ABOUT A SPECIAL METHOD IN S-MATRIX THEORY WITH THE p^4 MOMENTUM REPRESENTATION

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A regularization method in the Kadyshevsky momentum representation for the boson-boson and boson-photon interactions is applied.

1. The Bogolyubov renormalization method

Let us consider a system of interacting fields with the interaction Lagrange operator $\mathscr{L}_I(x)$. The standard formula for the S-operator describing this interaction, derived as an iterative solution of the Schrödinger equation in the interaction representation, can be expressed as follows:

$$S = 1+i\sum_n \mathscr{T}_n \equiv$$

$$\equiv 1+i\sum_n \frac{i^{n-1}}{n!}\iint S_n(x_1 \dots x_n)d^4x_1 \dots d^4x_n \tag{1.1}$$

where

$$S_n = T[\mathscr{L}_I(x_1)\dots\mathscr{L}_I(x_n)] \tag{1.2}$$

or in the symbolic form

$$S = T\exp\,[i\int\mathscr{L}_I(x)d^4x]. \tag{1.1'}$$

In order to interpret some singularities we meet in the calculation of S-matrix elements, Bogoliubov tried to give the most general form of the S-operator terms which satisfy the causality, unitarity, covariance and symmetry conditions. A method was found [1], which enabled a definition to be given of the $S_n(x_1\dots x_n)$ term, when $S_1\dots S_{n-1}$ are given. (With the choice $S_1(x) = \mathscr{L}_I(x)$.)

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These four conditions however do not determine the S_n-operator uniquely; we can add to it any hermitian operator $\Lambda_n(x_1 \ldots x_n)$, which satisfies two following conditions:

1. It must be symmetrical in the $x_1 \ldots x_n$ variables (this is connected with the symmetry condition for S_n).
2. It must be equal to zero except for those cases, when all variables are equal

$$x_1 = x_2 = \ldots = x_n$$

(following from the causality condition for the S-operator).
Such operators are called quasilocal. Their most general form can be expressed as a sum, each term of which must be a normal product of field operators with a "function coefficient" of the form:

$$Z(\ldots \partial x_i \ldots)\delta(x_1 - x_2)\ldots\delta(x_{n-1} - x_n),$$

where $Z(\ldots \partial x_i \ldots)$ is a symmetric polynomial of partial derivatives with constant coefficients. Hence, the quasilocal operator can be expressed as follows:

$$\Lambda_n(x_1 \ldots x_n) = \sum_r Z_r(\ldots \partial x_i \ldots)\delta(x_1 - x_2)\ldots \delta(x_{n-1} - x_n) : \varphi_{r_1}(x_i) \ldots \varphi_{r_j}(x_i) : \qquad (1.3)$$

where the indices $r_1 \ldots r_j$ are connected with several kinds of field operators in the r-term. The form of the Z_r polynomials and the structure of the normal products may be chosen freely.

In his work [1] Bogolyubov explained the connection between these quasilocal operators and the S-operator renormalization procedure.

It was shown there, that the addition of the quasilocal operators to the S-operator terms can be replaced by a modification of the $\mathscr{L}_I(x)$ operator as follows:

$$\overline{\mathscr{L}}_I(x) = \mathscr{L}_I(x) + \sum_{\nu=1}^{\infty} \frac{1}{\nu!} \int \Lambda_\nu(x, x_1, \ldots, x_{\nu-1}) d^4x_1 \ldots d^4x_{\nu-1}. \qquad (1.4)$$

Inserting $\overline{\mathscr{L}}_I(x)$ into (1.1), and collecting the terms with equal number of integrations we obtain a general expression for the S_n-operator ([1], page 159), which satisfies all the required conditions.
For example:

$$S_2(x_1, x_2) = iT[\mathscr{L}_I(x_1)\mathscr{L}_I(x_2)] + \Lambda_2(x_1, x_2), \qquad (1.5)$$

$$S_3(x_1, x_2, x_3) = -T[\mathscr{L}_I(x_1)\mathscr{L}_I(x_2)\mathscr{L}_I(x_3)] + iT[\mathscr{L}_I(x_1)\Lambda_2(x_2, x_3)] + $$
$$+ iT[\mathscr{L}_I(x_2)\Lambda_2(x_1, x_3)] + iT[\mathscr{L}_I(x_3)\Lambda_2(x_1, x_2)] + \Lambda_3(x_1, x_2, x_3). \qquad (1.6)$$

