ON THE STRONG COUPLING MODEL WITH ISOSPIN-HYPERCHARGE SYMMETRY

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The static baryon model with the symmetry group $SU(2)_J \otimes SU(2)_I \otimes U(1)_Y$ is investigated in the strong coupling limit. In the algebraic (or — equivalently — in bootstrap) formulation of that model the reduced meson-baryon coupling constants can be found from the sum rules involving the static angular momentum and isospin crossing matrices, both proportional to $6j$-symbols. This fact enables one to use the identities between $3nj$-symbols of the group SU(2) in order to find two types of solutions (Eqs (10) and (23)) for the coupling constants and the isobar states spectrum.

In this note we want to present several solutions to the static baryon model with internal symmetry $SU(2)_I \otimes U(1)_Y$, where I is the isospin and Y the hypercharge. We will use the static model in the strong coupling limit as formulated first in reference [1]; to a large extent this formulation is equivalent to the static bootstrap model (see the discussion in [2], where further references are given).

We shall consider the scattering of mesons (*e.g.* of pseudoscalar mesons) $\Pi_{\iota\lambda v}$ on the static baryons B_{ijy}, where i, j, y (ι, λ, v) are the isospin, spin and hypercharge of the baryon (meson), respectively. The s-channel scattering process is

$$B_{ijyr}+\Pi_{\iota\lambda v\varrho} \rightarrow B_{i'j'y'r'}+\Pi_{\iota'\lambda'v'\varrho'}, \tag{1}$$

while the u-channel scattering is

$$B_{ijyr}+\overline{\Pi}_{\iota'\lambda'-v'\bar{\varrho}'} \rightarrow B_{i'j'y'r'}+\overline{\Pi}_{\iota\lambda-v\bar{\varrho}};$$

the difference between them is that initial and final mesons are interchanged and replaced by their antiparticles, with the reversed sign of hypercharge. Because we cannot exclude in advance the possibility that there is more than one baryon (resp. meson) with quantum numbers i, j, y (resp. ι, λ, v), we add the repetition label r (resp. ϱ, or for antimeson $\bar{\varrho}$) in order to distinguish between otherwise similar particles. The interaction is assumed to be invariant with respect to the symmetry group $K = SU(2)_J \otimes SU(2)_I \otimes U(1)_Y$.

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The spectrum of the baryon states (isobars) and the reduced meson-baryon coupling constants are to be found from the set of equations

$$C_{su}\gamma_u = \gamma_s. \tag{2}$$

Here the total u-to-s channel crossing matrix C_{su} is equal to the direct product of the spin and isospin crossing matrices: $C_{su} = C^J_{su} \otimes C^I_{su}$, where C^J_{su} and C^I_{su} have the following matrix elements, respectively

$$C^{(j+\lambda\to j'+\lambda')}_{JJ'} = (-1)^{2J+\lambda+\lambda'}(2J'+1)\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix} \tag{3}$$

(λ and λ' take integer values — they are, for instance, orbital angular momenta of pseudoscalar mesons),

$$C^{(i+\iota+i+\iota')}_{II'} = (-1)^{2I+\varphi_\iota+\varphi_{\iota'}}(2I'+1)\begin{Bmatrix} i & \iota & I \\ i' & \iota' & I' \end{Bmatrix}, \tag{4}$$

where $\varphi_\iota = \iota(2\iota)$ for ι integer (half-integer) and $\{:::\}$ is the $6j$-symbol of the group SU(2). The upper indices of the matrices C in Eqs (3) and (4) specify the s-channel reaction. $\gamma_s(\gamma_u)$ in Eq. (2) is the sum of residua of one-baryon poles in the partial wave scattering amplitude, which follow from the s-channel (resp. u-channel) intermediate states with quantum numbers I, J, Y (resp. I', J', Y'):

$$(\gamma_s)_{IJY} = \sum_R \gamma^{ijyr}_{IJYR}(\iota\lambda v\varrho)^* \gamma^{i'j'y'r'}_{IJYR}(\iota'\lambda' v'\varrho')$$

$$(\gamma_u)_{I'J'Y'} = \sum_{R'} \gamma^{ijyr}_{I'J'Y'R'}(\iota'\lambda' - v'\bar{\varrho}')^* \gamma^{i'j'y'r'}_{I'J'Y'R'}(\iota\lambda - v\bar{\varrho}). \tag{5}$$

In the last equations the reduced coupling constant $\gamma^{ijyr}_{IJYR}(\iota\lambda v\varrho)$ for the decay $B_{IJYR} \to$ $\to B_{ijyr} + \Pi_{\iota\lambda v\varrho}$ is defined as follows

$$\langle II_3JJ_3YR|A^{\iota\lambda v\varrho}_{\iota_3\lambda_3}|ii_3jj_3yr\rangle = \begin{bmatrix} i & \iota & I \\ i_3 & \iota_3 & I_3 \end{bmatrix}\begin{bmatrix} j & \lambda & J \\ j_3 & \lambda_3 & J_3 \end{bmatrix}\delta_{y+v,Y}\gamma^{ijyr}_{IJYR}(\iota\lambda v\varrho)^*. \tag{6}$$

