THE PROBLEM OF EXISTENCE OF A SCATTERING S-MATRIX. III

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Instead of describing the scattering processes in terms of asymptotic fields at minus infinity and plus infinity in time, we express them in terms of ingoing and outgoing (concentric and excentric) waves from and to infinity in space. This involves a quite different boundary problem as compared to the traditional initial-condition-problem. The system consisting of fields streaming in (out) from (to) spatial infinity is not a well isolated physical system, hence a description in terms of a state obeying a Schrödinger equation is not applicable. The scattering problem is soluble in the Heisenberg Picture whereby the necessary boundary informations concern the ingoing fields crossing the surface of the domain. This type of information is subject to an uncertainty relation between energy and time.

1. *Introduction*

In the parts I and II the traditional approach to the scattering problems by means of an S-matrix derivable from the quantum field theory, was criticised [1]. In order to see that the traditional approach is, from the very starting point, incorrect and incompatible with the principles of quantum theory, it is enough to discuss the simplest case of interaction-free systems (fields). In this case every traditional S-operator formalism yields a unit operator $S = 1$ and this was regarded as an obvious and correct result with the explanation that in the interaction-free case "nothing happens" because there is no scattering at all. But is it really so? If the initial state at a remote past consists in some wave packets then these packets undergo, in general, a change and, in the course of time, become more and more smeared out so that the final state at a far distant future is by no means identical with the initial state. The operator describing the spontaneous dissolving of wave packets in the course of time is, according to quantum theory, obviously exp $iH_0(t_2-t_1)$ where H_0 denotes the interaction-free Hamiltonian. If one tries to perform the limit transitions $t_1 \to -\infty$, $t_2 \to \infty$, one does not get as a results a unit operator (as suggested by the S-matrix formalism) but the limit transition simply does not exist.

From the above discussion it does not follow that in the framework of quantum field theory no scattering matrix can be defined and transitions between asymptotic states

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cannot be computed at all. The above discussion only shows that the traditional approach to scattering problems in terms of asymptotic states defined for $t \to \pm\infty$ was particularly clumsy and that the scattering problem should be reformulated and expressed in more physical terms. We have to reject the concept of asymptotic states at $t \to \pm\infty$ as inadequate and replace them by a description of the scattering process in terms of stationary concentric waves coming in from spatial infinity and excentric ones spreading out towards spatial infinity.

2. *Scattering of classical waves*

In classical field theory it is possible to replace the usual initial conditions[1] for the whole space at $t = t_0$ by another type of initial-boundary conditions determining uniquely the solution in a finite volume V and in an arbitrary time interval. Namely, it is possible to assume the usual initial conditions at $t = t_0$ in a finite domain V and to prescribe, as a boundary condition, the flux of the field through the surface elements of V into its interior for $t > t_0$. From a four-dimensional point of view, we have to do with a boundary problem on a three-dimensional hypersurface consisting of a space-like part Σ_0 and a cy-

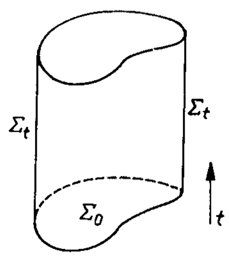

Fig. 1

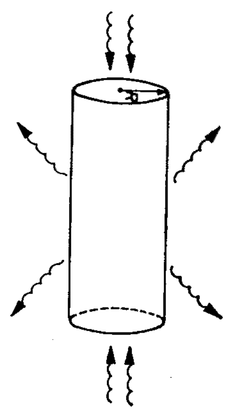
Fig. 2

lindrical timelike hypersurface Σ_t (Fig. 1). The information about the initial values of the field at Σ_0 together with the information about the flux of the field from the exterior through Σ_t into the interior of the cylinder determines the solution in it. Consequently, also the flux of the field streaming out of the interior of the cylinder will be determined.

