Vol. B 3 (1972) ACTA PHYSICA POLONICA No 4

ON THE NEW DIRAC EQUATIONS

By E. Kapuścik

Institute of Nuclear Physics, Cracow*

(Received December 23, 1971)

A general class of wave equations is considered which as a special case contains the new wave equations proposed by Dirac. It is shown that it is possible to introduce the interaction with an external electromagnetic field if we choose different members of the considered class. A particular case of such wave equations is treated in some detail.

1. *Introduction*

Recently, Dirac proposed a new relativistic wave equation which is not symmetrical between positive and negative energies. This new equation strongly resembles the usual Dirac equation but its physical consequences are very different. In the new scheme the wave function transforms according to some infinite dimensional representation of the Lorentz group and it gives only integral values for spin.

In connection with the new Dirac equation, many interesting problems arise. Among them is the prime problem of introducing into the equation the interaction with other fields, in particular with an external electromagnetic field. Dirac has shown that for the new wave equation it is impossible to solve this problem in the usual way by replacing the derivatives ∂_μ in the equation by $\partial_\mu - ieA_\mu$. Therefore, if we wish to preserve this manner we must change other elements of the considered theory.

In the present paper we show that the new Dirac equation may be obtained from a general class of wave equations as a consequence of some particular assumption. Rejecting this assumption we then get various examples of wave equations which also are not symmetrical between positive and negative energies and for which the interaction with an electromagnetic field can be introduced in the usual way. One such example is discussed in some detail at the end of this paper.

2. *The general class of wave equations and their consistency*

Let us consider vector-valued wave functions $\psi(x)$ defined on the Minkowski space M^4 and taking on values in some separable Hilbert space H. We shall assume that the wave function $\psi(x)$ is strongly continuous and has strong derivatives up to the second order.

* Address: Instytut Fizyki Jądrowej, Kraków 23, Radzikowskiego 152, Poland.

In addition, we shall assume that for each x, $\psi(x)$ belongs to the domains of some number of selfadjoint operators Q_a^μ and P_a where the indices μ and a run from 0 to 3 and from 1 to n, respectively. These assumptions must ensure the correctness of all performed operations.

The general class of wave equations which we shall discuss is of the form

$$(Q_a^\mu \partial_\mu + P_a)\psi(x) = 0; \; a = 1, 2, \dots n. \tag{2.1}$$

Since the wave function satisfies more than one differential equation it is necessary for them to satisfy some consistency condition. This consistency condition is obviously of the form

$$[Q_a^\mu \partial_\mu + P_a, Q_b^\nu \partial_\nu + P_b]\psi(x) = 0 \tag{2.2}$$

and leads to some second order differential equations for $\psi(x)$. We make the assumption that these equations are essentially the Klein-Gordon equation with mass m. Thus, we assume that

$$[Q_a^\mu \partial_\mu + P_a, Q_b^\nu \partial_\nu + P_b] = \Delta_{ab}(\Box + m^2) \tag{2.3}$$

where Δ_{ab} are elements of some non-singular numerical matrix Δ whose form has to be determined. A simple calculation then shows that the operators Q_a^μ and P_a must satisfy the following commutation relations:

$$[Q_a^\mu, Q_b^\nu] + [Q_a^\nu, Q_b^\mu] = 2g^{\mu\nu}\Delta_{ab} \tag{2.4}$$

$$[Q_a^\mu, P_b] + [P_a, Q_b^\mu] = 0 \tag{2.5}$$

$$[P_a, P_b] = m^2 \cdot \Delta_{ab}. \tag{2.6}$$

It is easy to show that this set of commutation relations has many inequivalent representations. Not all of them lead to physical theories satysfying the requirement of relativistic invariance, providing conserved density current four vectors $j^\mu(x)$ with positive definite density $j^0(x)$ and allowing the introduction of the interaction with other fields.

