CYLINDRICAL SYMMETRY IN EINSTEIN'S UNIFIED FIELD THEORY. III

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A cylindrically symmetric solution of Einstein's unified field equations is derived for the case when neither of the fields tentatively interpreted as electro- and magnetostatic vanishes. The form of the solution suggests a new interpretation of the geometry which will be proposed in a separate article.

1. Introduction

This is the third of the articles in which we discuss static, cylindrically symmetric solutions of the field equations arising in Einstein's nonsymmetric unified field theory (Ref. [1]). In the case of the symmetry considered, the difference between the so-called strong and weak field equations is slight and can be conveniently ignored. Consequently, we shall be concerned mainly with the former set of equations, namely,

$$g_{\underset{+}{\mu}\underset{-}{\nu};\lambda} \equiv g_{\mu\nu,\lambda}-\Gamma^{\sigma}_{\mu\lambda}g_{\sigma\nu}-\Gamma^{\sigma}_{\lambda\nu}g_{\mu\sigma} = 0, \tag{1a}$$

$$R_{\mu\nu} \equiv \Gamma^{\sigma}_{\mu\nu,\sigma}-\Gamma^{\sigma}_{\mu\sigma,\nu}-\Gamma^{\sigma}_{\mu\varrho}\Gamma^{\varrho}_{\sigma\nu}+\Gamma^{\sigma}_{\mu\nu}\Gamma^{\varrho}_{\sigma\varrho} = 0, \tag{1b}$$

$$*\mathfrak{g}^{\underset{\vee}{\mu\nu}}{}_{,\nu} = 0. \tag{1c}$$

Here $g_{\mu\nu}$ is the (nonsymmetric) fundamental tensor, $\Gamma^{\lambda}_{\mu\nu}$, the (likewise nonsymmetric) affine connection, $*g^{\mu\nu}$, the inverse tensor of $g_{\mu\nu}$, $*\mathfrak{g}^{\mu\nu} = \sqrt{-g}*g^{\mu\nu}$, the associated tensor density (g being the determinant of $g_{\mu\nu}$) and $*\mathfrak{g}^{\underset{\vee}{\mu\nu}}$, its skew symmetric part.

We anticipated in a previous article (Ref. [2], paper II) a discussion of a static, electromagnetic field, in which neither the electric vector $\boldsymbol{E}$, nor the magnetic vector $\boldsymbol{H}$ are zero. It was expected that this more general solution would lead to a formulation of an analogue of Ohm's law and a unified field theoretic description of steady currents. It turns out, however, that the peculiar form of the only possible solution (as shown below) suggests rather a radical interpretation of the theory with reference, in particular, to what should be regarded as the electromagnetic field.

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We consider a cylindrically symmetric field (Ref. [2], paper I) with the fundamental tensor

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\alpha & E & 0 \\ 0 & -E & -\beta & -H \\ 0 & 0 & H & \gamma \end{bmatrix}, \tag{2}$$

in which α β, γ, E and H are functions of the radial distance from the axis of symmetry, r, only. This corresponds to an electromagnetic field given by

$$\mathbf{E} = \frac{E}{r}\,\hat{\mathbf{r}}, \qquad \mathbf{H} = \frac{H}{r}\,\hat{\boldsymbol{\theta}}, \tag{3}$$

since, in the standard interpretation (Ref. [3]), $g_{\underset{\vee}{\mu\nu}}$ is either taken as the electromagnetic field, or bears a simple, algebraic relation to the latter. As usual, we assume that

$$g = -\alpha(\alpha\beta\gamma-\alpha H^2+\gamma E^2) = -\alpha^2\beta\gamma(1+p-q) \neq 0, \tag{4}$$

$p = H^2/\alpha\beta$, $q = H^2/\beta\gamma$, anywhere in the space-time manifold.

