}dV = -\int_{\bar{V}} M^{0}_{0}dV. \tag{4.11}$$

Again, even if M_K is unknown, (4.10) determines M^{ν}_{μ} to within a constant factor for a given "superpotential" $b^{\nu\lambda}_{\mu}$. Thus we have constructed an infinite number of complexes M^{ν}_{μ} (when $\varepsilon \neq 0$), each one satisfying (3.8). The above procedure is invalidated when $\varepsilon = 0$, and in this case we must return to the method of Section 3.

Since the complex (3.11) is constructed directly from the variational principle without the introduction of arbitrary elements ((3.11) is uniquely determined whereas (4.10) leads to an infinite number of such quantities) we can in some sense consider the M^{ν}_{μ} of (3.11) "canonical". We now give an application of the theory.

5. *Application to General Relativity*

If we translate the above formalism into that of General Relativity, we have $\Psi_A \to g^{\mu\nu}$, $\mathscr{L} \to \mathscr{L}^E$ where $\mathscr{L}^E$ is the Einstein Lagrangian:

$$\mathscr{L}^E = \sqrt{-g}\,g^{\mu\nu}\left[\begin{Bmatrix}\beta\\ \beta\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \nu\mu\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \beta\nu\end{Bmatrix}\begin{Bmatrix}\beta\\ \alpha\mu\end{Bmatrix}\right] \tag{5.1}$$

where $g = \det(g_{\mu\nu})$ and $\begin{Bmatrix}\mu\\ \nu\lambda\end{Bmatrix}$ are the usual Christoffel symbols. Also $L^A \to \sqrt{-g}\,G_{\mu\nu}$. $P^A \to \sqrt{-g}\,T_{\mu\nu} \overset{\text{defn}}{=} \mathscr{T}_{\mu\nu}$ where $G_{\mu\nu}$, $T_{\mu\nu}$ are the Einstein tensor and matter tensor respectively, and $P = -1$, $K = -\chi$ (where χ is the Einstein gravitational constant). Then we have:

$$F^{B\varrho}_{A\sigma} \to F^{\mu\nu\varrho}_{\alpha\beta\sigma} = \tfrac{1}{2}[\delta^{\mu}_{\alpha}\delta^{\varrho}_{\beta}\delta^{\nu}_{\sigma}+\delta^{\varrho}_{\alpha}\delta^{\nu}_{\beta}\delta^{\mu}_{\sigma}+\delta^{\nu}_{\alpha}\delta^{\varrho}_{\beta}\delta^{\mu}_{\sigma}+\delta^{\varrho}_{\alpha}\delta^{\mu}_{\beta}\delta^{\nu}_{\sigma}]. \tag{5.2}$$

We then must choose $n = 2$ so as to make our θ^{ν}_{μ} coincide with the usual canonical complex in General Relativity:

$$\theta^{\nu}_{\mu} = \mathscr{T}^{\nu}_{\mu} + \frac{1}{2\chi}\left(\frac{\partial\mathscr{L}^E}{\partial g^{\alpha\beta}_{,\nu}}g^{\alpha\beta}_{,\mu} - \delta^{\nu}_{\mu}\mathscr{L}^E\right). \tag{5.3}$$

The "canonical" superpotential $S_\mu^{\nu\lambda}$ is the is the Tolman expression, [2], [1] and the "usual" choice of anti-symmetric superpotential $h_\mu^{\nu\lambda}$is the Freud expression [7]:

$$S_\mu^{\nu\lambda} = \frac{1}{\chi}\frac{\partial \mathscr{L}^E}{\partial g_{,\lambda}^{\mu\beta}} g^{\nu\beta} \qquad h_\mu^{\nu\lambda} = \frac{g_{\mu\alpha}}{2\chi\sqrt{-g}}[(-g)(g^{\nu\alpha}g^{\lambda\beta}-g^{\lambda\alpha}g^{\nu\beta})]_{,\beta}. \tag{5.4}$$