In order to establish the relativistic invariance of the theory let us assume that under the Lorentz transformation

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu \tag{2.7}$$

the wave function $\psi(x)$ transforms according to some infinite dimensional linear representation $T(\Lambda)$ of the Lorentz group. By standard considerations we then find that the operators Q_a^μ and P_a have to be replaced in the wave equations by new operators $Q_a'^\mu$ and P_a' satisfying the same commutation relations as Q_a^μ and P_a do and related to them by

$$P_a' = S_{ab}(\Lambda)V(\Lambda)P_b T^{-1}(\Lambda) \tag{2.8}$$

and

$$Q_a'^\mu = S_{ab}(\Lambda)\Lambda^\mu_\nu V(\Lambda)Q_b^\nu T^{-1}(\Lambda) \tag{2.9}$$

where $S(\Lambda)$ and $V(\Lambda)$ are two representations of the Lorentz group, the former being finite dimensional and the latter infinite dimensional. In general $V(\Lambda)$ is not connected with $T(\Lambda)$, but for simplicity we shall assume that they coincide. Then the condition that the primed operators satisfy the same commutation relations as the unprimed gives us a relation between the matrices Δ and S in the form

$$\Delta = S(\Lambda)\Delta S^T(\Lambda) \tag{2.10}$$

In addition, if we assume that the representation $T(\Lambda)$ is unitary the elements of the matrix $S(\Lambda)$ must be real.

At this point we wish to stress the difference in treating the requirement of relativistic invariance in the above scheme and in the usual wave equations. In our case the operators Q_a^μ and P_a play the role of numerical constants in the usual wave equations but the concrete representations of these operators may vary with the coordinate systems while usually the numerical constants remain unchanged.

Performing now the usual manipulations with the wave equations we get many conserved quantities. Among them we shall consider only those which are of the form

$$j_\Omega^\mu(x) = (\psi(x), \Omega_{ab}P_a Q_b^\mu \psi(x)) \tag{2.11}$$

where the numerical coefficients Ω_{ab} satisfy the condition

$$\Omega_{ab} = -\Omega_{ba} \tag{2.12}$$

It is easy to check that as a consequence of the wave equations (2.1) and the commutation relation (2.5) this generalized density current satisfies the conservation law

$$\partial_\mu j_\Omega^\mu(x) = 0 \tag{2.13}$$

but does not in general transform like a fourvector. In order to get a fourvector we have to impose the condition

$$S^T(\Lambda)\Omega S(\Lambda) = \Omega \tag{2.14}$$

A further restriction on Ω comes from the requirement that

$$j_\Omega^0(x) \geqslant 0 \tag{2.15}$$

Suppose now that we want to introduce into the wave equations the interaction with an external electromagnetic field $A_\mu(x)$ by the usual replacement

$$\partial_\mu \to \partial_\mu - ieA_\mu(x). \tag{2.16}$$

It is clear that we must first change the consistency equation (2.3). An obvious generalization of (2.3) is then

$$[Q_a^\mu(\partial_\mu - ieA_\mu) + P_a, Q_b^\nu(\partial_\nu - ieA_\nu) + P_b] =$$
$$= \Delta_{ab}\{(\partial_\mu - ieA_\mu)(\partial^\mu - ieA^\mu) + m^2 + \Omega^{\mu\nu}F_{\mu\nu}\} \tag{2.17}$$

where $F_{\mu\nu}$ is the tensor of the electromagnetic field and $\Omega^{\mu\nu}$ are some operators which are skew in μ and ν. Performing then the corresponding commutations in addition to the relations (2.4)–(2.6) we get the following equality

$$\frac{-ie}{4}(\{Q_a^\mu, Q_b^\nu\}_+ - \{Q_a^\nu, Q_b^\mu\}_+)F_{\mu\nu} = \Delta_{ab}\Omega^{\mu\nu}F_{\mu\nu} \tag{2.18}$$

from which we infer that $F_{\mu\nu}$ may be different from zero only if

$$\frac{-ie}{4}(\{Q_a^\mu, Q_b^\nu\}_+ - \{Q_a^\nu, Q_b^\mu\}_+) = \Delta_{ab}^r \Omega^{\mu\nu} \tag{2.19}$$

($\{\cdot\}_+$ here denotes the anticommutator).

This condition is a restriction on the possible representations of the operators Q_a^μ since if it is not satisfied in the given representation we cannot have there non-zero $F_{\mu\nu}$. It is certainly not satisfied if the matrix Δ has too many zero elements, as happens in the Dirac case which we will describe in the next section.

The new operators $\Omega^{\mu\nu}$ play the role of operators of intrinsic magnetic and electric moments. This fact shows that our wave equations describe (if any) particles with non-trivial internal structure.