$A^{\iota\lambda v\varrho}_{\iota_3\lambda_3}$ are the commuting meson source operators acting in the isobar space, $[:::]$ denotes the C.-G. (Clebsch-Gordan) coefficient of the group SU(2), $\delta_{y+v,Y}$ results from the hypercharge conservation. In addition to Eq. (2) the reduced coupling constants have to satisfy the vertex crossing (or vertex symmetry) relation

$$\gamma^{ijyr}_{IJYR}(\iota\lambda v\varrho) = (-1)^{i+j-I-J+\Phi_\iota}\left[\frac{(2i+1)\,(2j+1)}{(2I+1)\,(2J+1)}\right]^{1/2}\gamma^{IJYR}_{ijyr}(\iota\lambda - v\bar{\varrho})^* \tag{7}$$

with $\Phi_\iota = \varphi_\iota + \iota = 0\ (-\iota)$ for ι integer (half-integer).

We shall look for the solutions of Eq. (2) by exploiting the identities connecting the $3nj$-symbols of the group SU(2). This method was used first in the strong coupling model of P-wave pion scattering [3] and was later generalized in reference [4]. The main idea of such an approach is the following: our aim is to find the matrix representation of the algebra of operators $A_q^{(\kappa)}$ which are the irreducible tensor operators with respect to a symmetry group K. The individual (physical) properties of the given tensor operator $A_q^{(\kappa)}$ are fully characterized by its reduced matrix elements. Hence by separating out these elements *via* the Wigner-Eckart theorem like in Eq. (6) and collecting the C.-G. coefficients (which describe the geometrical properties of the tensor operator) in the form of a $6j$-symbol, for instance (in the SU(2) case), one reduces the commutator of the type $[A_q^{(\kappa)}, A_{q'}^{(\kappa')})] = B_{qq'}$, to a set of equations for the reduced matrix elements of operators A, B. When $A_q^{(\kappa)}$'s commute, as is the case in the strong coupling limit, these equations are just of the type of Eq. (2).

As a first simple example we shall apply the well known identity between the $3j$-symbols (or equivalently C.-G. coefficients) and a $6j$-symbol (*e.g.* [5], Eq. (21.29)), which can be written as follows

$$\sum_{J'} (-1)^{2J}(2J'+1)\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix}(2J+1)^{-1/2}\begin{bmatrix} j & \lambda' & J' \\ m & -\mu' & M' \end{bmatrix}(2J+1)^{-1/2}\begin{bmatrix} j' & \lambda & J' \\ m' & -\mu & M' \end{bmatrix} =$$

$$= (-1)^{\lambda-\mu+\lambda'-\mu'}(2J+1)^{-1/2}\begin{bmatrix} j & \lambda & J \\ m & \mu & M \end{bmatrix}(2J+1)^{-1/2}\begin{bmatrix} j' & \lambda' & J \\ m' & \mu' & M \end{bmatrix}. \tag{8}$$

Combining Eq. (8) with the similar identity containing $\left\{\begin{smallmatrix} i & \iota & I \\ i' & \iota' & I' \end{smallmatrix}\right\}$ and using the definitions (3) and (4), we can write

$$\sum_{I'J'} C_{II'}^{(i+\iota\to i'+\iota')}C_{JJ'}^{(j+\lambda\to j'+\lambda')}[(2I'+1)(2J'+1)]^{-1/2}\begin{bmatrix} i & \iota' & I' \\ n & -\nu' & N' \end{bmatrix}\begin{bmatrix} j & \lambda' & J' \\ m & -\mu' & M' \end{bmatrix}\times$$

$$\times[(2I'+1)(2J'+1)]^{-1/2}\begin{bmatrix} i' & \iota & I' \\ n' & -\nu & N' \end{bmatrix}\begin{bmatrix} j' & \lambda & J' \\ m' & -\mu & M' \end{bmatrix} =$$

$$= (-1)^{\Phi_\iota-\nu-\mu+\Phi_\iota'-\nu'-\mu'}[(2I+1)(2J+1)]^{-1/2}\begin{bmatrix} i & \iota & I \\ n & \nu & N \end{bmatrix}\begin{bmatrix} j & \lambda & J \\ m & \mu & M \end{bmatrix}[(2I+1)(2J+1)]^{-1/2}\times$$

$$\times\begin{bmatrix} i' & \iota' & I \\ n' & \nu' & N \end{bmatrix}\begin{bmatrix} j' & \lambda' & J \\ m' & \mu' & M \end{bmatrix}. \tag{9}$$

