NON-RELATIVISTIC LIMIT OF AN AMPLIFIED DIRAC EQUATION

BY W. HANUS AND R. MRUGAŁA

Institute of Physics, Nicholas Copernicus University, Toruń*

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An amplified Dirac equation has been approximately reduced to the subspace of positive energy states following the "elimination" and "transformation" methods generalized to this case, successively. Their mutual equivalence has been verified by explicit calculations accomplished to the third order of approximation.

1. *Introduction*

Some general problems related to the amplified Dirac equation (including higher-order tensor terms combined in a Lorentz- and gauge-invariant way) have been investigated in our previous paper (Hanus, Mrugała [1]) where the importance of passing over to the Hamiltonian formulation of this equation has been stressed in connection with its quantum-mechanical applications. A particular form of the amplified equation describing radiative corrections (see Bethe and Salpeter [2], p. 136) has been discussed in more detail. Possible methods of its subsequent reduction to the subspace of positive energy states have also been shortly outlined. Explicit calculations dealing with generalizations of the two ("elimination" and "transformation") methods well established and widely used for the Dirac equation will be displayed, successively, in this paper to terms of the third order in the fine structure constant $\alpha = e^2/\hbar c = 1/137$. This accuracy, of one order higher than that usually required for the approximate treatment of the Dirac equation, seems to be indispensable in our case, as radiative corrections (although in a simplified, phenomenological way) are to be taken into account. According to [1] we start from the equation

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi, \quad H = mc^2\varrho_3 + c\varrho_1\boldsymbol{\sigma}\cdot\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right) + eA_0 -$$

$$-g_1\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot(\varrho_3\boldsymbol{B} + \varrho_2\boldsymbol{E}) - g_2\frac{4\pi e^2}{m^2c^2}\left(\varrho - \frac{1}{c}\varrho_1\boldsymbol{\sigma}\cdot\boldsymbol{j}\right), \tag{1}$$

* Address: Instytut Fizyki, Uniwersytet Mikołaja Kopernika, Toruń, ul. Grudziądzka 5, Poland.

(where $\boldsymbol{j}$ and ϱ denote the current and charge density related to the external, stationary electromagnetic field $\boldsymbol{E} = -\text{grad } A_0$, $\boldsymbol{B} = \text{rot } A$, the remaing symbols having their standard meaning)[1].

Numerical values of the dimensionless constants g_1 and g_2 are of the order of unity for heavy fermions, but — in accordance with the known result of quantum electrodynamics — of the order of $\alpha = 1/137$ for the electrons. Adapting our further calculations mainly to the latter alternative we take into account this order of magnitude of g_1 and g_2 in all our estimations.

2. *The elimination method*

Similarly as in the case of the Dirac equation, the method consists in the elementary procedure of eliminating the small bi-spinor components[2], but supplemented by a non--unitary transformation removing non-Hermitian terms from the so obtained "formal Hamiltonian", in accordance with the idea of Akhieser and Berestetskiǐ ([4], p .125). Hence, applying to (1) the unitary transformation

$$U^{(m)} = e^{-\frac{i}{\hbar}mc^2t} \tag{2}$$

(introducing the distinction between the large and the small bi-spinor components denoted by φ and χ, respectively) and introducing the standard representation of the Dirac operators we obtain the set of equations for φ and χ ($\boldsymbol{\sigma}$ now standing for the Pauli spin operator)

$$i\hbar\frac{\partial\varphi}{\partial t} = c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e}{c}A\right)\chi+eA_0\varphi-g_1\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}\varphi+$$

$$+g_1\frac{ie\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{E}\chi-g_2\frac{4\pi e\hbar^2}{m^2c^2}\varrho\varphi+g_2\frac{4\pi e\hbar^2}{m^2c^3}\boldsymbol{\sigma}\cdot\boldsymbol{j}\chi \tag{3}$$

$$i\hbar\frac{\partial\chi}{\partial t} = -2mc^2\chi+c\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e}{c}A\right)\varphi+eA_0\chi+g_1\frac{e\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{B}\chi-$$

$$-g_1\frac{ie\hbar}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{E}\varphi-g_2\frac{4\pi e\hbar^2}{m^2c^2}\varrho\chi+g_2\frac{4\pi e\hbar^2}{m^2c^3}\boldsymbol{\sigma}\cdot\boldsymbol{j}\varphi. \tag{4}$$

We begin with the well known procedure of the step by step elimination of χ by means of (4), in order to obtain from (3) the respective approximate equation for φ. For g_1,

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[1] see formula (7) of [1]. All symbols used in our present paper are the same as those defined there (with the exception only that we put now H, ψ instead of H', ψ' used in [1]). The order of magnitude of particular terms will be estimated — similarly as in [1] — in terms of powers of c^{-1}. A more detailed list of references may also be found in [1].