3. The Dirac new wave equations

The Dirac new wave equations follow from our general case if we make the assumption that only the operators P_a are independent operators while

$$Q_a^\mu = \sum_{b=1}^{n} \varrho_{ab}^\mu P_b \tag{3.1}$$

where ϱ_{ab}^μ are some real numbers. From the basic commutation relations we then get the restrictions on the possible forms of matrices ϱ^μ constituted from ϱ_{ab}^μ. They are

$$\varrho^\mu \Delta \varrho^{\nu T} + \varrho^\nu \Delta \varrho^{\mu T} = \frac{2}{m^2} g^{\mu\nu} \Delta \tag{3.2}$$

$$\varrho^\mu \Delta + \Delta \varrho^{\mu T} = 0. \tag{3.3}$$

Substituting (3.3) into (3.2) we see that

$$\varrho^\mu \varrho^\nu + \varrho^\nu \varrho^\mu = -\frac{2}{m^2} g^{\mu\nu} \tag{3.4}$$

from which we find that ϱ^μ are equal to $\pm \frac{1}{m}\gamma^\mu$ where γ^μ are the usual Dirac matrices. The reality of ϱ^μ restricts the possible representations of γ^μ to those in which all non-zero elements of γ^μ are real. The condition (3.3) now becomes

$$\gamma^\mu \Delta + \Delta \gamma^{\mu T} = 0. \tag{3.5}$$

Introducing the matrix B by

$$\gamma^{\mu T} = B \gamma^\mu B^{-1} \tag{3.6}$$

we easily find that

$$\Delta = \lambda \gamma^5 B^{-1} \tag{3.7}$$

with an arbitrary number λ. This result shows that the form of Δ is completely fixed up to a similarity transformation. Simultaneously we see that the condition (2.19) is never satisfied and consequently we cannot introduce the electromagnetic field into the equations in the conventional way.

4. *A different solution of the basic commutation relations*

In this section we shall consider the solution of the basic commutation relations augmented by an additional assumption that

$$[Q^{\mu}_{a}, Q^{\nu}_{b}] = g^{\mu\nu}\Delta_{ab}. \tag{4.1}$$

The first commutation relation (2.4) is then automatically satisfied. In order to satisfy the remaining two relations (2.5) and (2.6) we put

$$P_a = \sum_{b=1}^{n} (\lambda^T_{\mu})_{ab} Q^{\mu}_{b} \tag{4.2}$$

where $(\lambda_{\mu})_{ab}$ are constants and summation over μ is understood. For the matrices λ_{μ} constituted from $(\lambda_{\mu})_{ab}$ we obtain the following conditions

$$\lambda_{\mu}\Delta + \Delta\lambda^T_{\mu} = 0 \tag{4.3}$$

$$g^{\mu\nu}\lambda_{\mu}\lambda^T_{\nu} = -m^2. \tag{4.4}$$

We shall treat these two conditions as conditions on the matrices λ_{μ} and not on the matrix Δ. It is then possible to choose the lowest dimensional matrix Δ which is in fact two dimensional. Without loss of generality we may take

$$\Delta = i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.5}$$

and we find that the matrices λ_{μ} are of the form

$$\lambda_{\mu} = \begin{pmatrix} a_{\mu}, & b_{\mu} \\ c_{\mu}, & -a_{\mu} \end{pmatrix} \tag{4.6}$$

with

$$a_{\mu}a^{\mu} + b_{\mu}c^{\mu} = -m^2. \tag{4.7}$$

In the present case the formalism is therefore not so restrictive as before and we still have a large freedom in the choice of the parameters a_{μ}, b_{μ} and c_{μ}. In order to reduce this arbitrariness we take the general expression for the density current four-vector (2.11) and observe that for the positivity of $j^0(x)$ it is sufficient to assume that λ_{μ} are skew matrices. Then the conditions (4.6) and (4.7) tell us that

$$\lambda_{\mu} = b_{\mu}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.8}$$

where

$$b_\mu b^\mu = m^2. \tag{4.9}$$

Therefore we may choose the coordinate system in such a way that $b_\mu = (\pm m, 0, 0, 0)$. If we then take the matrix Ω as $(\lambda_0^T)^{-1}$ we get

$$j^0(x) = (\psi(x), [(Q_1^0)^2+(Q_2^0)^2]\psi(x) \tag{4.10}$$

which is obviously positive.