This equation is identical to Eq. (2) if one takes the reduced coupling constant in the form

$$\gamma_{IJNM}^{ijnm}(\iota\lambda\nu\mu) = \left[\frac{(2i+1)(2j+1)}{(2I+1)(2J+1)}\right]^{1/2}\begin{bmatrix} i & \iota & I \\ n & \nu & N \end{bmatrix}\begin{bmatrix} j & \lambda & J \\ m & \mu & M \end{bmatrix}\gamma(\iota\lambda\nu\mu) \tag{10}$$

(we neglect for the moment the dependence on hypercharges). The unknown factors depending only on the meson parameters have to satisfy the additional restriction following from Eq. (2)

$$\gamma(\iota'\lambda'-\nu'-\mu')^*\gamma(\iota\lambda-\nu-\mu) = (-1)^{\Phi_\iota-\nu-\mu+\Phi_\iota'-\nu'-\mu'}\gamma(\iota\lambda\nu\mu)^*\gamma(\iota'\lambda'\nu'\mu'). \tag{11}$$

The free parameters (N, M), (n, m) play the role of repetition indices R, r and similarly $(\nu, \mu) \equiv \varrho$, $(-\nu, -\mu) \equiv \bar{\varrho}$; the sums over R, R' in Eq. (9) each amount to only one term due to the properties of C.-G. coefficients. The vertex symmetry condition (7) determines the normalization factor $[(2i+1)(2j+1)]^{1/2}$ already included in Eq. (10) and moreover imposes the requirement

$$\gamma(\imath\lambda-\nu-\mu) = (-1)^{\Phi_\imath-\nu-\mu}\gamma(\imath\lambda\nu\mu)^*, \tag{12}$$

under which Eq. (11) is automatically fulfilled. Therefore the coupling constants (10) with the constraint (12) give the particular solution of Eq. (2). The baryon spectrum so found contains the infinite baryon sets B_{IJNM}, each characterized by parameters (N, M), with $I \geqslant |N|$, $J \geqslant |M|$. The values of I and J are uncorrelated and hence the solution (10) has too many particles to be physically interesting.

We shall try to find more economical solutions to Eq. (2) by taking into consideration the special examples of the reaction (1). In order to simplify the treatment we shall bypass for the time being the complications due to the hypercharge quantum number by assuming that mesons are nonstrange, that is $v = 0 = v'$ (and hence $\imath$, $\imath'$ are integer), and therefore all the baryons have an equal hypercharge $y = y' = Y = Y'$. This hypercharge label of the coupling constants will be omitted (in fact, as we will show later, the coupling constant does not depend on the common baryon hypercharge).

Let us set in reaction (1) $\imath = \lambda$, $\imath' = \lambda'$; then the relation (A.1) from Appendix A could be identified with Eq. (2), if we adopt the following form for the coupling constants

$$\gamma^{ij}_{IJ}(\lambda)_v = (-1)^{i+J+v+\lambda}[(2i+1)(2j+1)]^{1/2}\begin{Bmatrix} I & J & v \\ j & i & \lambda \end{Bmatrix}\gamma(\lambda)_v \tag{13}$$

with the real parameter $\gamma(\lambda)_v$. The normalization and the phase factor in Eq. (13) are taken in order to fulfil the condition (7). The label v takes on the values 0,1/2, 1, 3/2, ...; for each value of v we have an infinite isobar series with the baryon spin and isospin restricted by the triangle inequality $|I-v| \leqslant J \leqslant I+v$. The simplest baryon series one obtains for $v = 0$, when $i = j$, $I = J$ and the coupling constants (13) amount to

$$\gamma^j_J(\lambda) = \left(\frac{2j+1}{2J+1}\right)^{1/2}\gamma(\lambda).$$

Now let us consider the scattering of mesons with arbitrary $\imath\lambda$, $\imath'\lambda'$ on the baryons belonging to this simplest series; that is, we put in Eq. (1) $i = j$, $i' = j'$. In order to construct a solution of Eq. (2) for that case we use an identity proved by Wigner [6], which in our notation and with definitions (3) and (4) can be written as follows

$$\sum_{I'J'} C^{(j+\imath\to j'+\imath')}_{II'} C^{(j+\lambda\to j'+\lambda')}_{JJ'} \begin{Bmatrix} I' & J' & V \\ \lambda' & \imath' & j \end{Bmatrix}\begin{Bmatrix} I' & J' & V \\ \lambda & \imath & j' \end{Bmatrix} =$$

$$= (-1)^{\imath+\imath'+\lambda+\lambda'}\begin{Bmatrix} I & J & V \\ \lambda & \imath & j \end{Bmatrix}\begin{Bmatrix} I & J & V \\ \lambda' & \imath' & j' \end{Bmatrix}. \tag{14}$$