2. *The field equations*

Using the form (2) of the fundamental tensor, we can easily verify that the equations (1c) are identically satisfied. The components of the affine connection are determined from the equations (1a) and for the sake of the subsequent solution, we find it convenient to express the latter as follows. Let us define the functions U, V, A, B, J, K and L (of r), by

$$\frac{\alpha\beta'}{2E} - \frac{E\alpha'}{2\alpha} - \frac{E^2-\alpha\beta}{2\alpha\beta}E' + \frac{EH}{2\beta\gamma}H' - E\left(\frac{H^2}{\beta\gamma} - \frac{E^2}{\alpha\beta} + \frac{\alpha\beta}{E^2}\right)U - \frac{\alpha H}{E}V = 0, \tag{5a}$$

$$-\frac{H}{2\gamma}\gamma' - \frac{EH}{2\alpha\beta}E' + \frac{1}{2}\left(1+\frac{H^2}{\beta\gamma}\right)H' - H\left(\frac{H^2}{\beta\gamma} - \frac{E^2}{\alpha\beta}\right)U - \gamma V = 0, \tag{5b}$$

$$A = 2U - \frac{E'}{2E} - \frac{H'}{2H}, \tag{5c}$$

$$B = U - \frac{H'}{2H}, \tag{5d}$$

$$J = \frac{\beta'}{2\beta} + \frac{\gamma'}{2\gamma} + p\frac{E'}{2E} - (1+q)\frac{H'}{2H} - (p-q+1)U, \tag{5e}$$

$$K = \frac{\alpha'}{2\alpha} + \frac{\beta'}{2\beta} - (1-p)\frac{E'}{2E} - q\frac{H'}{2H} - (p-q+1)U, \tag{5f}$$

$$L = \frac{\beta'}{2\beta} - U, \tag{5g}$$

where dashes denote differentiation with respect to r. Eliminating V from the first two of these equations we find that the co-factor of U becomes

$$-\frac{g}{\alpha^2 H}\left(2+\frac{g}{\alpha^2\beta\gamma}\right),$$

and this is nonzero under the well-known conditions of Hlavaty ($\bar{g}(\bar{g}-2) \neq 0$, where $\bar{g} = g/\det(g_{\underline{\mu\nu}})$ and $g_{\underline{\mu\nu}}$ denotes as usual the symmetric part of the fundamental tensor, for the fields of the second class, Ref. [3], p. XV). It follows that our functions are well determined. We may observe also that

$$-\frac{pK-qJ}{p-q+1}+L+U = 0, \tag{6}$$

and

$$V = -\frac{H}{\gamma}(J-L-U). \tag{7}$$

With these definitions we can express the non-vanishing components of the affine connection as

$$\Gamma^1_{11} = (p-1)A-(p+q-1)B+K-L,$$

$$\Gamma^1_{22} = (p+1)A-(p-q+1)B-K+L,$$

$$\Gamma^1_{33} = \frac{\beta}{\alpha}L,$$

$$\Gamma^1_{44} = \frac{\gamma}{\alpha}\{pA-(p-q+1)B+J-L\},$$

$$\Gamma^1_{\underset{\vee}{23}} = \frac{E}{\alpha}(K-L), \quad \Gamma^1_{\underline{24}} = \frac{EH}{\alpha\beta}A, \quad \Gamma^1_{\underset{\vee}{34}} = \frac{H}{\alpha}(-J+L),$$

$$\Gamma^2_{\underline{12}} = -A-(q-1)B+K-L$$

$$\Gamma^2_{\underset{\vee}{13}} = \frac{E}{\alpha}(K-L-U), \quad \Gamma^2_{\underline{14}} = -\frac{EH}{\alpha\beta}(A-B),$$

$$\Gamma^3_{\underset{\vee}{12}} = -\frac{E}{\beta}(A-B), \quad \Gamma^3_{\underline{13}} = U, \quad \Gamma^3_{\underset{\vee}{14}} = -\frac{H}{\beta}B,$$

$$\Gamma^4_{\underline{12}} = \frac{EH}{\beta\gamma}B, \quad \Gamma^4_{\underset{\vee}{13}} = -\frac{H}{\gamma}(J-L-U),$$