As it is well known, the calculation of the $\mathscr{T}_n$ terms (1.1) is connected with certain mathematical difficulties arising from the existence of singularities of the integrals over the momenta of the internal lines of the Feynman diagrams. After the regularization of Pauli and Villars we separate the regular parts of these integrals, but their singular parts need interpretation.

In this point we can make use of the Bogolyubov quasilocal operators, because it happens that the singular parts have always a quasilocal form. We insert that part with an opposite sign into the modified $\bar{\mathscr{L}}_I(x)$ operator (1.4), and now the calculation of the $\mathscr{T}_n$ term with the application of the modified operator $\bar{\mathscr{L}}_I(x)$ will certainly lead to a regular result.

Another problem is the connection between the $\mathscr{L}(x)$ n-order modification and the regularization of the other $\mathscr{T}_{n'}$ terms (where $n' > n$). It is intuitively clear that each n-order diagram is a construction part of some n'-order diagrams. Bogolyubov showed that the modification of the operator $\mathscr{L}_I(x)$ coming from the n-order diagrams gives automatically a regularization of those singularities in the n'-order terms, which are connected with the construction parts mentioned above.

However, in order to obtain a full regularization of the $\mathscr{T}_{n'}$ term, we must modify the $\bar{\mathscr{L}}(x)$ operator by a new, characteristic to that order of approximation, quasilocal operator $\Lambda_n(x_1 \ldots x_{n'})$ *etc.*

For example, let us consider the $S_3(x_1, x_2, x_3)$ operator (1.6). Without the modification of the $\mathscr{L}_I(x)$ operator it would have a form identical with the first term: $-T[\mathscr{L}_I(x_1)\mathscr{L}_I(x_2)\mathscr{L}_I(x_3)]$. Its integration in the momentum representation (after the application of the Wick theorem) gives the clasic singularities connected with some terms of the Wick expansion (with some Feynman diagrams). The second, third and fourth terms will eliminate singularities arising from these diagrams. In order to get free from the singularities arising from the characteristic third order diagrams, we must insert the Λ_3 term (the fifth addend).

2. The Kadyshevsky momentum representation

The S-operator in the form similar to (1.1′) after the application of the Wick theorem can be interpreted as usual by the Feynman diagrams. The singularities we meet, while calculating the S-operator terms, are connected with the infinit integration ranges of the integrals over the propagator moments and this forces us to renormalize the masses and coupling constants of the fields represented by the propagators mentioned above.

The momentum representation we are about to explain introduces some modifications to the Feynman interpretation, and therefore to the renormalization process. Let us consider the term $\mathscr{T}_n$ (1.1) in its "primary" form, *i.e.* with the application of Heaviside $\Theta(x^0)$ function:

$$\mathscr{T}_n = i^{n-1} \iint \Theta(x_1^0 - x_2^0) \ldots \Theta(x_{n-1}^0 - x_n^0)\mathscr{L}_I(x_1) \ldots \mathscr{L}_I(x_n) d^4x_1 \ldots d^4x_n \tag{2.1}$$

where

$$\Theta(x^0) = 0 \quad \text{for} \quad x^0 < 0$$

$$\Theta(x^0) = 1 \quad \text{for} \quad x^0 > 0$$

or in the equivalent form

$$\mathscr{T}_n = i^{n-1} \iint \Theta(\lambda \cdot (x_1 - x_2)) \ldots \Theta(\lambda \cdot (x_{n-1} - x_n))\mathscr{L}_I(x_1) \ldots \mathscr{L}_I(x_n) d^4x_1 \ldots d^4x_n \tag{2.1'}$$

where λ is a 4-vector on the upper mass shell $\lambda^2 = 1$, $\lambda^0 > 0$.