The initial hypersurface Σ_0 may be shifted towards minus (or plus) infinity. Assuming for $t_0 \to -\infty$ a vacuum state of the field in the domain V, the field equations determine the solution in V for any t provided the ingoing waves through the boundary of V are given (for any t). Thus, it is possible to determine the field uniquely in the interior of the cylinder extending from $-\infty$ to ∞. In this case the solution is given in the form of re-

[1] Initial conditions are usually supplemented by the requirement of vacuum at infinity, *i. e.* of a sufficiently rapid decrease of the field for points going off to infinity in space-like directions so that the total flux through an infinitely remote surface vanishes and the system may be regarded as closed.

tarded potentials. If, on the other hand, the field would satisfy the condition of becoming a vacuum in V for $t \to \infty$, the solution would be given in the form of advanced potentials.

Inasmuch as the field equations supplemented by prescribing the waves ingoing into the domain V (together with the additional information that initially there was vacuum) determine the solution uniquely, we may compute the outgoing part of the field (from the knowledge of the ingoing waves crossing the boundary).

If a real scalar field φ is scattered by an external scatterer of a small extension (or by an external short ranged potential) we may assume the domain V to be a sphere with a large radius R with the target situated at its centre. The waves ingoing and outgoing across the surface of the sphere may be represented asymptotically in the form

$$\varphi = \varphi^{\text{in}} + \varphi^{\text{out}} \tag{1}$$

where

$$\varphi^{\text{in}} = \frac{1}{r} \sum_{k>0} f(k, \vartheta, \varphi) e^{i(kr+\omega t)} + \text{c.c.} \tag{1'}$$

and

$$\varphi^{\text{out}} = \frac{1}{r} \sum_{k>0} g(k, \vartheta, \varphi) e^{i(kr-\omega t)} + \text{c.c.} \tag{1''}$$

with $\omega = \sqrt{k^2+m^2}$. The outgoing waves result in consequence of both the spontaneous spread out of the ingoing waves and their interaction with the scatterer. With the knowledge of the ingoing waves the field may be computed from the field equation in the whole domain V. Consequently, the outgoing part of the field will be also determined.

If, on the other hand, we have to do with the problem of two colliding wave packets propagating along, say, the z-axis we may, more conveniently, choose the domain according to the symmetry of the problem: take a very long cylinder (Fig. 2) and use cylindrical coordinates ϱ, φ, z. The ingoing waves come from minus and plus infinity along the z-axis and their initial intensity at $z = \pm Z$ may be assumed to be different from zero only in the immediate neighbourhood of the z-axis ($\varrho \ll R$). If the length of the cylinder is very large ($Z \gg R$) the outgoing waves are, practically, only those crossing the cylindrical surface S. The asymptotic outgoing wave may be assumed to be of the form

$$\varphi^{\text{out}}_{(\varrho,\varphi,z)} = \frac{1}{\sqrt{\varrho^2+z^2}} \sum_{k_\varrho>0} \sum_{k_z} g(k_\varrho, k_z, \varphi) e^{i(k_\varrho \varrho + k_z z - \omega t)} \tag{2}$$

at the cylindrical surface $\varrho = R$ where

$$\omega = \sqrt{k_\varrho^2 + k_z^2 + m^2}. \tag{2'}$$

With the knowledge of the ingoing waves the outgoing waves (the function g) may be computed. It may be assumed that the intensity of the ingoing waves is small and constant in time. Then we have to do with a stationary scattering process: the flux across the cylindrical surface $\varrho = R$ will also be constant in time (if averaged over $z \sim 1/m$).

3. *The scattering problem in quantum field theory*

For reasons of correspondence between classical and quantum theories it must be possible to formulate the scattering problem in a similar way also in quantum field theory. However, trying to formulate the quantum analogue of the above described classical problem we are faced with a situation never yet considered in quantum theories. Hitherto the states were referred exclusively to space-like hypersurfaces. However, the problem of determining the state at an arbitrary space-like hypersurface if the state at Σ_0 is given, is not a pure initial-condition problem because the concept of the state applies to isolated physical systems which means (among others) the fact that the system is localised in a finite region beyond which there is vacuum. Thus, the traditional problem of initial conditions is, in fact, a mixed problem of initial and boundary conditions which may be pictured with the help of Fig. 1 as follows: Besides the information about the state at Σ_0 there is an additional information about the system, *viz.* that there is no flux of the field across the cylindrical part Σ_t of the boundary. The hypersurface Σ_t was (tacitly) removed to infinity and the requirement of a vanishing flux across it was replaced by the requirement that the state of the field sufficiently quickly approaches a vacuum state for points going off to infinity is space-like directions so as to avoid any streaming of energy, momentum, angular momentum and charge, to or from infinity.

