A REVISED NONSYMMETRIC UNIFIED FIELD THEORY

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A further development of the Einstein-Kaufman, nonsymmetric unified field theory is discussed. Static, spherically symmetric solutions of the field equations are considered. It is shown that there does not exist a solution corresponding to a magnetic monopole. In the purely electric case, one of Papapetrou's solutions is recovered and a new "cosmological" solution is found in which the space-time metric is that of a flat Minkowski world but a diverging electric field is present. It is pointed out that the theory may be significant as an account of charged matter.

1. *Introduction*

In his last attempt to formulate a nonsymmetric unified field theory, Einstein (in collaboration with Kaufman, Ref. [1]) proposed to use as one set of variational parameters a linear combination

$$U^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} \tag{1}$$

of the components of the affine connection $\Gamma^{\lambda}_{\mu\nu}$ (Greek indices going from 1 to 4). The object was to ensure that the Ricci tensor $R_{\mu\nu}$ should become automatically Hermitian symmetric (transposition invariant) with respect to the new variables. The Ricci tensor is of course, Hermitian symmetric with respect to the affine connection but only under the additional condition

$$\Gamma_{\mu}(\Gamma^{\sigma}_{\underset{\vee}{\mu\sigma}} = \tfrac{1}{2}(\Gamma^{\sigma}_{\mu\sigma} - \Gamma^{\sigma}_{\sigma\mu})) = 0 \tag{2}$$

The other set of variational parameters are the components of the fundamental tensor density $\mathfrak{g}^{\mu\nu}$.

The field equations of the theory of Einstein and Kaufman have never been solved and one aim of this article is to investigate their solution in the case of spherical symmetry. Actually, we shall solve the field equations of an amended version of the theory.

It has been pointed out recently (Ref. [2]) that the U-substitution is not unique. In particular, if we write

$$\Gamma^{\lambda}_{\mu\nu} = V^{\lambda}_{\mu\nu} - \tfrac{1}{3}V^{\sigma}_{\mu\sigma}\delta^{\lambda}_{\nu} - \tfrac{1}{3}V_{\nu}\delta^{\lambda}_{\mu}, \tag{3}$$

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where

$$V_\nu = V^\sigma_{\underset{\vee}{\nu\sigma}} = \tfrac{1}{2}(V^\sigma_{\nu\sigma} - V^\sigma_{\sigma\nu}), \tag{4}$$

then $R_{\mu\nu}$ still remains Hermitian symmetric with respect to the new variables $V^\lambda_{\mu\nu}$. Moreover, one of the sets of the field equations acquires the form of the second set of Maxwell's equations for, although V_ν is not a vector, its curl is a tensor. There are some complications in regarding V_ν as an electromagnetic potential "pseudo-vector". Indeed, the static, spherically symmetric situation which alone we consider herein, is not the best for investigating the nature of such potentials which is only fully brought out in the time-dependent, dynamic conditions.

The components of the affine connection in General Relativity and in the Einstein--Straus theory (Ref. [3]) are, in general, uniquely determined in terms of the fundamental tensor and its first derivatives. However, in the present versions of the nonsymmetric theory, the equations from which $U^\lambda_{\mu\nu}$ or $V^\lambda_{\mu\nu}$ are to be found, happen to be the same. Consequently a unique solution is impossible except under certain additional conditions which will be found below. It seems as if the logical elegance of previous theories was being sacrificed to an excessive insistence on Hermitian symmetry.

On the other hand (Ref. [4]), Hermitian symmetry represents the only hypothesis linking the abstract structure of the theory to physics. Hlavaty recalled Einstein's inspired guess that the property gives, in fact, charge conjugation invariance, being therefore indispensible to a theory purporting to unify the macrophysics of gravitation and electromagnetism. In addition, there are good reasons (Ref. [5]) for regarding nonsymmetric theories as an extension of General Relativity necessary for explaining its own limitation.

Finally, recent cylindrically symmetric solutions of the Einstein-Straus field exquations (Ref. [6]) have reinforced an early suggestion (of Einstein) that in defining the electromagnetic field tensors, the parts played respectively by the electric and magnetic vectors should be reversed. This identification is of great importance to the interpretation of the results which follow.

