AMPLIFIED DIRAC EQUATION AND ITS REDUCTION TO THE PAULI FORMALISM

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Covariant and Hamiltonian forms of the Dirac equation (for a charged particle in an external electromagnetic field) amplified by the Pauli terms, have been investigated in connection with the problem of reduction to the subspace of positive energy states.

1. Introduction

The possibility of modifying the inhomogeneous Dirac equation in a Lorentz- and gauge-invariant way (preserving also the spinor character of the wave function) by introducing additional terms proportional to the external field, has been pointed out by Pauli ([1], p. 233, see also [2], p. 203). The most characteristic feature of this equation is that an "intrinsic" magnetic moment of the related particle can be described in terms of it. The further development of the idea of Pauli has followed two, rather distinct paths:

1) Relativistic effects caused by the magnetic moments of the nucleons have been — for lack of a satisfactory theory of nuclear forces — approximately described by means of this phenomenological method (extended also to the case of other external fields)[1]

2) For the electrons a "semi-phenomenological" description of radiative corrections may be achieved in this way, as only numerical values of the dimensionless constants appearing in this equation are taken from quantum electrodynamics. (For details see Bethe and Salpeter [18], p. 136 and 175). Explicit calculations given in [16] for the case of the hydrogen atom have shown the usefulness of this method, implying its more general use.

The amplified Dirac equation in its most general form offers thus a wide mathematical scheme suitable for a simplified treatment of various problems of fermion particles (these problems being defined, from the beginning, in the Hilbert space of the first quantization).

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[1] Among many publications on this subject the following may be quoted: Furry [3], Margenau [4], Caldirola [5], [6], [7], Breit [8], [9], Petiau [10], [11], Sachs [12], Foldy [13], [14], [15], Barker and Chraplyvy [16], Bethe [17].

There are also possibilities of extending this method to many-particles problems[2]. We restrict our present considerations to the one-particle problem taking into account the special form of the amplified Dirac equation quoted in [18].

2. *Covariant and Hamiltonian forms of the amplified Dirac equation*

The manifestly covariant forms of the (unmodified) Dirac equation, as well as those of its amplified counterpart, correspond to the fundamental role played by the relativity postulates in deducing these equations. Hence one usually starts from the well known formulae[3] ((2) being the previously mentioned form [18] of the amplified equation).

$$\Omega\psi = 0, \qquad \Omega = \gamma_\mu\left(p_\mu - \frac{e}{c}A_\mu\right) - imc, \tag{1}$$

$$\Omega'\psi' = 0, \qquad \Omega' = \Omega + g_1\frac{eh}{4mc^2}\gamma_\mu\gamma_\nu F_{\mu\nu} - g_2\frac{eh^2}{m^2c^3}\gamma_\mu\Box A_\mu, \tag{2}$$

where

$$F_{\mu\nu} = \partial A_\nu/\partial x_\mu - \partial A_\mu/\partial x_\nu. \tag{3}$$

(A_μ denoting four-potential of the electromagnetic field and the "primed" quantities referring to the amplified equation). Besides the proper Pauli terms (proportional to $F_{\mu\nu}$) still other (containing $\Box A_\mu$) are present in (2) and they correspond too to the radiative corrections. The formula

$$\Box A_\mu = -\frac{4\pi}{c}j_\mu, \tag{4}$$

may also be taken into account, j_μ standing for the current-charge four-vector related to the source of the external field. It is possible to treat the right-hand side of (2) as the lowest-order approximation to an infinite expansion in a power series containing $\Box^n A_\mu$ and $\Box^n F_{\mu\nu}$ (where $n = 0, 1, 2$ ...while $\Box^0$ denotes the unit operator). Such a series (for details see *e. g.* [15]) can be formally constructed if derivatives of all orders of A_μ are admitted in their linear combinations with the products of γ_μ (the algebraic properties of the latter being taken into account). The approximation assumed in (2) (discussed in [18]) seems to be sufficient for all practical applications and it will be adopted in our further considerations.

Irrespective of the fundamental importance of the manifestly covariant formulation, it is well known that the consequent quantum-mechanical interpretation ("the first quantization") of the Dirac equation is much more closely related to its Hamiltonian form[4]

$$ih\frac{\partial\psi}{\partial t} = H\psi, \qquad H = \beta mc^2 + c\alpha\cdot\left(p - \frac{e}{c}A\right) + eA_0, \tag{5}$$

[2] See Breit and Meyerott [19], Chraplyvy and Glover [20], Barker and Glover [21].

[3] The standard symbols of the Dirac electron theory and the summation convention have been introduced in (1) and (2) and used throughout this paper (the charge of the particle is denoted by e).

[4] There is also u possibility of a covariant Hamiltonian formulation of the unmodified as well, as of the amplified Dirac equation, *e.g.* following the idea displayed by one of us (Hanus [22], [23]) for the case of the free Dirac particle.