On the other hand, the initial-boundary problem for scattering phenomena is quite different: contrary to the assumption of the absence of a flux across Σ_t (or of a vacuum at space-like infinity) we have to admit that there is, indeed, a flux of ingoing and outgoing waves across Σ_t whereby, in order to determine the field in the whole interior of the domain pictured in Fig. 1, it is necessary and sufficient to specify (besides the initial state at Σ_0) the ingoing part of the field at Σ_t. The state at Σ_0 is specified, as usually, by pointing out the eigenvalues of a complete set of observables at Σ_0 whereas at the time-like part of the boundary Σ_t we need to prescribe the eigenvalues of an incomplete set of observables: only the numbers and detailed characteristics of the ingoing particles. The information about the numbers of outgoing particles is not dynamically independent of the former but is determined by the dynamics of the physical system.

In connection with the above-described situation there appears the following problem: how is it possible that measurements determining the number and detailed characteristics of particles ingoing into the 3-dimensional domain V through its surface S do not interfere with each other inasmuch as, obviously, they constitute a set of measurements at a time--like hypersurface? First of all let us recall that measurements of two events situated one inside the light cone of the other need not necessarily disturb each other. The question whether they do or do not disturb each other depends upon the type of measurements to be performed.[2]

It is intuitively clear that a measurement aiming at a determination of the numbers and detailed characteristics of the ingoing particles but yielding no information about

[2] In particular, two subsequent measurements of the same set of observables performed soon one after the other do not disturb each other and have the character of measurements only confirming the previous results according to the axiom of reproducibility of measurements.

w h e n the respective particles cross the surface of the domain V, is possible without running into an internal contradiction because in this case the problem of a mutual disturbance of different acts of measurements performed at different time instances does not appear at all. A contradiction could only emerge if we tried to determine the temporal details of the process of streaming in (or out) of particles across the surface. A state where any information concerning the question w h e n this or that particle crosses the surface is lacking, denotes a stationary state with well defined energies of the respective particles.

Now, it is not difficult to guess that it is also possible to obtain some information about the temporal details of the streaming in (or out) of particles a t t h e c o s t of a loss of information about their energies. Thus, an uncertainty relation between time and energy (whose existance was suspected since the days of birth of quantum mechanics) appears, at last, in a concrete shape: it is connected with the problem of measurements on time-like hypersurfaces: Only these informations about particles crossing the cylindrical hypersurface Σ_t (whose normal is space-like) are attainable for which

$$\Delta E \cdot \Delta t > 1 \tag{3}$$

where ΔE means the inexactitude of the energy of the incoming (outgoing) particle and Δt the inexactitude of the time instant when the particle crosses the surface of the domain in question.

A relation between the ingoing and outgoing fields cannot be obtained from the Schrödinger equation because the latter is only applicable to isolated physical systems whereas the field in the interior of V does not constitute a closed system *i. e.* is not isolated from that in the exterior of V. Nevertheless, the operators describing the ingoing and outgoing waves must be related by a unitary transformation

$$\varphi^{\text{out}} = S\varphi^{\text{in}}S^{-1} \tag{4}$$

provided no other sources of the outgoing waves are present except for the mutual interaction and diffraction of the ingoing wave packets. The S-operator should be independent of any particular assumptions about what actually happened at the boundary. But, of course, the S-operator will be most conveniently computed if the scattering process is stationary.

The operator S defined by (4) is an analogue of the traditional S-operator but, in view of a different definition of the ingoing and outgoing fields, it is not identical with the traditional operator (1). In the traditional approach the scattering operator mapped the whole Hilbert space upon itself. Now, inasmuch as φ^{in} and φ^{out} span two different subspaces of the Hilbert space, the S-operator maps one of them upon the other.

REFERENCES

[1] J. Rayski, *Acta Phys. Polon.*, **A37**, 269 (1970); **B1**, 31 (1970).