As far as the notation is concerned, a comma indicates ordinary partial differentiation. Also we use Einstein's notation for the symmetric and the skew-symmetric parts of a two index object:

$$A_{\underline{\mu\nu}} = \tfrac{1}{2}(A_{\mu\nu} + A_{\nu\mu}),$$

$$A_{\underset{\vee}{\mu\nu}} = \tfrac{1}{2}(A_{\mu\nu} - A_{\nu\mu}).$$

2. *The field equations*

The equations (3) which define th affine connection in terms of the new variables $V^\lambda_{\mu\nu}$ can be solved for the latter (this is a property of the substitution necessary for its usefulness in a physical theory, Ref. [2]). Thus, from (3)

$$\Gamma^\varrho_{\mu\varrho} = -\tfrac{1}{3} V^\varrho_{\mu\varrho} - \tfrac{1}{3} V_\mu,$$

$$\Gamma^\varrho_{\varrho\mu} = V^\varrho_{\varrho\mu} - \tfrac{1}{3} V^\varrho_{\mu\varrho} - \tfrac{4}{3} V_\mu,$$

whence

$$V^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \tfrac{1}{12}\,(10\Gamma^{\varrho}_{\mu\varrho} - \Gamma^{\varrho}_{\varrho\mu})\delta^{\lambda}_{\nu} - \tfrac{1}{12}\,(\Gamma^{\varrho}_{\varrho\nu} + 2\Gamma^{\varrho}_{\nu\varrho})\delta^{\lambda}_{\mu}, \tag{5}$$

The Ricci tensor is defined by

$$R_{\mu\nu} = -\Gamma^{\sigma}_{\mu\nu,\sigma} + \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\varrho}_{\mu\sigma}\Gamma^{\sigma}_{\varrho\nu} - \Gamma^{\varrho}_{\mu\nu}\Gamma^{\sigma}_{\varrho\sigma}. \tag{6}$$

In terms of $V^{\lambda}_{\mu\nu}$, it becomes

$$R_{\mu\nu} = -V^{\sigma}_{\mu\nu,\sigma} + V^{\varrho}_{\mu\sigma}V^{\sigma}_{\varrho\nu} - \tfrac{1}{3}\,V^{\sigma}_{\mu\sigma}V^{\varrho}_{\varrho\nu} + \tfrac{1}{3}\,(V_{\nu,\mu} - V_{\mu,\nu}). \tag{7}$$

The field equations are derived from a variational principle

$$\delta \int_{C} \mathscr{H}\, d\tau = 0 \tag{8}$$

where $d\tau = dx^1dx^2dx^3dx^4$ and $\mathscr{H}$ is a scalar density.

Following Einstein, we select

$$\mathscr{H} = \mathfrak{g}^{\mu\nu}R_{\mu\nu}, \tag{9}$$

where

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\; g^{\mu\nu},$$

g is the nonzero determinant of the fundamental (not metric, since we have no *a priori* laws of geometrical measurement in the nonsymmetric theory) tensor $g_{\mu\nu}$, and $g^{\mu\nu}$ is defined as usual by

$$g_{\mu\sigma}g^{\nu\sigma} = \delta^{\nu}_{\mu} = g_{\sigma\mu}g^{\sigma\nu}.$$

Carrying out the variation with respect to $\mathfrak{g}^{\mu\nu}$ and $V^{\lambda}_{\mu\nu}$ under the standard assumption that all integrated-once parts vanish on the boundary of the region C.

$$\delta \int \mathfrak{g}^{\mu\nu}R_{\mu\nu}d\tau = \int \{R_{\mu\nu}\delta\mathfrak{g}^{\mu\nu} + [\mathfrak{g}^{\mu\nu}{}_{,\sigma} + \mathfrak{g}^{\mu\varrho}(V^{\nu}_{\sigma\varrho} - \tfrac{1}{3}\,V^{\alpha}_{\alpha\varrho}\delta^{\nu}_{\sigma}) +$$

$$+ \mathfrak{g}^{\varrho\nu}(V^{\mu}_{\varrho\sigma} - \tfrac{1}{3}\,V^{\alpha}_{\varrho\alpha}\delta^{\mu}_{\sigma}) + \tfrac{1}{3}\,(\mathfrak{g}^{\underset{\vee}{\mu\varrho}}{}_{,\varrho}\delta^{\nu}_{\sigma} - \mathfrak{g}^{\underset{\vee}{\mu\varrho}}{}_{,\varrho}\delta^{\mu}_{\sigma})]\delta V^{\sigma}_{\mu\nu}\}d\tau.$$