The equivalence of these two formulas follows from the commutation of the $\mathscr{L}_I(x)$ operators:

$$[\mathscr{L}_I(x), \mathscr{L}_I(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0.$$

Let us convert to the momentum representation:

$$\mathscr{L}_I(x) = \frac{1}{(2\pi)^4} \int e^{ip\cdot x} \tilde{\mathscr{L}}(p) d^4 p. \tag{2.2}$$

We express the Heaviside function as a Fourier transformate:

$$\Theta(\lambda \cdot x) = -\frac{1}{2\pi i} \int \frac{e^{-i\tau\lambda\cdot x}}{\tau + i\varepsilon} d\tau,$$

$$\lambda \cdot x \equiv \lambda^0 x^0 - \vec{\lambda} \cdot \vec{x},$$

$$p \cdot x \equiv p^0 x^0 - \vec{p} \cdot \vec{x}.$$

It is now easy to show, that:

$$\mathscr{T}_1 = \tilde{\mathscr{L}}_I(0)$$

$$\mathscr{T}_2 = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{\mathscr{L}}_I(\lambda\tau) \frac{d\tau}{\tau + i\varepsilon} \tilde{\mathscr{L}}_I(-\lambda\tau)$$

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$$\mathscr{T}_n = \left(-\frac{1}{2\pi}\right)^{n-1} \int_{-\infty}^{+\infty} \tilde{\mathscr{L}}_I(\lambda\tau_1) \frac{d\tau_1}{\tau_1 + i\varepsilon} \tilde{\mathscr{L}}_I(-\lambda\tau_1 + \lambda\tau_2)\ldots$$

$$\ldots \frac{d\tau_{n-1}}{\tau_{n-1} + i\varepsilon} \tilde{\mathscr{L}}_I(-\lambda\tau_{n-1}). \tag{2.3}$$

In order to show the Kadyshevsky graphical interpretation we must use a definite interaction model, for example:

$$\mathscr{L}_I(x) = e : \varphi^4(x): \tag{2.4}$$

which can be converted into momentum representation:

$$\tilde{\mathscr{L}}_I(\lambda\tau) = \frac{1}{\sqrt{2\pi}} \int \delta(\lambda\tau - \sum_{i=1}^{4} k_i) :\varphi(k_1)\varphi(k_2)\varphi(k_3)\varphi(k_4): \times$$

$$\times d^4 k_1 \ldots d^4 k_4. \tag{2.5}$$

We give a graphical interpretation to this expression:

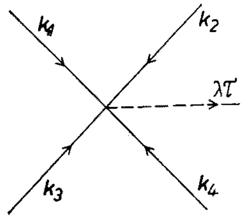

Fig. 1

where full lines represent the physical fields $\varphi(k)$, and the dotted line with momentum $\lambda\tau$ follows from the argument of the δ — Dirac function (2.5). Let us call it a quasiparticle line. Thus for $\mathscr{T}_1 = \tilde{\mathscr{L}}(0)$ we obtain the diagram:

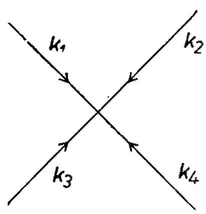

Fig. 2

and for $\mathscr{T}_2$ (after the application of the Wick theorem):