Comparison of Eqs (14) and (2) shows that the solution of Eq. (2) in this case is

$$\gamma^{jj}_{IJ}(\imath\lambda)_V = (-1)^{j+I+V+\lambda}\sqrt{2j+1}\begin{Bmatrix} I & J & V \\ \lambda & \imath & j \end{Bmatrix}\gamma(\imath\lambda)_V, \tag{15}$$

with the restriction

$$\gamma(\imath'\lambda')^*_V\gamma(\imath\lambda)_V = (-1)^{\imath+\lambda+\imath'+\lambda'}\gamma(\imath\lambda)^*_V\gamma(\imath'\lambda')_V. \tag{16}$$

Next we assume that in the reaction (1) only the initial baryon belongs to the series $v = 0$ and hence satisfies $i = j$. To find the solution of Eq. (2) for that case we can exploit an identity between $6j$ and $9j$-symbols (*e.g.* Eq. (24,37) in [5]), which in our notation is

$$\sum_{I'J'}(2I'+1)(2J'+1)(-1)^{I'+\lambda'+\lambda-I-v'+2\imath'}\begin{Bmatrix} j & \imath & I \\ i' & \imath' & I' \end{Bmatrix}\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix}\begin{Bmatrix} I' & J' & V' \\ \lambda' & \imath' & j \end{Bmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & V \\ I' & J' & V' \end{Bmatrix} =$$

$$= \begin{Bmatrix} I & J & V \\ \lambda & \imath & j \end{Bmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & V' \\ I & J & V \end{Bmatrix}. \tag{17}$$

With the help of Eqs (3), (4) and (15) the above relation reads

$$\sum_{I'J'} C^{(j+\imath\to i'+\imath')}_{II'}C^{(j+\lambda\to j'+\lambda')}_{JJ'}\gamma^{jj}_{I'J'}(\lambda')^*_{V'}(-1)^{v'-V-V'}\begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & V \\ I' & J' & V' \end{Bmatrix}\gamma(\imath\lambda)_V =$$

$$= \gamma^{jj}_{IJ}(\lambda)^*_V\begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & V' \\ I & J & V \end{Bmatrix}\gamma(\imath'\lambda')_{V'}, \tag{18}$$

where the constants $\gamma(\imath\lambda)_V$ have to satisfy a constraint being the generalization of Eq. (16) with $V \neq V'$

$$\gamma(\imath'\lambda')^*_{V'}\gamma(\imath\lambda)_V = (-1)^{\imath+\lambda+\imath'+\lambda'}\gamma(\imath\lambda)^*_V\gamma(\imath'\lambda')_{V'}. \tag{19}$$

Eq. (18) suggests that the coupling constant describing the general case of transition between baryons belonging to sets with different parameters $v \neq V$ should be expressible in the form

$$\gamma^{ijv}_{IJV}(\imath\lambda\varphi) = \gamma^v_V(\varphi)\begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix}\gamma^{ijv}\gamma(\imath\lambda\varphi) \tag{20}$$

with the unknown factor $\gamma^v_V(\varphi)$ which cancels out from both sides of Eq. (18) and has the property that $\gamma^v_V(\varphi)$ is proportional to $(-1)^{v-V-\varphi}\gamma^V_v(\varphi)$.

Our guess (20) is confirmed if one rewrites the same identity in a different way, adopting it to the case when in reaction (1) $\imath' = \lambda'$ (and in general $i \neq j$, $i' \neq j'$). Multiplying both sides by $[(2i+1)(2j+1)(2i'+1)(2j'+1)]^{1/2}\gamma(\lambda')$ with real $\gamma(\lambda')$ and using definitions (3) and (4), we can write the identity as follows

$$\sum_{I'J'} C_{II'}^{(i+\imath\to i'+\lambda')} C_{JJ'}^{(j+\lambda\to j'+\lambda')} \gamma_{I'J'}^{ij}(\lambda')_v^* (-1)^{v-v'+\varphi} \begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & \varphi \\ I' & J' & v \end{Bmatrix} [(2i'+1)(2j'+1)]^{1/2} =$$

$$= (-1)^{\imath+\lambda} \gamma_{IJ}^{i'j'}(\lambda)_{v'} \begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & v' \end{Bmatrix} [(2i+1)(2j+1)]^{1/2}, \tag{21}$$

where $\gamma_{IJ}^{ij}(\lambda)_v$ is given by Eq. (13) with $\gamma(\lambda)$ replacing $\gamma(\lambda)_v$, which is an indication that the constant $\gamma(\lambda)_v$ in Eq. (13) in fact do not depend on v. Eq. (21) has the form of Eq. (2), and similarly as Eq. (18), implies the validity of the supposed general formula (20) for the coupling between baryons with $i \neq j$, $I \neq J$, $v \neq 0 \neq V$. The factor depending on meson parameters only should satisfy $\gamma(\imath\lambda\varphi)^* = (-1)^{\imath+\lambda}\gamma(\imath\lambda\varphi)$ and the factor $(-1)^{v-v'+\varphi}$ in Eq. (21) is due to the phase difference between factors $\gamma_v^{v'}(\varphi)^*$ and $\gamma_{v'}^{v}(\varphi)$ cancelled out from both sides of Eq. (21).

