A NEW INTERPRETATION OF EINSTEIN'S UNIFIED FIELD

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It is proposed that the electromagnetic field in the nonsymmetric unified field theory should be related to the skewsymmetric part of the fundamental tensor by a second grade differential operation. The form of this relation suggested leads to a correction factor in the Coulomb law.

1. *Introduction*

In the standard interpretations of Einstein's unified field geometries (Refs [1] and [2]) the skewsymmetric part, $\underset{\vee}{g}_{\mu\nu}$, of the fundamental tensor $g_{\mu\nu}$ (which replaces the metric of General Relativity), or a linear combination of its components, represents the electromagnetic field. This is not surprising in the early development of the nonsymmetric unified field theory because of the very few exact solutions of the field equations discovered and because it was a unification of the symmetric gravitational field with the skewsymmetric Maxwell field that Einstein set out to find.

However, we have suggested in a previous article (Ref. [3]) that it may be now necessary to revise the above interpretation. The reason for this is that the solution we have found in the case of cylindrically symmetric fields led to an unlikely form of the electromagnetic field. Since a polynomial expression for the latter in terms of the distance r from the axis of symmetry is unique, it seemed sensible to try to lower the exponents and this can be achieved only by some process of invariant differentiation.

2. *The electromagnetic field*

Let us consider first a weak field approximation for which

$$g_{\mu\nu} = \eta_{\mu\nu}+g_{\mu\nu}^{(1)}, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, diag $(-1, -1, -1, +1)$ and $g_{\mu\nu}^{(1)}$ is a small correction term whose squares and higher powers are neglected in the field equations.

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It may well be inappropriate to place too much reliance on such an approximation in a unified field theory, that is to treat both the gravitational and the electromagnetic fields as weak in the above sense. It is known, for example (Ref. [4]), that the approximation splits the field equations in such a way that it is impossible to derive from them the equations of motion of a charged particle (in other words, the Lorentz force). A comparison of the relative strengths of the two fields (or, for that matter, of the gravitational field with any other field known to the physics) suggests that we should look instead at a fundamental tensor of the form

$$\eta_{\mu\nu}+g^{(1)}_{\underline{\mu\nu}}+g_{\underset{\vee}{\mu\nu}} \tag{2}$$

with only the correction $g^{(1)}_{\underline{\mu\nu}}$ to the symmetric part of the tensor regarded as small. This, however, would almost irrevocably commit us to interpreting the symmetric part of $g_{\mu\nu}$ as the gravitational field and even such an interpretation may not be warranted in an ultimate unified field theory based on de-symmetrization of general relativistic geometry. We employ the weak field approximation here, merely as a heuristic device to suggest to us an alternative form of the electromagnetic field tensor.

If we write

$$g_{\sigma\mu}(\eta^{\sigma\lambda}+{}^{*}g^{(1)\sigma\lambda}) = \delta^{\lambda}_{\mu}, \tag{3}$$

the relevant weak field equations (Ref. [4]) are well known to take the form

$$^{*}g^{(1)\underset{\vee}{\mu\nu}}{}_{,\nu} = 0, \tag{4a}$$

$$\Box(g^{(1)}_{\underset{\vee}{\dot{\mu}\dot{\nu}},\dot{\lambda}}) = 0, \tag{4b}$$

where $\Box$ is the D'Alembertian operator,

$$\Box \equiv \eta^{\varrho\sigma}\frac{\partial}{\partial x^{\varrho}}\frac{\partial}{\partial x^{\sigma}}, \tag{5}$$

and dots under the indices in the equations (4b) signify cyclic summation. We have omitted $\sqrt{-\det(g_{\mu\nu})}$ in the equations (4a) (and were left with tensors rather than with tensor densities) since the density factor is equal to unity to the order of the approximation considered. Let us now identify $\Box g^{(1)}_{\underset{\vee}{\mu\nu}}$ as the electromagnetic field tensor

$$f_{\mu\nu} = -f_{\nu\mu} = \Box g^{(1)}_{\underset{\vee}{\mu\nu}}. \tag{6}$$

The field equations (4) then acquire the form of Maxwell's equations in vacuum:

$$f^{\mu\nu}{}_{,\nu} = 0, f_{\dot{\mu}\dot{\nu},\dot{\lambda}} = 0 \tag{7}$$

but again only to the desired order of approximation. The tensor indices are now raised and lowered with the help of Minkowski's $\eta_{\mu\nu}$.

Expression (6) is, of course, not generally covariant and we have considerable freedom in choosing a corresponding covariant formula. The guiding principle in selecting the latter is that whatever we should adopt ought to reduce to (6) for genuinely weak fields and, moreover, that the expression for $f_{\mu\nu}$ should be transposition invariant with respect to $g_{\mu\nu}$ and to the affine connection which must be introduced at this stage. A suitable candidate is, in Einstein's notation

$$f_{\mu\nu} = {}^*g^{\alpha\beta} g_{\underset{+-}{\mu\nu};\alpha\beta}, \tag{8}$$

where ${}^*g^{\alpha\beta}$ is the tensorial inverse of $g_{\alpha\beta}$. We now postulate that this should in fact give the electromagnetic field tensor, when referred to a cartesian coordinate system (the signficance of this last remark is apparent from Ref. [5]).