Hence the field equations become

$$R_{\mu\nu} = 0, \tag{10}$$

or

$$-V^{\sigma}_{\underline{\mu\nu},\sigma} + V^{\varrho}_{\underline{\mu\sigma}}V^{\sigma}_{\underline{\varrho\nu}} + V^{\varrho}_{\underset{\vee}{\mu\sigma}}V^{\sigma}_{\underset{\vee}{\varrho\nu}} - \tfrac{1}{3}\,V^{\sigma}_{\underline{\mu\sigma}}V^{\varrho}_{\underline{\varrho\nu}} + \tfrac{1}{3}\,V_{\mu}V_{\nu} = 0, \tag{11}$$

and

$$-V^{\sigma}_{\underset{\vee}{\mu\nu},\sigma} + V^{\varrho}_{\underline{\mu\sigma}}V^{\sigma}_{\underset{\vee}{\varrho\nu}} + V^{\varrho}_{\underset{\vee}{\mu\sigma}}V^{\sigma}_{\underline{\varrho\nu}} + \tfrac{1}{3}\,(V^{\sigma}_{\underline{\mu\sigma}}V_{\nu} - V^{\sigma}_{\underline{\nu\sigma}}V_{\mu}) + \tfrac{1}{3}\,(V_{\nu,\mu} - V_{\mu,\nu}) = 0, \tag{12}$$

together with

$$\mathfrak{g}^{\mu\nu}{}_{,\sigma}+\mathfrak{g}^{\mu\varrho}(V^{\nu}_{\sigma\varrho}-\tfrac{1}{3}V^{\alpha}_{\alpha\varrho}\delta^{\nu}_{\sigma})+\mathfrak{g}^{\varrho\nu}(V^{\mu}_{\varrho\sigma}-\tfrac{1}{3}V^{\alpha}_{\varrho\alpha}\delta^{\mu}_{\sigma})+$$
$$+\tfrac{1}{3}(\mathfrak{g}^{\underset{\vee}{\mu\varrho}}{}_{,\varrho}\delta^{\nu}_{\sigma}-\mathfrak{g}^{\underset{\vee}{\nu\varrho}}{}_{,\varrho}\delta^{\mu}_{\sigma}) = 0. \tag{13}$$

Contracting (13) first respect to ν and σ, and then with respect to μ and σ, and subtracting, we find that

$$\mathfrak{g}^{\underset{\vee}{\mu\sigma}}{}_{,\sigma}+\tfrac{1}{3}\mathfrak{g}^{\underline{\mu\varrho}}V_{\varrho}-\tfrac{1}{3}\mathfrak{g}^{\underset{\vee}{\mu\varrho}}V^{\sigma}_{\underline{\varrho\sigma}} = 0,$$

or

$$\mathfrak{g}^{\underset{\vee}{\mu\sigma}}{}_{,\sigma} = \tfrac{1}{6}\mathfrak{g}^{\mu\varrho}V^{\sigma}_{\sigma\varrho}-\tfrac{1}{6}\mathfrak{g}^{\varrho\mu}V^{\sigma}_{\varrho\sigma} \tag{14}$$

Substituting this into (13) and skew symmetrizing with respect to μ and ν, we obtain

$$\mathfrak{g}^{\mu\nu}{}_{,\sigma}+\tfrac{1}{2}(\mathfrak{g}^{\mu\varrho}V^{\nu}_{\sigma\varrho}+\mathfrak{g}^{\varrho\nu}V^{\mu}_{\varrho\sigma}-\mathfrak{g}^{\nu\varrho}V^{\mu}_{\sigma\varrho}-\mathfrak{g}^{\varrho\mu}V^{\alpha}_{\alpha\varrho})-$$
$$-\tfrac{1}{9}[\delta^{\nu}_{\sigma}(\mathfrak{g}^{\mu\varrho}V^{\alpha}_{\alpha\varrho}-\mathfrak{g}^{\varrho\mu}V^{\alpha}_{\varrho\alpha})+\delta^{\mu}_{\sigma}(\mathfrak{g}^{\varrho\nu}V^{\alpha}_{\varrho\alpha}-\mathfrak{g}^{\nu\varrho}V^{\alpha}_{\alpha\varrho})] = 0,$$

which, on contraction with respect to ν and σ, gives

$$\mathfrak{g}^{\underset{\vee}{\mu\sigma}}{}_{,\sigma} = -\tfrac{1}{6}\mathfrak{g}^{\mu\varrho}V^{\sigma}_{\sigma\varrho}+\tfrac{1}{6}\mathfrak{g}^{\varrho\mu}V^{\sigma}_{\varrho\sigma}. \tag{15}$$

Equations (14) and (15) now imply that

$$\mathfrak{g}^{\underset{\vee}{\mu\sigma}}{}_{,\sigma} = 0, \tag{16}$$

so that equation (13) becomes, as in the Einstein-Kaufman theory

$$\mathfrak{g}^{\mu\nu}{}_{,\sigma}+\mathfrak{g}^{\mu\varrho}(V^{\nu}_{\sigma\varrho}-\tfrac{1}{3}V^{\alpha}_{\alpha\varrho}\delta^{\nu}_{\sigma})+\mathfrak{g}^{\varrho\nu}(V^{\mu}_{\varrho\sigma}-\tfrac{1}{3}V^{\alpha}_{\varrho\alpha}\delta^{\mu}_{\sigma}) = 0. \tag{17}$$