[2] such calculations for the amplified Dirac equation may be found in the paper of Petiau [3].

$g_2 \sim 1/137$ we obtain

$$\chi^{(1)} = \frac{1}{2mc}\,\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)\varphi, \tag{5}$$

$$i\hbar\frac{\partial\varphi^{(1)}}{\partial t} = H^{(1)}\varphi^{(1)}, \qquad H^{(1)} = H^0 + H'_p, \tag{6}$$

$$H^0 = \frac{1}{2m}\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right)^2 + eA_0, \tag{7}$$

$$H'_p = -\frac{e\hbar}{2mc}\,\boldsymbol{\sigma}\cdot\boldsymbol{B}, \tag{8}$$

$$\chi^{(2)} = \left\{\frac{1}{2mc}\,\boldsymbol{\sigma}\cdot\left(\boldsymbol{p}-\frac{e}{c}\boldsymbol{A}\right) + \frac{eA_0}{4m^2c^3}\,\boldsymbol{\sigma}\cdot\boldsymbol{p} - \frac{i\hbar}{4m^2c^3}\,\boldsymbol{\sigma}\cdot\boldsymbol{p}\frac{\partial}{\partial t}\right\}\varphi, \tag{9}$$

$$i\hbar\frac{\partial\varphi^{(2)}}{\partial t} = H^{(2)}\varphi^{(2)}, \qquad H^{(2)} = H^0 + (1+g_1)H'_p + H''_h + H''_a, \tag{10}$$

$$H''_h = H''_s + H''_t + H''_d, \tag{11}$$

$$H''_s = -\frac{p^4}{8m^3c^2}, \tag{12}$$

$$H''_t = -\frac{e\hbar}{8m^2c^2}\,\boldsymbol{\sigma}(\boldsymbol{E}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{E}), \tag{13}$$

$$H''_d = -\frac{e\hbar^2}{8m^2c^2}\,\mathrm{div}\,E, \tag{14}$$

$$H''_a = \frac{e}{8m^2c^2}\,[A_0,\,p^2]. \tag{15}$$

The first-order approximation (formulae (6)–(8)) gives thus the well known Pauli equation, while (12)–(14) represent the second-order corrections of Sommerfeld, Thomas-Frenkel and Darwin, respectively. H''_a is anti-Hermitian, but it can be transformed away by means of the Akhieser-Berestetskiĭ transformation

$$\Theta^{(2)} = 1 + \frac{p^2}{8m^2c^2}. \tag{16}$$

Simple calculations give

$$H_h^{(2)} = \Theta^{(2)}H^{(2)}(\Theta^{(2)})^{-1} = H^0 + (1+g_1)H'_p + H''_h. \tag{17}$$

The result differs from that for the Dirac equation merely by the additive contribution, proportional to g_1, to the Pauli term H'_p. Extending this iterative procedure to the third order of approximation we obtain after elementary, although somewhat lengthy calculations

$$\chi^{(3)} = \left\{ \frac{1}{2mc} \sigma \cdot \left(p - \frac{e}{c} A \right) + \frac{eA_0}{4m^2c^3} \sigma \left(p - \frac{e}{c} A \right) - \right.$$

$$\left. - \frac{i\hbar}{4m^2c^3} \sigma \cdot \left(p - \frac{e}{c} A \right) \frac{\partial}{\partial t} - g_1 \frac{ie\hbar}{4m^2c^3} \sigma \cdot E \right\} \varphi, \tag{18}$$

$$i\hbar \frac{\partial \varphi^{(3)}}{\partial t} = H^{(3)} \varphi^{(3)}, \quad H^{(3)} = H^0 + (1+g_1) H'_p +$$

$$+ H''_s + (1+2g_1) H''_t + (1+2g_1) H''_d + H''_a + H'''_h + H'''_a, \tag{19}$$