The simplest choice for the quantity $\gamma_V^v(\varphi)$ depending on 3 parameters which fulfil the vector addition rule is $3j$-symbol of SU(2); all the preceding results strongly indicate that there should exist an identity connecting $3j$, $6j$ and $9j$-symbols, with Eqs (A.1), (14) and (21) as its special cases. In fact there is such an identity (B.1) which we prove in Appendix B; it can be rewritten in terms of C.-G. coefficients rather than $3j$-symbols and with the general crossing matrices (3) and (4) in the following way

$$\sum_{I'J'} C_{II'}^{(i+\imath\to i'+\imath')} C_{JJ'}^{(j+\lambda\to j'+\lambda')} \sum_{V'} \begin{bmatrix} v & \varphi' & V' \\ v_3 & -\varphi_3' & V_3' \end{bmatrix} \begin{Bmatrix} i & j & v \\ \imath' & \lambda' & \varphi' \\ I' & J' & V' \end{Bmatrix} \begin{bmatrix} v' & \varphi & V' \\ v_3' & -\varphi_3 & V_3' \end{bmatrix} \begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & \varphi \\ I' & J' & V' \end{Bmatrix} =$$

$$= (-1)^{\varphi_\imath+\lambda+\varphi-\varphi_3+\varphi_\imath'+\lambda'+\varphi'-\varphi'_3} \sum_{V} \begin{bmatrix} v & \varphi & V \\ v_3 & \varphi_3 & V_3 \end{bmatrix} \begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix} \begin{bmatrix} v' & \varphi' & V \\ v_3' & \varphi_3' & V_3 \end{bmatrix} \begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & \varphi' \\ I & J & V \end{Bmatrix}. \tag{22}$$

The above equation has the form of Eq. (2) for the general scattering process (1) with $v \neq 0 \neq v'$ (and $\imath$, $\imath'$ integer or half-integer) and provides the following solution for the coupling constants

$$\gamma_{IJYVV_3}^{ijyvv_3}(\imath\lambda v\varphi\varphi_3) = [(2i+1)(2j+1)(2v+1)]^{1/2} \begin{bmatrix} v & \varphi & V \\ v_3 & \varphi_3 & V_3 \end{bmatrix} \begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix} \gamma(\imath\lambda\varphi\varphi_3;\, Yvy). \tag{23}$$

The repetition labels are $R \equiv (V, V_3)$, $r \equiv (v, v_3)$, $\varrho \equiv (\varphi, \varphi_3)$ (and in Eq. (22) $\bar{\varrho} \equiv (\varphi, -\varphi_3)$). The constants (23) obey Eq. (22) with the restriction

$$\gamma(\imath'\lambda'\varphi' - \varphi_3';\, Y' - v'y)^* \gamma(\imath\lambda\varphi - \varphi_3;\, Y' - vy') =$$

$$= (-1)^{\varphi_\imath+\lambda+\varphi-\varphi_3+\varphi_\imath'+\lambda'+\varphi'-\varphi'_3} \gamma(\imath\lambda\varphi\varphi_3;\, Yvy)^* \gamma(\imath'\lambda'\varphi'\varphi_3';\, Yv'y'). \tag{24}$$

The vertex relation (7) determines the factor standing before the C.-G. coeficient in Eq. (23) and leads to the condition

$$\gamma(\iota\lambda\varphi-\varphi_3;\ y-vY) = (-1)^{\iota+\lambda+\varphi+\varphi_3+\Phi_\iota}\gamma(\iota\lambda\varphi\varphi_3;\ Yvy)^*. \tag{25}$$

Putting Eqs (24) and (25) together, we have

$$\frac{\gamma(\iota'\lambda'\varphi'\varphi_3';\ yv'Y')}{\gamma(\iota'\lambda'\varphi'\varphi_3';\ Yv'y')} = \frac{\gamma(\iota\lambda\varphi\varphi_3;\ Yvy)^*}{\gamma(\iota\lambda\varphi\varphi_3;\ y'vY')^*}. \tag{26}$$

Setting $v = 0$ in Eq. (26) and making use of the hypercharge conservation: $y = Y, y' = Y'$, we see that the left-hand side in the above relation becomes unity, which results in

$$\gamma(\iota\lambda\varphi\varphi_3;\ y'0y') = \gamma(\iota\lambda\varphi\varphi_3;\ y0y). \tag{27}$$

This means that the free parameters $\gamma(\iota\lambda\varphi\varphi_3;\ y0y)$ in fact do not depend on the baryon hypercharge y.