The symmetry considerations of Papapetrou (Ref. [7]) are not affected by the cho:ce of $V^{\lambda}_{\mu\nu}$ as a variational parameter instead of $\Gamma^{\lambda}_{\mu\nu}$. We shall describe below a number of static, spherically symmetric solutions in the two cases:

i) when the nonzero components of $g_{\mu\nu}$ are

$$g_{11}, g_{22\cdot} = -r^2, g_{33} = g_{22}\sin^2\theta, g_{44} \text{ and } g_{14} = -g_{41},$$

and

ii) when only

$$g_{11}, g_{22}, g_{33} = g_{22}\sin^2\theta, g_{44} \text{ and } g_{23} = -g_{32}$$

are not identically zero. In the first case, we shall find that the field equations are incompatible, a fact of considerable importance in a physical interpretation of the theory.

3. The first, or magnetic case

Let us write for the moment

$$g_{\mu\nu} = \begin{bmatrix} -a & 0 & 0 & w \\ 0 & -b & 0 & 0 \\ 0 & 0 & -b\sin^2\theta & 0 \\ -w & 0 & 0 & c \end{bmatrix}, \tag{18}$$

where a, b, c and w are functions of $x' = r$ only. Then

$$\mathfrak{g}^{\mu\nu} = \begin{bmatrix} -\dfrac{bc\sin\theta}{\sqrt{ac-w^2}} & 0 & 0 & -\dfrac{wb\sin\theta}{\sqrt{ac-w^2}} \\ 0 & -\sqrt{ac-w^2}\sin\theta & 0 & 0 \\ 0 & 0 & -\dfrac{\sqrt{ac-w^2}}{\sin\theta} & 0 \\ \dfrac{wb\sin\theta}{\sqrt{ac-w^2}}. & 0 & 0 & \dfrac{ab\sin\theta}{\sqrt{ac-w^2}} \end{bmatrix}, \tag{19}$$

where, by the equation (16),

$$\frac{wb}{\sqrt{ac-w^2}} = l, \quad \text{a constant.} \tag{20}$$

The solution of the sixtyfour algebraic equations (17) is more complicated than in Papapetrou's case because of the occurrence of the terms $(V^{\sigma}_{\mu\sigma})$. Only twelve of the $V^{\lambda}_{\mu\nu}$'s can be seen to vanish without any more conditions. If, however, we also have

$$\mathfrak{g}^{11}\mathfrak{g}^{44} \neq \mathfrak{g}^{14^2}, (ac+w^2 \neq 0) \tag{21}$$

then we find that the nonzero components of the pseudo-connection V are

$$V^{2}_{\underline{12}} = V^{3}_{\underline{13}} = -\tfrac{1}{2}\mathfrak{g}^{22},{}_{1}/\mathfrak{g}^{22},\ V^{4}_{\underline{14}} = \tfrac{1}{2}\mathfrak{g}^{44},{}_{1}/\mathfrak{g}^{44},$$

$$V^{1}_{11} = -\tfrac{3}{4}D'/D + V^{2}_{\underline{12}} - V^{4}_{\underline{14}},\ \mathfrak{g}^{22}V^{1}_{22} = \mathfrak{g}^{33}V^{1}_{33} = -\tfrac{1}{4}D'/\mathfrak{g}^{44},$$

$$\mathfrak{g}^{44}V^{1}_{44} = \mathfrak{g}^{11}V^{1}_{11} + \frac{1}{2}D'/\mathfrak{g}^{44}, \quad V^{1}_{\check{4}1} - V^{2}_{\check{4}2} = \frac{3}{4}\frac{\mathfrak{g}^{14}}{\mathfrak{g}^{44}}\frac{D'}{D'}, \quad V^{2}_{\check{4}2} = V^{3}_{\check{4}3}$$

$$V^{2}_{12} = V^{3}_{13} = V^{4}_{14} = \tfrac{1}{3}V_1 \text{ say,}$$

$V^3_{\underline{23}} = \frac{1}{2}\cot\theta$, $V^2_{22} = -\cot\theta$ (the condition (21) is not needed for the results so far)

$$V^1_{\underline{12}} = V^4_{\underline{42}} = -\tfrac{1}{2}\cot\theta, \qquad V^2_{33} = -\sin\theta\cos\theta,$$

$$V^1_{\check{21}} = V^3_{\check{23}} = V^4_{\check{24}} = \tfrac{1}{3}V_2, \qquad V^1_{\check{31}} = V^2_{\check{32}} = V^4_{\check{34}} = \tfrac{1}{3}V_3 \text{ (say)}, \tag{22}$$