$$\Gamma^4_{\underline{14}} = pA-(p+1)B+J-L.$$

With the help of these expressions we can now work out the components of the Ricci tensor $R_{\mu\nu}$ and hence write down the field equations to be solved. After some straightforward manipulation the latter (the strong field equations) become

$$A'+A[-(2p-2q+3)B+J-2K-L+U]+B[2(p-q+1)B-2J+2K] = 0 \tag{8a}$$

$$B'+B[-2(1+p)A+(2p+2q+1)B-J-L+U] = 0, \tag{8b}$$

$$J'+J[-3B+J+L+3U]-\frac{2pq}{p-q+1}(J-K)^2-\frac{2p}{p-q+1}K(J-K) = 0, \tag{8c}$$

$$K'+K[-2(A-2B)-3B+J+L+3U]-\frac{2pq}{p-q+1}(J-K)^2-\frac{2q}{p-q+1}J(J-K) = 0, \tag{8d}$$

$$L'+L[-B+J-L+U]-\frac{2pq}{p-q+1}(J-K)^2+\{[2K(L+U)-\frac{2q}{p-q+1}J(J-K)], \tag{8e}$$

$$\text{or } \left[2J(L+U)-\frac{2p}{p-q+1}K(J-K)\right]\} = 0, \tag{8f}$$

and

$$U'+U(-B+J-L+U) = 0,$$

together with (6) and the easily verified relation

$$p(A-B)^2+(A-B)B-qB^2 = 0, B \neq 0. \tag{9}$$

3. *Integration of the field equations*

We find immediately from (6), (8b) and (8f) that

$$B = \frac{\sqrt{\beta\gamma}}{EH}\frac{c_2}{\sqrt{p-q+1}}, \tag{10}$$

$$U = \frac{c_3}{\sqrt{\beta\gamma}\sqrt{p-q+1}}. \tag{11}$$

and, by subtracting (8a) and (8b), that

$$A-B = \frac{\alpha\beta}{EH}\frac{c_1}{\sqrt{\beta\gamma}\sqrt{p-q+1}}. \tag{12}$$

c_1, c_2 and c_3 are arbitrary, nonzero constants (since when $A = B$ we have $B = 0$). The equations for J, K and L can be simplified by the substitutions

$$J = L+M, K = L+N, Z = \Gamma^1_{22}, W = -B+M+U,$$

and

$$X = (p-1)(A-B)-(q+1)B+M+N+U. \tag{13}$$

We then get two more first integrals in the form

$$Z+W = \frac{2c_4}{\sqrt{\beta\gamma}\sqrt{p-q+1}}, \tag{14}$$

$$X = \frac{2c_5}{\sqrt{\beta\gamma}\sqrt{p-q+1}}, \tag{15}$$

c_4 and c_5 being two more constants of integration.

Let now

$$c_6 = c_5+c_4-c_3,\ c_7 = c_5-c_4, \tag{16}$$

and

$$Q = \frac{1}{\sqrt{\beta\gamma}\sqrt{p-q+1}} \tag{17}$$

Since

$$\beta = \frac{c_2}{c_1}\frac{H^2}{\alpha} - \frac{c_1}{c_2}\frac{E^2}{\gamma},$$

we now arrive at the complete set of first order equations:

$$\frac{\alpha'}{2\alpha} = \left(c_7+c_1\frac{E}{H}\right)Q, \tag{18a}$$

$$\frac{E'}{2E} = \left(c_3-c_2\frac{H}{E}+\frac{c_1^2}{c_2}\frac{\alpha E}{\gamma H}\right)Q, \tag{18b}$$

$$\frac{H'}{2H} = \left(c_3-\frac{c_2^2}{c_1}\frac{\gamma H}{\alpha E}+c_1\frac{E}{H}\right)Q, \tag{18c}$$