Thus we have been successful in proving that the coupling constants (23) with the conditions (25), (26) provide the solution to Eq. (2) for the reaction (1) and describe the decay $B_{IJYVV_3} \to B_{ijyvv_3} + \Pi_{\iota\lambda v\varphi\varphi_3}$; if $v = 0$, the coupling constant does not depend on $y = Y$. When $v = 0$, we get from Eq. (23) the special cases found earlier: setting $\varphi_3 = 0 = \varphi$ gives Eq. (13) with $v = V$ and $\gamma(\lambda)_v = \gamma(\lambda\lambda 00)/(2\lambda+1)^{1/2}$ (and hence $\gamma(\lambda)_v$ indeed does not depend on v); when $v_3 = 0 = v$, we obtain Eq. (15) with $\varphi = V$, $\varphi_3 = V_3$ and $\gamma(\iota\lambda)_V = \gamma(\iota\lambda VV_3)/(2V+1)^{1/2}$.

The solution (23) describes the transitions between bands of isobar states B_{IJYVV_3} labelled by integer or half-integer index V_3; each band contains an infinite series of baryons for each value of the parameter V which runs over the range $V \geqslant |V_3|$. As opposed to the solutions (10), the baryons of the solution (23) display the triangular type correlation between I and J: $|I-V| \leqslant J \leqslant I+V$, because otherwise the $9j$-symbol in Eq. (23) would vanish. If we insist that there is only one meson with quantum numbers ι, λ, v which are sufficient to specify the meson state, then the solution will be given by the sum of expressions (23) over all the allowed values of the label φ.

The special cases of the solution (23) for $\varphi_3 = 0$, $v = 0$ were found previously by means of the group-theoretical methods in reference [7] for the case $\iota = 1$, $\varphi = \lambda-1$ and in reference [8] for any ι, φ. The possible physical realisations of the solution (23) as well as the problem of the isobar mass spectrum will be discussed in a future publication.

APPENDIX A

In order to prove the identity

$$(-1)^{2I+2J}\sum_{I'J'}(2J'+1)(2I'+1)\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix}\begin{Bmatrix} i & \lambda & I \\ i' & \lambda' & I' \end{Bmatrix}\begin{Bmatrix} I' & J' & v \\ j & i & \lambda' \end{Bmatrix}\begin{Bmatrix} I' & J' & v \\ j' & i' & \lambda \end{Bmatrix} =$$
$$= \begin{Bmatrix} I & J & v \\ j & i & \lambda \end{Bmatrix}\begin{Bmatrix} I & J & v \\ j' & i' & \lambda' \end{Bmatrix}, \tag{A.1}$$

we first use expansion of an $9j$-symbol in terms of $6j$-symbols (*e.g.* Eq. (24.33) in [5]), which allows us to perform the sum over I' on the left-hand side of Eq. (A.1)

$$\sum_{I'}(2I'+1)\begin{Bmatrix} i & \lambda & I \\ i' & \lambda' & I' \end{Bmatrix}\begin{Bmatrix} I' & J' & v \\ j & i & \lambda' \end{Bmatrix}\begin{Bmatrix} I' & J' & v \\ j' & i' & \lambda \end{Bmatrix} = (-1)^{2I}\begin{Bmatrix} i' & \lambda' & I \\ v & j & i \\ j' & J' & \lambda \end{Bmatrix}.$$

Next, summation over J' with the help of the relation (24.35) (Ref. [5]) yields

$$\sum_{J'}(2J'+1)\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix}\begin{Bmatrix} i' & \lambda' & I \\ v & j & i \\ j' & J' & \lambda \end{Bmatrix} = (-1)^{2J}\begin{Bmatrix} I & J & v \\ j & i & \lambda \end{Bmatrix}\begin{Bmatrix} I & J & v \\ j' & i' & \lambda' \end{Bmatrix},$$

which is (up to the cancelling phase factor) just the right-hand side of Eq. (A.1).