where $D = \mathfrak{g}^{11}\mathfrak{g}^{44} + \mathfrak{g}^{14^2}$ ($\neq 0$) and D' is the derivative of D with respect to r. The functions V_1, V_2, V_3 are arbitrary but, in accordance with the symmetry conditions employed, they may be assumed to be functions of, at most, r and θ only. The field equations (11) and (12) which are not identically satisfied are

$$\begin{gathered}
-V^1_{11,1} + (V^1_{11})^2 + 2(V^2_{\underline{12}})^2 + (V^4_{\underline{14}})^2 - \tfrac{1}{3}(V^1_{11} + 2V^2_{\underline{12}} + V^4_{\underline{14}})^2 = 0, \\
-V^1_{22,1} + 2V^2_{\underline{12}}V^1_{22} - 1 = 0, \\
-V^1_{44,1} + 2V^4_{\underline{14}}V^1_{44} = 0, \\
V_{2,1} - V_{1,2} = 0, \quad V_{3,1} = 0, \quad V_{4,1} = 0, \text{ where } V_4 = V^\sigma_{\check{4\sigma}}, \\
V_{3,2} = 0, \quad V_{4,2} = 0,
\end{gathered} \tag{23}$$

together with the equation (20).

Thus, we immediately have

$$V_3 = 0 = V_4 \text{ and } V^1_{\check{41}} = -2V^2_{\check{42}} = \frac{1}{4}\frac{\mathfrak{g}^{14}}{\mathfrak{g}^{44}}\frac{D'}{D}, \tag{24}$$

and, from the third of the equations (23),

$$\mathfrak{g}^{11}V^1_{11} + \frac{1}{2}\frac{D'}{\mathfrak{g}^{44}} = k\sin\theta, \qquad k \text{ a constant.} \tag{25}$$

In what follows we can clearly omit the θ-dependence and, moreover, we can set up the coordinate system in such a way (Ref. [7]) that

$$b = r^2. \tag{26}$$

Let us write also $u = w/a$. Then, by (20), $D = -r^4$,

$$\mathfrak{g}^{11} = -\frac{lc}{w}, \qquad \mathfrak{g}^{22} = -\sqrt{ac - w^2} = -\frac{wr^2}{l}, \qquad \mathfrak{g}^{44} = \frac{l}{u}$$

$$V^2_{\underline{12}} = -\frac{1}{2}\frac{w'}{w} - \frac{1}{r}, \qquad V^4_{\underline{14}} = \frac{1}{2}\frac{u'}{u}, \qquad \frac{c}{w} = \frac{u}{2}(l^2 + r^4),$$

$$V^1_{11} = -\frac{4}{r} - \frac{1}{2}\frac{w'}{w} - \frac{1}{2}\frac{u'}{u},$$

and equation (25) gives on differentiation

$$(V_{11})' = -\frac{6r^2}{l^2+r^4} - \frac{2r^3}{l^2+r^4}\frac{u'}{u} - \left(\frac{u'}{u} + \frac{4r^3}{l^2+r^4}\right) V^1_{11}. \tag{27}$$

Substituting (27) into the first of the equations (23), we find

$$-\frac{5r^2}{l^2+r^4} + \frac{3}{r^2} - \frac{r^3}{l^2+r^4}\frac{w'}{w} = 0,$$

whence

$$w = Ar^{-2}\exp\left(-\tfrac{3}{4}\, l^2 r^{-4}\right), \qquad A \text{ a constant.} \tag{28}$$

Hence, from (25)

$$u' + \left(\frac{6}{r} + \frac{3l^2}{r^5} - \frac{4r^3}{l^2+r^4}\right) u = \frac{2kl}{l^2+r^4},$$

the integrating factor of which is

$$p = \frac{r^6}{l^2+r^4}\exp\left(-\frac{3}{4}\, l^2 r^{-4}\right),$$

so that

$$(pu)' = \frac{2klr^6}{(l^2+r^4)^2}\exp\left(-\frac{3}{4}\, l^2 r^{-4}\right).$$

But, the second of the equations (23)

$$(\mathfrak{g}^{22} V^1_{22})' = -\mathfrak{g}^{22},$$

becomes

$$(r^3 u)' = wr^2 = A\exp\left(-\tfrac{3}{4}\, l^2 r^{-4}\right).$$

It is readily seen that these equations are not compatible so that there is no solution of the "magnetic case". We shall return to a possible interpretation of this result below.