(with $\gamma_\mu = (-i\beta\boldsymbol{\alpha}, \beta)$, $A_\mu = (\boldsymbol{A}, iA_0)$) than to the equation (1). The same argument holds (or even becomes more obligatory) in the case of the amplified Dirac equation, as its meaning is, by definition, restricted to the quantum-mechanical domain. Indeed, more general field-theoretical considerations dispense alltogether with the notion of any amplified Dirac equation. Therefore, as a first step in our investigations, we transform the equation (2) into its respective Hamiltonian form. Simple calculations using (4) give

$$i\hbar \frac{\partial \psi'}{\partial t} = H'\psi', \quad H' = H - g_1 \frac{e\hbar}{2mc} \beta(\boldsymbol{\sigma} \cdot \boldsymbol{B} - i\boldsymbol{\alpha} \cdot \boldsymbol{E}) - \\ - g_2 \frac{4\pi e^2}{m^2c^2}\left(\varrho - \frac{1}{c}\boldsymbol{\alpha} \cdot \boldsymbol{j}\right), \tag{6}$$

where $\boldsymbol{B}$ and $\boldsymbol{E}$ denote the magnetic and electric parts of the tensor $F_{\mu\nu}$, respectively, $j_\mu = (\boldsymbol{j}, ic\varrho)$, while $\boldsymbol{\sigma}$ stands for the Dirac spin operator. The Hamiltonian (6) and, in particular, its Pauli terms gain a form more suitable for physical interpretation, if the Dirac operators $\boldsymbol{\sigma}$, $\boldsymbol{\varrho}$ are used[5] instead of $\boldsymbol{\alpha}$ and β ($\beta = \varrho_3$, $\boldsymbol{\alpha} = \varrho_1\boldsymbol{\sigma}$). We obtain in this way

$$H' = mc^2\varrho_3 + c\varrho_1\boldsymbol{\sigma} \cdot \left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) + eA_0 - \\ - g_1 \frac{e\hbar}{2mc} \boldsymbol{\sigma} \cdot (\varrho_3\boldsymbol{B} + \varrho_2\boldsymbol{E}) - g_2 \frac{4\pi e^2}{m^2c^2}\left(\varrho - \frac{1}{c}\varrho_1\boldsymbol{\sigma} \cdot \boldsymbol{j}\right). \tag{7}$$

For simplicity, $\boldsymbol{E}$ and $\boldsymbol{B}$ are assumed to be stationary. The dimensionless constants g_1 and g_2 are of the order of $\alpha = 1/137$ for the electron (in accordance with the results of quantum electrodynamics) and of the order of unity for the nucleons. In both cases the additional terms are small enough to be treated by the perturbation calculus. The order of magnitude of particular terms of the Hamiltonians (5) or (6) is usually characterized by the powers of c^{-1}(even if the atomic unit system $\hbar = e = 1$, $c = 137$ has been not used explicitly).

Following the, rather obvious, analogy in treating the equations (5) and (6) we arrive at the important problem (well known for (5)) — that of a consequent and unambiguous reduction of (6) to the subspace of positive energy states, in order to obtain the approximate "effective" Hamiltonian expressed within the Pauli formalism.

3. Decoupling of positive and negative energy states

This problem has been solved for the Dirac equation by means of the two alternative, "elimination" and "transformation" methods, respectively, while their equivalence has been subsequently proved by de Vries and Jonker ([24], [25], [26], [27])[6]. The impor-

[5] The danger of a confusion between the components of $\boldsymbol{\varrho}$ and the charge density ϱ present in (6) is not too serious.

[6] For details concerning these two main methods of reduction see Akhieser and Berestetskii [28], Foldy and Wouthuysen [29], Eriksen [30] and the quoted papers [24]-[27] where an extensive historical review of the problem may also be found.

tance of this strict result (suggested by their earlier, approximate calculations performed with the help of a computer to the 12-th order of approximation) seems to rely upon the fact that an unambiguous sense (independent of the method used) can be attributed to the formal definition of the reduction of (5) to the subspace of positive energy states. An analogical question arises for each modification of the amplified Dirac equation, the existing calculations being rather incomplete in this case[7]. The general solution of the problem would be to prove the strict formal equivalence between the elimination and transformation methods properly generalized, in order to include all possible modifications of the amplified Dirac equation. Leaving aside such a complicated task, we propose to begin by verifying this equivalence, by explicit calculations performed to a required order of approximation, in particular for the equation (6). Hence, the elimination method (of Pauli-Akhieser-Berestetskii) ought to be, tentatively, generalized in order to be used for this equation and the so obtained, reduced Hamiltonian compared with that resulting, in the same order of approximation, after applying to (6) a set of unitary transformations removing "odd" terms from it. Instead of the original transformation method (of Foldy-Wouthuysen-Eriksen) we shall use a modified procedure based on the transformation of Case [31], as this procedure has been successfully used for several other problems (see Garszczyński and Hanus [32], Hanus and Janyszek ([33], [34]) as well as Janyszek [35]). The case of the amplified Dirac equation will offer a new possibility to verify the efficiency of this procedure.

Practical applications of the reduced Dirac Hamiltonian are restricted, in general, to the second order of approximation — mainly with regard to the circumstance that contributions from the radiative corrections would be of the same order of magnitude, as those given by the next step in the reduction procedure[8]. For the same reason, the third order of approximation must be expected to be meaningful in the case of the reduced amplified Dirac equation, as additional terms giving a simplified "surrogate" of the radiative corrections are just of this order of magnitude.

Detailed, rather lengthy, calculations following the two generalized methods of reduction successively, and leading to a relatively simple and unambiguously established form of the third-order reduced Hamiltonian related to the equation (6), will be given soon. General remarks on the decoupling problem shortly summarized in this Chapter remain valid for all possible modifications of the amplified Dirac equation.

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[7] For use of the elementary elimination procedure introducing non-Hermitian terms, see [11], while calculations (to the second order of approximation) following the FW-iterative procedure have been given in [16], [20] and [21].

[8] For few calculations going beyond the second-order approximation see Bactavatsalou [36], Garszczyński and Hanus [37] and Passegi *et al.* [38].

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