APPENDIX B

Our aim is to obtain the following identity between $3j$, $6j$ and $9j$-symbols

$$\sum_{I'J'}(2I'+1)(2J'+1)\begin{Bmatrix} i & \imath & I \\ i' & \imath' & I' \end{Bmatrix}\begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix}\sum_{V'}(2V'+1)\begin{pmatrix} v & \varphi' & V' \\ v_3 & -\varphi'_3 & -V'_3 \end{pmatrix}\times$$
$$\times\begin{pmatrix} v' & \varphi & V' \\ v'_3 & -\varphi_3 & -V'_3 \end{pmatrix}\begin{Bmatrix} i & j & v \\ \imath' & \lambda' & \varphi' \\ I' & J' & V' \end{Bmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & \varphi \\ I' & J' & V' \end{Bmatrix} = (-1)^{\varphi+\varphi_3+\varphi'+\varphi'_3}\sum_{V}(-1)^{2V}(2V+1)\times$$
$$\times\begin{pmatrix} v & \varphi & V \\ v_3 & \varphi_3 & -V_3 \end{pmatrix}\begin{pmatrix} v' & \varphi' & V \\ v'_3 & \varphi'_3 & -V_3 \end{pmatrix}\begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & \varphi' \\ I & J & V \end{Bmatrix}, \tag{B.1}$$

where $(:::)$ is the $3j$-symbol. We will prove Eq. (B.1) by the straightforward summation of its left-hand side over the variables I', J', V'. To do this we use the following expansions (*e.g.* [9], Eq. (C.40c))

$$\begin{pmatrix} v & \varphi' & V' \\ v_3 & -\varphi'_3 & -V'_3 \end{pmatrix}\begin{Bmatrix} i & j & v \\ \imath' & \lambda' & \varphi' \\ I' & J' & V' \end{Bmatrix} = \sum_{\substack{m_1m_2m_3m_4 \\ M_{12}M_{34}}}\begin{pmatrix} i & \imath' & I' \\ m_1 & m_2 & M_{12} \end{pmatrix}\begin{pmatrix} j & \lambda' & J' \\ m_3 & m_4 & M_{34} \end{pmatrix}\times$$
$$\times\begin{pmatrix} i & j & v \\ m_1 & m_3 & v_3 \end{pmatrix}\begin{pmatrix} \imath' & \lambda' & \varphi' \\ m_2 & m_4 & -\varphi'_3 \end{pmatrix}\begin{pmatrix} I' & J' & V' \\ M_{12} & M_{34} & -V'_3 \end{pmatrix},$$

$$\begin{pmatrix} v' & \varphi & V' \\ v'_3 & -\varphi_3 & -V'_3 \end{pmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath & \lambda & \varphi \\ I' & J' & V' \end{Bmatrix} = \sum_{\substack{m'_1m'_2m'_3m'_4 \\ M'_{12}M'_{34}}}\begin{pmatrix} i' & \imath & I' \\ m'_1 & m'_2 & M'_{12} \end{pmatrix}\begin{pmatrix} j' & \lambda & J' \\ m'_3 & m'_4 & M'_{34} \end{pmatrix}\times$$
$$\times\begin{pmatrix} i' & j' & v' \\ m'_1 & m'_3 & v'_3 \end{pmatrix}\begin{pmatrix} \imath & \lambda & \varphi \\ m'_2 & m'_4 & -\varphi_3 \end{pmatrix}\begin{pmatrix} I' & J' & V' \\ M'_{12} & M'_{34} & -V'_3 \end{pmatrix}, \tag{B.2}$$

and next apply the expansions (*e.g.* [9], Eq. (C.33))

$$\begin{pmatrix} i & \imath' & I' \\ m_1 & m_2 & M_{12} \end{pmatrix} \begin{Bmatrix} i & \imath & I \\ i' & \imath' & I' \end{Bmatrix} = \sum_{M_1 M_2 M_3} (-1)^{i'+\imath+I'+M_1+M_2+M_3} \begin{pmatrix} i' & \imath & I' \\ M_1 & -M_2 & M_{12} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \imath & I & i \\ M_2 & -M_3 & m_1 \end{pmatrix} \begin{pmatrix} I & i' & \imath' \\ M_3 & -M_1 & m_2 \end{pmatrix},$$

$$\begin{pmatrix} j & \lambda' & J' \\ m_3 & m_4 & M_{34} \end{pmatrix} \begin{Bmatrix} j & \lambda & J \\ j' & \lambda' & J' \end{Bmatrix} = \sum_{M_1' M_2' M_3'} (-1)^{j'+\lambda+J+M_1'+M_2'+M_3'} \begin{pmatrix} j' & \lambda & J' \\ M_1' & -M_2' & M_{34} \end{pmatrix} \times$$

$$\times \begin{pmatrix} \lambda & J & j \\ M_2' & -M_3' & m_3 \end{pmatrix} \begin{pmatrix} J & j' & \lambda' \\ M_3' & -M_1' & m_4 \end{pmatrix}. \tag{B.3}$$