4. The second, or electric case

We consider now

$$g_{\mu\nu} = \begin{Vmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & r^2 v \sin\theta & 0 \\ 0 & -r^2 v \sin\theta & -\beta \sin^2\theta & 0 \\ 0 & 0 & 0 & \gamma \end{Vmatrix},$$

so that

$$\mathfrak{g}^{\mu\nu} = \left\| \begin{matrix} -\sqrt{\frac{\gamma}{\alpha}(\beta^2+r^4v^2)}\sin\theta & 0 & 0 & 0 \\ 0 & -\beta\sqrt{\frac{\alpha\gamma}{\beta^2+r^4v^2}}\sin & r^2v\sqrt{\frac{\alpha\gamma}{\beta^2+r^4v^2}} & 0 \\ 0 & -r^2v\sqrt{\frac{\alpha\gamma}{\beta^2+r^4v^2}} & -\beta\sqrt{\frac{\alpha\gamma}{\beta^2+r^4v^2}}\frac{1}{\sin\theta} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{\alpha}{\gamma}(\beta^2+r^4v^2)}\sin\theta \end{matrix} \right\| \equiv$$

$$\equiv \left\| \begin{matrix} -a\sin\theta & 0 & 0 & 0 \\ 0 & -b\sin\theta & f & 0 \\ 0 & -f & -\frac{b}{\sin\theta} & 0 \\ 0 & 0 & 0 & c\sin\theta \end{matrix} \right\|, \tag{29}$$

where, as before, a, b, f and c are functions of r only and v corresponds to the magnitude of the (radial) electric field. As in the case of the Papapetrou solution, the equation (16) is now identically satisfied and does not furnish us with any information.

The additional condition under which a unique solution of the equations (17) can be obtained is

$$(\mathfrak{g}^{23})^2 - \mathfrak{g}^{22}\mathfrak{g}^{33} \neq 0, \tag{30}$$

when the nonzero components of $V^{\lambda}_{\mu\nu}$ become

$$V^2_{\underline{12}} = V^3_{\underline{13}} = -\frac{1}{2}\frac{b'}{b} - \frac{1}{2}\frac{xx'}{1+x^2}, \qquad V^4_{\underline{14}} = -\frac{1}{2}\frac{c'}{c},$$

$$V^1_{11} = -\frac{1}{4}\left(3\frac{a'}{a} + \frac{c'}{c} + 2\frac{b'}{b} + \frac{2xx'}{1+x^2}\right)$$

$$V^1_{22} = V^1_{33}\,\mathrm{cosec}^2\,\theta = -\frac{1}{4}\,\frac{a}{b(1+x^2)}\left(\frac{a'}{a}+\frac{c'}{c}+\frac{2xx'}{1+x^2}\right)$$

$$V^1_{44} = \frac{1}{4}\,\frac{a}{c}\left(\frac{a'}{a}\,\frac{c'}{c}+2\,\frac{b'}{b}+\frac{2xx'}{1+x^2}\right)$$

$$V^1_{\check{2}3} = \frac{1}{4}\,\frac{a}{b(1+x^2)}\left(x\left(\frac{a'}{a}+\frac{c'}{c}\right)-\frac{2x'}{1+x^2}\right)$$

$$V^3_{\check{1}2} = V^3_{\check{3}1}\,\mathrm{cosec}^2\,\theta = \frac{1}{2}\,\frac{x'\,\mathrm{cosec}\,\theta}{1+x^2}\,,$$

$$V^1_{\underline{12}} = V^4_{\underline{42}} = -V^3_{\underline{32}} = -\tfrac{1}{2}\cot\theta,\qquad V^2_{22} = -\cot\theta,\qquad V^2_{33} = -\sin\theta\cos\theta,$$

$$V^1_{\check{2}1} = V^3_{\check{2}3} = V^4_{\check{2}4} = \tfrac{1}{3}\,V_2,\qquad V^1_{\check{3}1} = V^2_{\check{3}2} = V^4_{\check{3}4} = \tfrac{1}{3}\,V_3,\ \text{say.} \tag{31}$$

Notice that we have $V_1 = V_4 = 0$.

Omitting θ-dependence, the relevant field equations are

$$-V^1_{11,1}+(V^1_{11})^2+2(V^2_{\underline{12}})^2+(V^4_{\underline{14}})^2-$$
$$-\tfrac{1}{3}(V^1_{11}+2V^2_{\underline{12}}+V^4_{\underline{14}})^2+2V^2_{\check{1}3}V^3_{\check{2}1} = 0$$
$$-V^1_{22,1}-1+2V^2_{\underline{12}}V^1_{22}+2V^1_{\check{2}3}V^3_{\check{1}2} = 0$$
$$-V^1_{44,1}+2V^4_{\underline{14}}V^1_{44} = 0$$
$$V_{2,1} = 0,\qquad V_{3,1} = 0$$
$$-V^1_{\check{2}3,1}+\tfrac{2}{3}\,V_{3,2}+2V^2_{\underline{12}}V^1_{\check{2}3}+2V^1_{22}V^2_{\check{1}3} = 0, \tag{32}$$