$$\frac{\gamma'}{2\gamma} = \left(c_6-c_2\frac{H}{E}\right)Q, \tag{18d}$$

and

$$\frac{Q'}{Q^2} = -c_6-c_3+c_1\frac{E}{H} \tag{18e}$$

Let further,

$$\varrho = -\frac{c_2}{c_1}\frac{E}{H}, \qquad \sigma = -\frac{c_2}{c_1}\frac{\gamma}{\alpha}, \tag{19}$$

and define new constants, b and c, by

$$bc = \frac{c_2^3}{c_1^3}, \qquad b+c = \frac{c_2}{c_1^2}(c_7-c_6) \tag{20}$$

Equation

$$R_{11} = 0,$$

then gives

$$\frac{d\sigma}{d\varrho} = \frac{\sigma^2(\varrho-b)(\varrho-c)}{\varrho(\sigma+1)(bc\sigma+\varrho^2)} \tag{21}$$

σ and ϱ cannot be constant since then $\beta = 0$ which is meaningless. It follows that

$$bc\sigma+\varrho^2 \neq 0.$$

There are two cases. Either

$$\frac{c_2}{c_1^2}(c_3-c_7) = a = 0,$$

or

$$bc(a+b+c) = 0.$$

The first of these does not satisfy the field equations and the second leads to the conclusion that

$$c_1 = c_2 = 0.$$

Accordingly we obtain two solutions, for which either

$$c_7+c_3 = 0$$

or not.

The first of these is an exponential solution which is hard to interpret physically. We reject it in the same way as the spurious solutions of paper I.

In the second case, we put

$$c_6+c_3 = l, \tag{22}$$

and deduce easily that

$$Q = (lr+m)^{-1}, \tag{23}$$

m being a constant of integration.

Let us write

$$R = lr+m \text{ and } c_3 = \lambda l.$$

Then with the help of (23) the final solution becomes

$$\alpha = R^{2\lambda(\lambda-1)},$$

$$\beta = R^{2\lambda}-m_3^2R^{2\lambda(3-\lambda)}+m_4^2R^{2(3\lambda-1)},$$

$$\gamma = R^{2(1-\lambda)},$$

$$E = m_3 R^{2\lambda},$$

and

$$H = m_4 R^{2\lambda}, \tag{24}$$

where m_3 and m_4 are constants of integration.

4. Discussion

It may well be thought that the solution (24) of the field equations is no more reasonable than the exponential solutions rejected both here and in I. The most likely value we can choose for λ is unity when both α and $\gamma = 1$. Unfortunately E and H are then proportional to R^2. The corresponding electric and magnetic field intensities are hard to visualize. The problem is similar to that encountered in the special solution of Papapetrou (Ref. [4)] where we have to envisage charge densities of a distribution depending on the radial coordinate. In view of (3) it would be best if we could lower the exponents of E and H by 2, or in other words, if we could change the interpretation of geometry in such a way that these quantities had to be differentiated twice before their identification with physical objects.

Now there is no compelling reason why we should regard $g_{\underset{\vee}{\mu\nu}}$ (or a combination of its components found by Hlavaty) as the electromagnetic field at all. It is true that we seek a geometrization of the latter and that, because of the skewsymmetry of the field we want to geometrize (that is to relate it by algebraic and analytical processes to the fundamental tensor and to the affine connection which define the geometry), we presume it to be expressed in terms of $g_{\underset{\vee}{\mu\nu}}$. However, any tensor function of the skew symmetric part of the fundamental tensor (in which the symmetric part may be likewise involved) which does not destroy symmetric properties and preserves transposition invariance (Ref. [1]) is *a priori* acceptable. What that function should be, must be dictated by reasons of expediency since, as Einstein put it, all that we can expect from a unified field theory of the kind considered is that a Maxwell field should somehow appear related to the geometrical structure proposed.

In view of what we want to do to the functions E and H found in the preceding section, we further expect the above relation to be a second order differential one. An explicit proposal for a radical re-interpretation of Einstein's theory will be made in a separate publication since it will be found that it applies to a more general symmetry than the cylindrical case considered herein.

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