$$H'''_h = \frac{e}{8m^3c^3} \{p^2, p \cdot A + A \cdot p + \hbar \sigma \cdot B\} - \frac{e^2\hbar}{4m^2c^3} \sigma \cdot (A \times E) -$$

$$- g_2 \frac{4\pi e \hbar^2}{m^2c^2} \varrho, \tag{20}$$

$$H'''_a = - \frac{e^2}{8m^2c^3} [A_0, A \cdot p + p \cdot A]. \tag{21}$$

The anti-hermitian term $H''_a + H'''_a$ disappears after transforming $H^{(3)}$ by

$$\Theta^{(3)} = 1 + \frac{1}{8m^2c^3} \left(p - \frac{e}{c} A \right)^2 \approx 1 + \frac{p^2}{8m^2c^2} - \frac{e}{8m^2c^3} (p \cdot A + A \cdot p). \tag{22}$$

Hence we obtain finally[3]

$$H_h^{(3)} = \Theta^{(3)} H^{(3)} (\Theta^{(3)})^{-1} = H^0 + (1+g_1) H'_p +$$

$$+ H''_s + (1+2g_1) H''_t + (1+2g_1) H''_d + H'''_h. \tag{23}$$

3. *The transformation method*

The well known "FW-method" of Foldy and Wouthuysen [5] supplemented by the condition of Eriksen [6] may be shortly denoted as the "transformation method". The proof of its equivalence with the previously discussed "elimination method" has been given by de Vries and Jonker ([7], [8], [9], [10]). Their result implies the analogical question for the amplified Dirac equation. Leaving aside the problem of the strict equivalence we shall verify by explicit calculations given in this chapter that — to the third order of approximation — the reduced Hamiltonian (23) is identical with that resulting from the Hamiltonian (1) transformed to the "even" form (separating positive and negative

[3] some additional terms would be present in the more general case of quasi-stationary fields.

energy states). It must be stressed that our procedure is a modification, rather than a simple generalization of the FW-method in its original version[4]. Instead of that we start from the transformation of Case [12] which brings to the strictly "even" form the operator of the kinetic energy of the Dirac Hamiltonian. The advantage of applying this transformation (introduced by Case, himself only for $A \neq 0$, $A_0 = 0$) when both, magnetic and electric fields are present, has been discussed by Garszczyński and Hanus [13]. Besides some new aspects of the physical interpretation, a considerable simplification of calculations has been achieved in this way, leading through an "intermediary" representation of the Dirac equation. Hence, we follow a similar path for the amplified Dirac equation. The unitary operator of Case reads

$$U = e^{iS}, \quad S = \frac{1}{2}\varrho_2 \operatorname{arc\,tg} \frac{z}{mc}, \quad z = \sigma\cdot\left(p - \frac{e}{c}A\right). \tag{24}$$

After simple calculations it can also be written down in the form

$$U = U_1 + i\varrho_2 U_2, \tag{25}$$

with

$$U_1 = \cos\frac{\varphi}{2} = \sqrt{\frac{E+mc^2}{2E}}, \quad U_2 = \sin\frac{\varphi}{2} = \sqrt{\frac{E-mc^2}{2E}} = \frac{p}{\sqrt{2E(E+mc^2)}}, \tag{26}$$

$$E = c\sqrt{mc^2 + z^2}. \tag{27}$$

Transforming the Hamiltonian (1) by means of U, expanding all expressions in power series of the small parameter z/mc and neglecting terms of the order higher than the third, we obtain, after somewhat tedious calculations

$$H_u = UHU^{-1} = K_0 + \sum_{j=1}^{3} \varrho_j K_j \tag{28}$$

with

$$K_0 = eA_0 - (1+2g_1)\frac{e\hbar}{8m^2c^2}(E\times p - p\times E) -$$

$$-(1+2g_1)\frac{e\hbar^2}{8m^2c^2}\operatorname{div} E - \frac{e^2\hbar}{4m^2c^3}\sigma\cdot(A\times E) - g_2\frac{4\pi e\hbar^2}{m^2c^2}\varrho, \tag{29}$$

$$K_3 = mc^2 + \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 - (1+g_1)\frac{e\hbar}{2mc}\sigma\cdot B -$$

$$-\frac{p^4}{8m^3c^2} + \frac{e}{8m^3c^3}\{p^2, p\cdot A + A\cdot p + \hbar\sigma\cdot B\}, \tag{30}$$