the latter becoming, on eliminating θ-dependence,

$$-V^1_{\check{2}3,1}+B-2V^1_{22}V^2_{\check{3}1}+2V^2_{\underline{12}}V^1_{\check{2}3} = 0$$

where B is a constant such that

$$V_3 = \tfrac{3}{2}\,B\cos\theta.$$

Since

$$xV^1_{22}+V^1_{\check{2}3} = -\frac{a}{b}\,V^2_{31},$$

$$xV^1_{\check{2}3}-V^1_{22} = \frac{1}{4}\,\frac{a}{b}\left(\frac{a'}{a}+\frac{c'}{c}\right),$$

the V^1_{22} and V^1_{23} equations may be rewritten in the simpler form

$$\left[a\left(\frac{a'}{a}+\frac{c'}{c}\right)\right]-4b(Bx+1) = 0,$$

and

$$(aV^2_{31})+b(B-x) = 0.$$

Also, from the third of the equations (32), we immediately find that

$$\frac{a'}{a} - \frac{c'}{c} + 2\frac{b'}{b} + \frac{2xx'}{1+x^2} = \frac{k}{a}, \tag{34}$$

where k is a constant, whence the first of equations (33) can be written as

$$(aV^1_{11})+2b(Bx+1) = 0,$$

giving, together with the first of (32) another integral of the second order equations in the form

$$-\frac{1}{4}\left(\frac{a'}{a}+\frac{c'}{c}\right)\left(\frac{k}{a}+\frac{1}{2}\frac{a'}{a}+\frac{1}{2}\frac{c'}{c}\right)+2\frac{b}{a}(Bx+1)+\frac{1}{2}\frac{x'^2}{(1+x^2)^2} = 0. \tag{35}$$

Equations (33), (34) and (35) are therefore four equations for four unknowns, a, b, c and x and are, in general, compatible.

Some particular solutions of these equations can be easily obtained and we consider three cases:

i) a Coulomb charge,

ii) a general relativistic form,

and *iii*) a cosmological case,

these names being based on the tentative interpretation of $\underline{g}_{\mu\nu}$ as the metric tensor and of $\check{g}_{\mu\nu}$ as the electromagnetic (electric) field.

5. *Some particular solutions*

i) The case of an inverse square electric field is characterized by the condition that

$$q = r^2 v$$

should be a constatnt. Hence we obtain an additional field equation

$$\frac{a'}{a} + \frac{c'}{c} + \frac{2x'}{x(1+x^2)} = 0. \tag{36}$$

We can eliminate c between the equations (36) and (34) to obtain

$$\frac{a'}{a} + \frac{b'}{b} + \frac{x'}{x} = \frac{k}{2a}, \tag{37}$$

while the first of the equations (32) now gives

$$\left(\frac{u'}{u}\right) + \frac{a'}{a}\frac{u'}{u} - \frac{1}{2}\frac{k}{a}\frac{u'}{u} = 0, \tag{38}$$

where

$$u = \frac{x}{\sqrt{1+x^2}}.$$

From (37) and (38), we get

$$b = \frac{Ax'}{x^2(1+x^2)}, \qquad A \text{ a constant,}$$

and we can easily convince ourselves that the remaining field equations cannot be satisfied except by the absurd $x' = 0 = b$.

Thus a Coulomb solution has no place in our theory.

ii) We call general relativistic the case when

$$\alpha = 1/\gamma.$$

Again all the field quantities can be easily eliminated except a and x for which we find the following three equations

$$aa'' - \frac{1}{2}a^{12} + \frac{1}{8}k^2 + \frac{1}{2}\left(\frac{ax'}{1+x^2}\right)^2 = 0,$$

$$a'' - 2\frac{Bx+1}{\sqrt{1+x^2}} = 0,$$

$$\left(\frac{ax'}{1+x^2}\right) + \frac{2(B-x)}{\sqrt{1+x^2}} = 0 \tag{39}$$

of which two are independent.

A particular solution of these equations is

$$\alpha = \left(1 - \frac{2m}{r}\right)^{-1} = \gamma^{-1}, \quad \beta = r^2, \quad q = \frac{k}{4m}r^2, \tag{40}$$

so that we have a Schwarzschild metric together with a constant radial electric field. This solution was obtained by Papapetrou for the field equations of the Einstein-Straus-Schrödinger theory. Its recovery here must be regarded as a strong indication that we are on the right track.