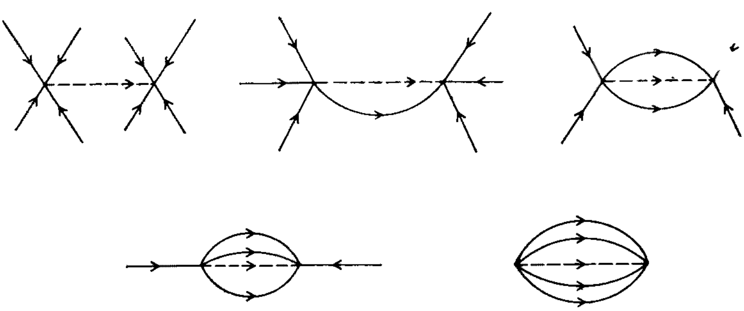

Fig. 3

The propagator function (the contraction of the field operators) for the scalar field $\varphi(k)$ is:

$$\underbrace{\varphi(k)\varphi(p)} = \delta(p+k)\Theta(k^0)\delta(k^2-m^2) \equiv \delta(p+k)D(k) \tag{2.6}$$

where k is the fourmomentum transferred on the propagator. The $\Theta(k^0)$ factor existing in the propagator function enables energy to be transferred in one direction along the internal line (for example from left vertex to the right; the numeration of the vertexes is identical with the succession of the $\mathscr{L}_I(x)$ operators in the $\mathscr{T}_2$ term). The particular kinematic analysis [2] shows that the momenta on the internal field lines are limited in that graphical interpretation, and that only the momenta of the quasiparticles can be infinite.

Thus we integrate over the internal lines, as follows. At first the integration over the internal physical lines (full lines) must be performed (these integrals have finite limits, so they are regular and become functions of $\lambda\tau$, *i.e.* of the quasiparticle line momentum). Then we integrate over the τ parameter and in general we meet singularities in this integral. The Pauli and Villars regularization, as always, brings this integral into a finite and a singular part $\tilde{\Lambda}_2$. The $\tilde{\Lambda}_2$ operator after rewriting it into the x-representation becomes a quasilocal operator and can be inserted into the modified $\bar{\mathscr{L}}_I(x)$ operator (1.4). The treatment of the next $\mathscr{T}_n$ terms is quite similar. The momentum Kadyshevsky representation therefore liberate us from the internal physical lines regularization, because all singularities are "pushed down" to the internal quasiparticle lines.

The authors of this work calculated the quasilocal operators in the second order of approximation for two models of interaction:

$$1.\quad \mathscr{L}_I(x) = e : \varphi^4(x):$$

$$2.\quad \mathscr{L}_I(x) = e : \overset{*}{\varphi}(x)A^\mu(x)A_\mu(x)\varphi(x): \tag{2.7}$$

where $\varphi(x)$ is connected with the boson field (charged in the second case) and $A^\mu(x)$ describes the electromagnetic field.

Let us consider only these terms of the S-matrix, which give a contribution to the cross-section.

Ad 1

The set of the modified Feynman diagrams [2] for that interaction in the second order of approximation has the form [3]:

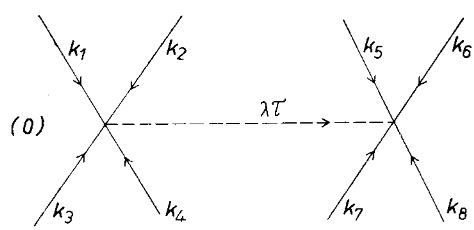

Fig. 4. $\lambda\tau = k_1+k_2+k_3+k_4 = -(k_5+k_6+k_7+k_8)$

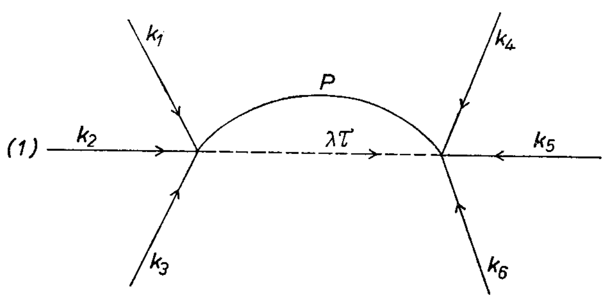

Fig. 5. $p = k_1+k_2+k_3-\lambda\tau$

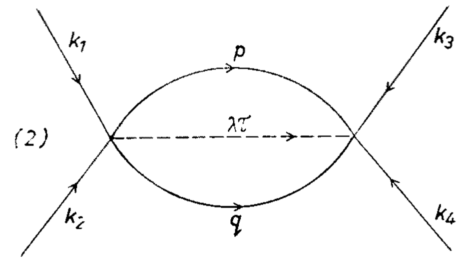

Fig. 6. $p = k_1+k_2-\lambda\tau-q$

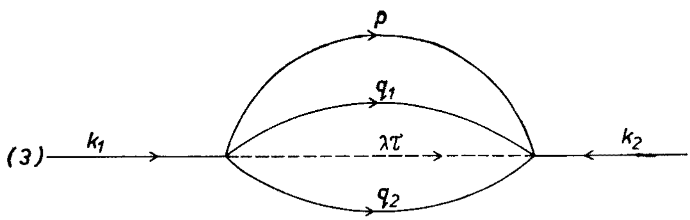

Fig. 7. $p = k_1-q_1-q_2-\lambda\tau$

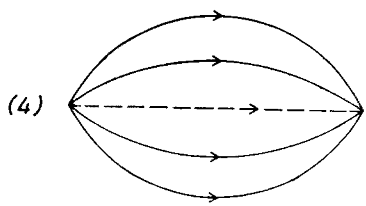

Fig. 8. No contribution to the cross-section

We have the matrix element of the second order as a sum of the parts connected with all diagrams:

$$\mathscr{T}_2 = \sum_{r=0}^{4} \mathscr{T}_2^{(r)}. \tag{2.8}$$

The calculation shows the necessity of regularization for the terms $\mathscr{T}_2^{(2)}$, $\mathscr{T}_2^{(3)}$ and $\mathscr{T}_{(2)}^4$ only.

After a regularization of singular integrals (Pauli and Villars):

$$\frac{1}{\tau+i\varepsilon} \to \frac{1}{\tau+i\varepsilon} - \left(\frac{1}{\tau+M+i\varepsilon}\right)^{r+2} \tag{2.9}$$

we separate the terms divergent with $M \to \infty$ and obtain for the quasilocal operators: for the diagram (2):

$$\Lambda_2^{(2)}(x_1, x_2) = a^{(2)}e\left[m - \frac{1}{4i}\left(\frac{\partial}{\partial x_1^0} - \frac{\partial}{\partial x_2^0}\right)\right] \times$$

$$\times\, \delta(x_1-x_2) : \varphi^2(x_1)\varphi^2(x_2) : \ln\left|\frac{M}{m}\right| \qquad (2.10$$

in the CM system ($\vec{k}_1+\vec{k}_2 = 0$), and with the λ-vector chosen as $\vec{\lambda} = 0$ in this system, and for the diagram (3)

$$\Lambda_2^{(3)}(x_1, x_2) = a^{(3)}e\left[a\left(\frac{\partial}{\partial x_1^0} - \frac{\partial}{\partial x_2^0}\right) + b\left(\frac{\partial^2}{\partial x_1^{02}} - \frac{\partial^2}{\partial x_2^{02}}\right) + \right.$$

$$\left. + c\left(\frac{\partial^3}{\partial x_1^{03}} - \frac{\partial^3}{\partial x_2^{03}}\right) + d\left(\frac{\partial^4}{\partial x_1^{04}} + \frac{\partial^4}{\partial x_2^{04}}\right) + f\right]\delta(x_1 - x_2) : \varphi(x_1)\varphi(x_2) : \tag{2.11}$$

in the $\vec{k}_1 = 0$ system and with $\vec{\lambda} = 0$ in this system, where

$$a = \frac{81}{2}\ln 3\left|\frac{M}{m}\right| + \frac{343}{2}\ln\left|\frac{M}{m}\right| + \frac{405}{2}\left(\frac{M}{m}\right)^2;$$

$$b = \frac{234}{2}\left(\frac{M}{m}\right); \quad c = -\frac{1}{2}\left(\frac{M}{m}\right)^2;$$

$$d = \frac{1}{2}\left(\frac{M}{m}\right); \quad f = -\left(\frac{M}{m}\right) + \frac{81}{2}\left(\frac{M}{m}\right)^2.$$