Now it can be shown that [1]:

$$S_\mu^{\nu\lambda}-h_\mu^{\nu\lambda} = \frac{1}{2\chi}[\delta_\mu^\lambda(\sqrt{-g}\,g^{\nu\sigma})_{,\sigma}-(\sqrt{-g}\,g^{\nu\lambda})_{,\mu}]. \tag{5.5}$$

It follows from (5.5) that $S_\alpha^{\alpha\lambda} = h_\alpha^{\alpha\lambda}$ and then (5.2) gives:

$$\frac{-1}{K}\left(\frac{\partial \mathscr{L}}{\delta \Psi_{A,\nu}}\Psi_A\right)_{,\mu} \to \frac{1}{\chi}\left(\frac{\partial \mathscr{L}^F}{\partial g_{,\nu}^{\alpha\beta}} g^{\alpha\beta}\right)_{,\mu} = S_{\alpha,\mu}^{\alpha\nu} = h_{\alpha,\mu}^{\alpha\nu}. \tag{5.6}$$

Then from (3.11), (5.4) and (5.6) if follows that:

$$M_\mu^\nu = 2\left[\mathscr{T}_\mu^\nu + \frac{1}{2\chi}\left(\frac{\partial \mathscr{L}^E}{\partial g_{,\nu}^{\alpha\beta}} g_{,\mu}^{\alpha\beta} - \delta_\mu^\nu \mathscr{L}^E\right) - \frac{1}{2}(\delta_\mu^\nu h_{\alpha,\beta}^{\alpha\beta} - h_{\alpha,\mu}^{\alpha\nu})\right] \tag{5.7}$$

where in obtaining (5.7) we have evaluated N_K for "asymptotically Galileian coordinates"; obtaining $N_K = -\frac{1}{2}$. The quantity M_μ^ν is clearly the Moller energy-momentum complex [1]. We note that since we have $\varepsilon \neq 0$ we could have used the method of Section 4. In fact if we put $b_\mu^{\nu\lambda} = h_\mu^{\nu\lambda}$ in (4.10) we obtain M_μ^ν as in (5.7), provided we evaluate M_K in "asymptotically Galileian coordinates", (obtaining $M_K = 0$ [4]).

In his original paper, Tolman [2] indirectly obtained $N_K = -\frac{1}{2}$ and his result now reads (from (3.7) or (4.7)):

$$E = -\int_V (\mathscr{T}_0^0 - \mathscr{T}_1^1 - \mathscr{T}_2^2 - \mathscr{T}_3^3)\,dV. \tag{5.8}$$

The expression (5.8) was first given by Nordström [8] but some doubt has been cast upon his proof [2]. A derivation of (5.8) differing from that of Tolman has been given by Papapetrou [9].

6. *Conclusion*

We have constructed an infinite number of energy-momentum complexes all satisfying (3.8). Of this infinite set, the one which may reasonably be called canonical turns out to be the Moller complex. These complexes arise naturally in an attempt to remedy the above mentioned criticism of Tolman's theorem. All of the constructed complexes M_μ^ν agree for the energy of a stationary system in all frames where $\Psi_{A,0} = 0$, and agree with the energy constructed from θ_μ^ν in the class of frames satisfying the conditions (3.6). The

[4] It is worthwhile to point out that in his original paper, Moller [1] constructed M_μ^ν at first to within a constant factor, this factor being determined when Moller equated $-\int M_0^0\,dV$ and $-\int \theta_0^0\,dV$ for asymptotically Galileian coordinates. This ties up with the fact that the constant factors in (3.11) and (4.10) are determined when M_K and N_K are given.

interpretation (if any) of the conditions $(1+N_K) \neq 0$, $\left(1+\frac{1}{\varepsilon n P}-M_K\right) \neq 0$, is not clear. As seen above they are satisfied in General Relativity, in an "asymptotically Galileian frame".

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