By means of Eqs (B.2) and (B.3) we expressed the left-hand side of Eq. (B.1) as a sum which involves the product of 14 $3j$-symbols; six of them can be summed over in pairs due to the orthogonality properties of $3j$-symbols:

$$\sum_{V_3'} (2V'+1) \begin{pmatrix} I' & J' & V' \\ M_{12} & M_{34} & -V_3' \end{pmatrix} \begin{pmatrix} I' & J' & V' \\ M_{12}' & M_{34}' & -V_3' \end{pmatrix} = \delta_{M_{12}M_{12}'} \delta_{M_{34}M_{34}'},$$

$$\sum_{I'M_{12}} (2I'+1) \begin{pmatrix} i' & \imath & I' \\ m_1' & m_2' & M_{12} \end{pmatrix} \begin{pmatrix} i' & \imath & I' \\ M_1 & -M_2 & M_{12} \end{pmatrix} = \delta_{m_1'M_1} \delta_{m_2',-M_2},$$

$$\sum_{J'M_{34}} (2J'+1) \begin{pmatrix} j' & \lambda & J' \\ m_3' & m_4' & M_{34} \end{pmatrix} \begin{pmatrix} j' & \lambda & J' \\ M_1' & -M_2' & M_{34} \end{pmatrix} = \delta_{m_3'M_1'} \delta_{m_4',-M_2'}.$$

Now the above δ-symbols enable us to simplify the sum over magnetic quantum numbers; changing signs of these numbers in four $3j$-symbols and collecting together the resulting phase factors as well as the phase factors from Eqs (B.3), we obtain for the left-hand side of Eq. (B.1) the expression

$$\sum_{M_3 M_3'} (-1)^{2v'-\varphi-\varphi'+2I+2J-\varphi_3-v_3'-M_3-M_3'} \Bigg[\sum_{m_1 m_3 m_2' m_4'} \begin{pmatrix} i & j & v \\ m_1 & m_3 & v_3 \end{pmatrix} \begin{pmatrix} \imath & \lambda & \varphi \\ m_2' & m_4' & \varphi_3 \end{pmatrix} \times$$

$$\times \begin{pmatrix} \lambda & J & j \\ m_4' & M_3' & m_3 \end{pmatrix} \begin{pmatrix} \imath & I & i \\ m_2' & M_3 & m_1 \end{pmatrix} \Bigg] \Bigg[\sum_{m_1' m_3' m_2 m_4} \begin{pmatrix} i' & j' & v' \\ m_1' & m_3' & v_3' \end{pmatrix} \begin{pmatrix} \imath' & \lambda' & \varphi' \\ m_2 & m_4 & \varphi_3' \end{pmatrix} \times$$

$$\times \begin{pmatrix} \lambda' & J & j' \\ m_4 & M_3' & m_3' \end{pmatrix} \begin{pmatrix} \imath' & I & i' \\ m_2 & M_3 & m_1' \end{pmatrix} \Bigg]. \tag{B.4}$$

The sum in the first square bracket can be written as

$$\sum_V (2V+1) \begin{pmatrix} I & J & V \\ M_3 & M_3' & -V_3 \end{pmatrix} \begin{pmatrix} v & \varphi & V \\ v_3 & \varphi_3 & -V_3 \end{pmatrix} \begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix};$$

this can be done by taking the first of the identities (B.2) (with primes and the minus sign before φ_3 omitted), multiplying both sides by $(2V+1)\begin{pmatrix} I & J & V \\ M_3 & M_3' & -V_3 \end{pmatrix}$ and using the orthogonality property of the $3j$-symbol with respect to summation over V. Similarly the second square bracket in Eq. (B.4) is equal to

$$\sum_{V''}(2V''+1)\begin{pmatrix} I & J & V'' \\ M_3 & M_3' & -V_3'' \end{pmatrix}\begin{pmatrix} v' & \varphi' & V'' \\ v_3' & \varphi_3' & -V_3'' \end{pmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & \varphi' \\ I & J & V'' \end{Bmatrix}.$$

Using the fact that $M_3+M'_3 = V''_3 = v'_3+\varphi'_3$ and the orthogonality relation

$$\sum_{M_3M_3'}(2V''+1)\begin{pmatrix} I & J & V \\ M_3 & M_3' & -V_3 \end{pmatrix}\begin{pmatrix} I & J & V'' \\ M_3 & M_3' & -V_3'' \end{pmatrix} = \delta_{VV''}\delta_{V_3V_3''},$$

we obtain finally as a result of summation of the left-hand side of Eq. (B.1)

$$(-1)^{\varphi+\varphi_3+\varphi'+\varphi_3'}\sum_{V}(-1)^{2V}(2V+1)\begin{pmatrix} v & \varphi & V \\ v_3 & \varphi_3 & -V_3 \end{pmatrix}\begin{Bmatrix} i & j & v \\ \imath & \lambda & \varphi \\ I & J & V \end{Bmatrix}\begin{pmatrix} v' & \varphi' & V \\ v_3' & \varphi_3' & -V_3 \end{pmatrix}\begin{Bmatrix} i' & j' & v' \\ \imath' & \lambda' & \varphi' \\ I & J & V \end{Bmatrix},$$

which is just the right-hand side of Eq. (B.1), Q.E.D.

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