Ad 2

We obtain the following Feynman diagrams:

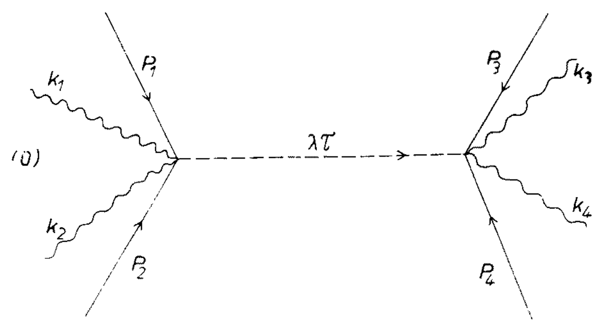

Fig. 9

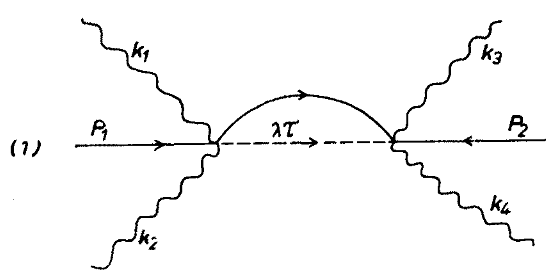

Fig. 10

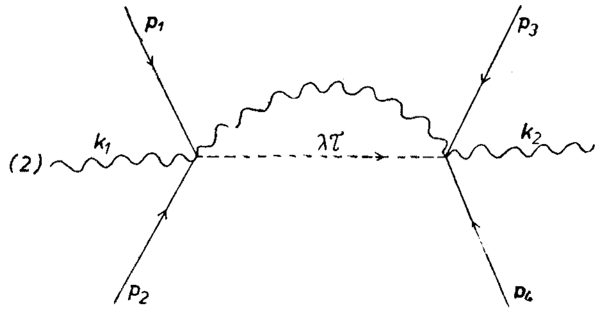

Fig. 11

The terms of $\mathscr{T}_2^{(0)}$, $\mathscr{T}_2^{(1)}$ and $\mathscr{T}_2^{(2)}$ are regular, so they do not need regularization. For other terms:

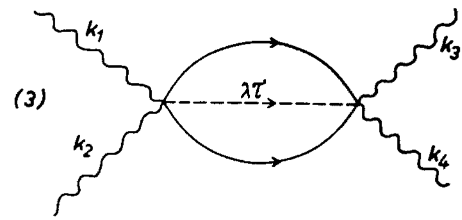

Fig. 12

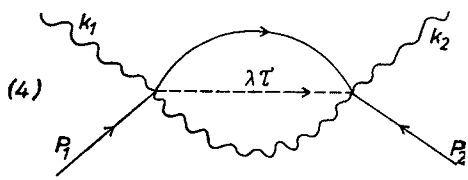

Fig. 13

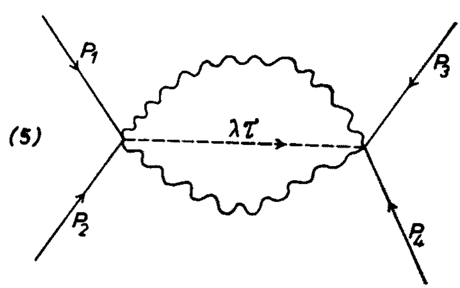

Fig. 14

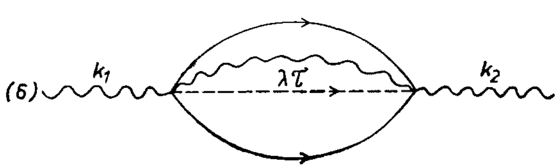

Fig. 15

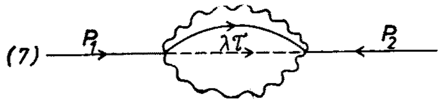

Fig. 16

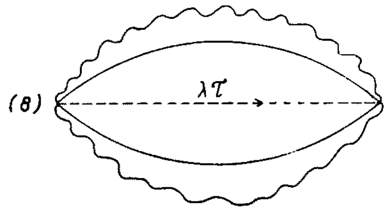

Fig. 17

we obtain after regularization:

$$\Lambda_2^{(3)}(x_1, x_2) = a^{(3)}\delta(x_1 - x_2) \ln \left| \frac{M}{m} \right| :A^{\mu}(x_1)A_{\mu}(x_2)A^{\nu}(x_1)A_{\nu}(x_2):$$

$$\Lambda_2^{(4)}(x_1, x_2) = a^{(4)}\delta(x_1 - x_2) \ln \left| \frac{M}{m} \right| :\overset{*}{\varphi}(x_1)\varphi(x_2)A^{\mu}(x_1)A_{\mu}(x_2):$$

$$\Lambda_2^{(5)}(x_1, x_2) = a^{(5)}\delta(x_1 - x_2) \ln \left| \frac{M}{m} \right| :\overset{*}{\varphi}(x_1)\varphi(x_1)\overset{*}{\varphi}(x_2)\varphi(x_2):$$

$$\Lambda_2^{(6)}(x_1, x_2) = a^{(6)}\delta(x_1 - x_2) \left[\left(\alpha + \beta \frac{\partial^2}{\partial x_1^{0 2}} \right) \ln \left| \frac{M}{m} \right| + \gamma \ln^2 \left| \frac{M}{m} \right| \right] :A^{\mu}(x_1)A_{\mu}(x_2):$$

$$\Lambda_2^{(7)}(x_1, x_2) = a^{(7)}\delta(x_1 - x_2) \left[\left(\delta + \eta \frac{\partial^2}{\partial x_1^{0 2}} \right) \ln \left| \frac{M}{m} \right| + \varrho \ln^2 \left| \frac{M}{m} \right| \right] :\overset{*}{\varphi}(x_1)\varphi(x_2):.$$

where

$$\alpha = 3\pi m^2 - i\frac{m^2}{4}; \quad \beta = \frac{i}{8}; \quad \gamma = -i\frac{m^2}{2}$$

$$\delta = \pi m^2; \quad \eta = -\frac{i}{2}; \quad \varrho = i\frac{m^2}{2}.$$

The terms $\Lambda_2^{(6)}$ and $\Lambda_2^{(7)}$ are calculated in the CM systems $\vec{P}_{\text{in}} = 0$ and with the choice $\vec{\lambda} = 0$ in those systems. For both models of interaction, the regularization process was performed in the momentum representation, and the results are rewritten to the x-representation. It can be expected [2] that the calculated quasilocal terms are the same as those defined by Bogolyubov, but a direct verification has not been so far performed.

The authors consider this work as a preliminary step to future developments.

REFERENCES

[1] N. N. Bogolyubov, D. V. Shirkov, *Vvedenie v teoriyu kvantovannykh poleǐ*, Moskva 1957.
[2] V. G. Kadyshevsky, *Zh. Eksper. Teor. Fiz.* (USSR), **46**, 654, 873 (1964).
[3] Z. Borelowski, *Acta Phys. Polon.*, **B1**, 155 (1970).
[4] H. A. Bethe, F. de Hoffmann, *Mesons and Fields*, Vol. 2, Row, Peterson and Co., Evanston Ill., 1955.
[5] B. V. Medvedev, *Dokl. Akad. Nauk SSSR*, **135**, 1087 (1960).
[6] B. V. Medvedev, *Zh. Eksper. Teor. Fiz.* (USSR), **40**, 826 (1961).
[7] B. V. Medvedev, *Zh. Eksper. Teor. Fiz.* (USSR), **41**, 1130 (1961).
[8] B. V. Medvedev, *Dokl. Akad. Nauk SSSR*, **143**, 1071 (1962).
[9] B. V. Medvedev, *Zh. Eksper. Teor. Fiz.* (USSR), **47**, 147 (1964).