3. *Consequences of the new interpretation*

The above formula fulfils our anticipated requirement (Ref. [3]) that the exponents of the fields found previously under restricted symmetry conditions should be lowered by 2. We can illustrate this explicitly considering particular solutions of the field equations.

a) General, cylindrically symmetric field

The cylindrically symmetric fundamental tensor has the form (Ref. [3])

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & & & \\ & -\alpha & E & \\ & -E & -\beta & H \\ & & -H & \gamma \end{bmatrix}, \tag{9}$$

with α, β, γ, E and H functions of only the radial distance from the axis of symmetry r. The solution of the strong field equations found in the above mentioned article was

$$\alpha = \gamma = 1, \beta = R^2 \{1-(m_3^2-m_4^2) R^2\},$$
$$E = m_3 R^2, H = m_4 R^2, \tag{10}$$

where

$$R = lr+m,$$

and l, m, m_3 and m_4 are constants. The components of the electromagnetic field can be calculated immediately from the expression (8). As expected, we find that its only non-vanishing components are

$$f_{23} = -10\,m_3, f_{34} = -10\,m_4, \tag{11}$$

in a cylindrical polar coordinate system. This then corresponds (Ref. [3]) to the classical fields

$$\underset{\sim}{E} = -\frac{10m_3}{R}\hat{\underset{\sim}{R}} \text{ (radial)}, \qquad \underset{\sim}{H} = -\frac{10m_4}{R}\hat{\underset{\sim}{\theta}} \text{ (transverse)}$$

surrounding a static line charge and a steady line current on the axis of symmetry.

b) Papapetrou's special electrostatic solution (Ref. [5]).
The fundamental spherically symmetric tensor of this solution is

$$\begin{bmatrix} -\alpha & & & \\ & -\beta & f\sin\theta & \\ & -f\sin\theta & -\beta\sin^2\theta & \\ & & & \gamma \end{bmatrix}, \tag{12}$$

with f/r^2 representing magnitude of the electrostatic field intensity. Papapetrou's special solution is (without the "cosmological" constant)

$$\gamma = \alpha^{-1} = 1 - \frac{2m}{r}, \quad \beta = r^2, \quad f = -cr^2, \tag{13}$$

where m and c are constants.

Let us now consider the result of calculating the field according to (8). At a great distance from the origin ($r = 0$) where the joint source of the field is situated, we have

$$\Box f = -2c,$$

so that the corresponding electrostatic field is given by the classical Coulomb formula

$$\boldsymbol{E} = -\frac{2c}{r^2}\hat{\boldsymbol{r}}. \tag{14}$$

Near the origin, however, we obtain a correction to the Coulomb law, although, of course, it is still only the f_{23} component thereof, which does not identically vanish. Since the components of the affine connection are given by

$$\Gamma^1_{11} = -\Gamma^4_{\underline{14}} = \frac{1}{2}\alpha^{-1}\frac{d\alpha}{dr}, \quad \Gamma^1_{22} = -r\alpha^{-1}, \quad \Gamma^3_{\underline{13}} = \Gamma^2_{\underline{12}} = r^{-1},$$

$$\Gamma^3_{23} = \cot\theta, \; \Gamma^2_{33} = -\sin\theta\cos\theta, \; \Gamma^1_{33} = -r\alpha^{-1}\sin^2\theta$$

$$\Gamma^1_{44} = -\frac{1}{2}\alpha^{-3}\frac{d\alpha}{dr}, \quad \Gamma^1_{23} = -\frac{1}{2}cr\alpha^{-1}\sin\theta,$$

a straightforward calculation shows that

$$f_{23} = \frac{4c}{1+c^2}\frac{\sin\theta}{\alpha},$$

whence the radial electrostatic field is of magnitude

$$\frac{4c}{1+c^2}\frac{\left(1-\dfrac{2m}{r}\right)}{r^2}. \tag{15}$$

4. *Discussion*

It follows from the formula (15) that, in the current interpretation, the nonsymmetric unified field theory implies a correction to the Coulomb inverse square law. The effect is very small, for if we take

$$e = \frac{4c}{1+c^2},$$

as the electronic charge and m as the Schwarzschild mass of an electron, the coefficient of the inverse cube term is

$$\frac{2Gme}{c_1^2} \sim 10^{-48} \text{ cm};$$

G being the Newtonian gravitational constant and c_1 the speed of light in a vacuum. Massive bodies with a large electric charge are not known but, if we were to assume the same correction factor for a magnetic field (we do not have yet a dipole solution!) the effect might well be observable in the vicinity, say, of a magnetic star (such as the recently discovered object of about the solar mass, terrestrial radius and a field of 10^7 gauss).

The form of the correction seems to create some difficulty in that it indicates an equilibrium surface

$$r = 2m$$

on which the electrostatic force vanishes. This, however, is only apparent for the following reason. Our solution is inapplicable for a two-body problem. Consequently, it predicts merely that an infinitely small test charge would rest in equilibrium on the above surface, the field being of course due to a point source at $r = 0$. Such a test charge has little physical reality. In other words expression (15) represents the electrostatic force between two charges at a distance apart large enough for the source to be regarded as a geometrical point and certainly for

$$r \gg 2m$$

Finally, we should recall that in the Einstein-Kaufman theory recently revised by one of us (Ref. [6]) there exists no solution corresponding to a Coulomb point charge (that is, a charge which would give exactly the inverse square law of force). The question of whether a corrected solution is possible in this version of the unified field theory is being investigated. However, Papapetrou's special solution is also a particular solution of the new theory and therefore we can expect the force formula found in the present article to be a valid, although perhaps only approximate, correction to the Coulomb law.

APPENDIX

Wave-like solutions

It may appear from equation (6) that the above interpretation excludes wave solutions. This, of course is not the case as may be easily seen from a one-dimensional example. Consider

$$\Box g(x, t) = f(x-ct)$$

If we let

$$u = x - ct \text{ and } v = x + ct,$$

then the above equation becomes

$$\frac{\partial^2}{\partial u \partial v} g = f(u),$$

whence, we can always find a solution in the form.

$$g = v \int_A^u f(u)\, du,$$

A being an arbitrary constant. It is clear that g will not, in general, satisfy the homogeneous wave equation, although

$$\Box f = 0.$$

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