iii) In the "cosmological" case we put

$$\gamma = 1 = b\sqrt{\frac{a}{c}(1+x^2)}.$$

Hence, from (34) we have

$$k = 0.$$

But, differentiating equation (35) with respect to r, and substituting from (33)

$$\frac{a'}{a}-\frac{c'}{c}+2\frac{b'}{b}+\frac{2xx'}{1+x^2}-\frac{2Bx'}{Bx+1}=0,$$

so that (since $B\neq 0$)

$$x'=0,$$

that is

$$x=B.$$

Then equations (33) and (35) become

$$\left(a\left(\frac{a'}{a}+\frac{c'}{c}\right)\right)-4\sqrt{1+B^2}\sqrt{\frac{c}{a}}=0.$$

and

$$-\frac{9}{8}\left(\frac{a'}{a}+\frac{c'}{c}\right)^2+2\sqrt{1+B^2}\sqrt{\frac{c}{a}}=0.$$

Eliminating $\sqrt{1+B^2}\sqrt{\frac{c}{a}}$, integrating and comparing the result with the last equation, we find

$$a=1.$$

Thus, finally we have

$$\left(\frac{c'}{c}\right)^2-16\sqrt{(1+B^2)}\sqrt{c}=0,$$

whence

$$c^{-1/4}=(\sqrt{1+B^2})^{1/4}(r_0-r),$$

where r_0 is a constant.

Hence the solution is

$$a=1,\quad b=\frac{1}{(1+B^2)(r_0-r)^2},\quad c=\frac{1}{(1+B^2)(r_0-r)^4},\quad f=Bb,$$

or

$$\alpha=\frac{1}{(1+B^2)(r_0-r)^4},\quad \beta=\frac{1}{(1+B^2)(r_0-r)^2},\quad \gamma=1,$$

$$q=\frac{B}{(1+B^2)(r_0-r)^2}. \tag{41}$$

Treating $g_{\underline{\mu\nu}}\, dx^{\mu}\, dx^{\nu}$ as the metric invariant we obtain for our space-time

$$dt^2 - \frac{dr^2}{(1+B^2)(r_0-r)^4} - \frac{1}{(1+B^2)(r_0-r)^2}\, d\Omega^2, \tag{42}$$

which, on putting

$$R = \frac{1}{\sqrt{1+B^2}\,(r_0-r)},$$

becomes the metric of a flat, Minkowski universe

$$ds^2 = dt^2 - dR^2 - R^2 d\Omega^2,$$

with a radial electric field of magnitude of the order $\sim R^4$.

The condition (30) on the solvability of the equations determining $V^{\lambda}_{\mu\nu}$ demands that

$$B^2 \neq 1.$$

6. *Discussion*

The theory discussed in the preceding sections differs considerably from that of Einstein and Straus in spite of our recovery of one of Papapetrou's solutions above. A particularly interesting feature of it is the non-existence, demonstrated in Section 3, of a magnetic solution. The Papapetrou symmetry conditions imposed represent additional constraints on the system of field equations. Hence, the incompatibility of the latter in a special case does not contradict their formal consistency derived from the variational principle (I am indebted to Prof. G. Szekeres, F.A.A. for pointing this out).

On the other hand, the case of a static, spherically symmetric field can imply only one physically meaningful conclusion. It is that isolated magnetic charges cannot exist. As far as I am aware, this is the only purely macroscopic theory in which this is a consequence of an *a priori* unrelated field structure. In Maxwell's electrodynamics, for example, the non-existence of magnetic monopoles can be regarded as axiomatically written into the theory.

It is impossible at this stage to read too much significance into the results obtained in the last Section. It is not at all certain that either $g_{\underline{\mu\nu}}$ should be interpreted as the metric tensor or $g_{\check{\mu\nu}}$ as the electromagnetic field. For example, if instead of the latter we took the tensor suggested by Hlavaty (Ref. [4]), the elastic field in the last of our particular solutions would turn out to be of the order R^2 instead of R^4. However, Hlavaty's electromagnetic field tensor was obtained from the Einstein-Straus theory and there is no reason to employ it in the present case. Moreover, the way in which our particular solutions are derived, that is, the simplifying assumptions made, depend on the interpretation adopted for the components of the fundamental tensor. Finally, it is of some interest to comment on the relation between our theory and that of Einstein and Kaufman. In spite of their formal similarity, we cannot conclude that our most striking result, the non--existence of magnetic monopoles, is valid also in the latter. It has been mentioned before that the static, spherically symmetric case is not the most likely one to throw light on the physical significance of the functions V_1, V_2, V_3 and V_4 whose presence distinguishes the theory considered herein from the Einstein-Kaufman one. If they were to have something to do with four-potentials, as the equation (12) suggests, it is most likely that this would only become apparent if electromagnetism were described by a theory of the type of Mie's rather than of Maxwell's in which only one field tensor has an independent significance. In other words, we should look for a physical interpretation of our functions in a theory of charged matter.

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