[4] Calculations closely following the latter method have been given by Barker and Chraplyvy [11], to terms of the order c^{-2}.

$$K_1 = g_1 \frac{eh}{4m^2c^2}(\boldsymbol{p}\cdot\boldsymbol{B}+\boldsymbol{B}\cdot\boldsymbol{p}), \tag{31}$$

$$K_2 = -(1+g_1)\frac{eh}{2mc}\boldsymbol{\sigma}\cdot\boldsymbol{E}+$$

$$+\frac{ie}{16m^3c^3}[3p^2(\boldsymbol{\sigma}\cdot\boldsymbol{p})A_0-p^2A_0(\boldsymbol{\sigma}\cdot\boldsymbol{p})+(\boldsymbol{\sigma}\cdot\boldsymbol{p})A_0p^2-3A_0(\boldsymbol{\sigma}\cdot\boldsymbol{p})p^2]. \tag{32}$$

The terms $\varrho_1 K_1+\varrho_2 K_2$ are "odd". Following the general idea of the iterative FW-method, it is not difficult to guess the explicit form of an additional unitary transformation causing the disappearance of these two terms. It sufficies to put

$$W = e^{iR}, \quad R = \frac{1}{2mc^2}\left[\varrho_2 K_1-\varrho_1\left(K_2+\frac{eh}{8m^3c^3}\{p^2,\boldsymbol{\sigma}\cdot\boldsymbol{E}\}\right)\right] \tag{33}$$

in order to obtain the final form of the Hamiltonian (1) transfomed to the "even" form in the required, third order of approximation. We have, namely

$$H^{(\text{even})} = WH_uW = K_0+\varrho_3 K_3. \tag{34}$$

The, somewhat controversial, problem of uniqueness of W does not matter in this approximation, R containing only sufficiently high powers of c^{-1}. For the same reason it is allowed to treat W as one unitary transformation or, alternatively, as the product of such transformations, with particular terms of R in the exponent. This possibility results immediately from the Hausdorff's formula.

The reduction of H^{even} to the subspace of positive energy states is trivial and consists in putting $\varrho_3 = +1$. The reduced Hamiltonian is identical with that given by (23), except for the absence of the term mc^2 in the latter (in consequence of the, previously applied to it, transformation (2), changing the normalization of energy).

4. *Conclusions*

Calculations given in Chapters 2 and 3 have verified the — at least approximate — equivalence between the "elimination method" (of Pauli-Akhieser-Berestetskiĭ) immediately generalized to the amplified Dirac equation (1) and the procedure, proposed by us, of applying the Case's transformation followed by a supplementary transformation, in order to obtain from (1) the "even" Hamiltonian in the same order of approximation. The latter procedure is a modification of that resulting from the original "transformation method" (of Foldy-Wouthuysen-Eriksen). The characteristic feature of the Case's transformation U is that the Hamiltonian (1) transformed to this "intermediary representation" contains all terms of the final "even" Hamiltonian so that the only role of the supplementary transformation W is to transform away the superfluous "odd" terms (of the order c^{-1} and higher), without spoiling the correct form of the "even" part. The simplification of calculations obtained in this way, as compared to those of the iterative FW-method (becoming very cumbersome already in the second-order approximation) is obvious.

The problem of a strict equivalence (in the sense assumed in the quoted papers of de Vries and Jonker) between various modifications of the transformation procedure as well, as between a uniquely defined transformation and elimination methods, respectively, remains open, as yet, for the amplified Dirac equation discussed in this paper. It seems, however, that the achieved accuracy (to the terms of the third order, *i.e.* to those proportional to c^{-3}, g_1c^{-2} and g_2c^{-2}, successively) is sufficient for many practical applications of (23). There are also possibilities of generalizations to the many-fermion problems.

It would be possible to adapt (23) to the case of heavy fermions, with $g_1, g_2 \sim 1$, however only in the approximation c^{-2}, after changing the ordering of particular terms, according to their order of magnitude. Another possibility — that of modifying all calculations, in order to include all terms of the order c^{-3} also in this case — seems to be rather superfluous for practical reasons.

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