The wave equations in the above coordinate system are of the form

$$(Q_1^\mu \partial_\mu \pm m Q_2^0)\psi(x) = 0$$

$$(Q_2^\mu \partial_\mu \mp m Q_1^0)\psi(x) = 0. \tag{4.11}$$

In order to obtain explicit solutions of these equations we realize our Hilbert space as a space of complex valued square integrable functions of four variables q^μ ($\mu = 0, 1, 2, 3$). We realize the operators as

$$(Q_1^\mu \psi)(q) = q^\mu \psi(q) \tag{4.12}$$

and

$$(Q_2^\mu \psi)(q) = -ig^{\mu\nu} \frac{\partial}{\partial q^\nu} \psi(q).$$

We may write the general solution of (4.11) as a Fourier integral

$$\psi(q, x) = \int e^{-ipy} \psi(q, p) d\varrho(p). \tag{4.13}$$

For the function $\psi(q, p)$ we then obtain two equations

$$\left(q^0 p^0 - \vec{p} \cdot \vec{q} \pm m \frac{\partial}{\partial q^0}\right) \psi(q, p) = 0$$

$$\left(p^0 \frac{\partial}{\partial q^0} + \vec{p} \cdot \vec{\nabla}_q \pm m q^0\right) \psi(q, p) = 0. \tag{4.14}$$

The consistency of these equations requires that $p^2 = m^2$ and the normalized solution is

$$\psi(q, p) = \frac{1}{\pi} \sqrt{\frac{p^0}{m}} \exp\left\{\mp \frac{p^0}{2m}(q_0^2+q_1^2+q_2^2+q_3^2) \pm \frac{\vec{p} \cdot \vec{q}}{m} q_0\right\}. \tag{4.15}$$

From this expression we see that for the upper signs in (4.11) the solution $\psi(q, p)$ belongs to our Hilbert space only for positive energies $p^0 > 0$ while for the lower signs it does so only for negative energies $p^0 < 0$. In this way we have either positive or negative energies but not both. The theory is, however, symmetric with respect to the sign of energy since both signs in (4.11) are admissible.

Assuming that the representations of the operators Q^μ_a are invariant under the Lorentz group, we get from (2.9) the representation of the infinitesimal generators of this group in the form

$$T^{\mu\nu} = Q^\mu_1 Q^\nu_2 - Q^\nu_1 Q^\mu_2. \tag{4.16}$$

Using these operators we see that the solution (4.15) in the rest frame of the particle is an eigenstate of the spin operators with the eigenvalue zero. The wave equations (4.11) therefore describe particles of zero spin.

The density current four-vector is

$$j^\mu(x) = \pm \frac{p^\mu}{m} \tag{4.17}$$

as is expected for the free particles.

The mean values of the operators $\Omega^{\mu\nu}$ turn out to be equal zero. This shows that our wave equations describe particles which may polarize in the presence of an external electromagnetic field.

5. *Conclusion*

We have shown that the above-described theory meets all the requirements for a physical theory. Nevertheless, it is only a special case of the general scheme developed in Section 2. We may equally well assume different realizations of the operators P_a in terms of the operators Q^μ_a, for example the quadratic one where the operators P_a are linear combinations of the three scalar operators

$$W_1 = g_{\mu\nu} Q^\mu_1 Q^\nu_1$$
$$W_2 = g_{\mu\nu} Q^\mu_2 Q^\nu_2 \tag{5.1}$$

and

$$W_3 = g_{\mu\nu}(Q^\mu_1 Q^\nu_2 + Q^\mu_2 Q^\nu_1)$$

In order to admit such a possibility we have only slightly to generalize the consistency condition (2.3). This generalization consists in replacing the mass term by an operator acting on the internal degrees of freedom. With an appropriate choice of coefficients we may take this operator to be of the form

$$M^2 = \alpha W_1 + \beta W_2. \tag{5.2}$$

The eigenvalues of this operator are given by

$$m_0^2(n_0 - n_1 - n_2 - n_3 - 1) \tag{5.3}$$

where m_0^2 is some positive constant and n_μ are integers. However, this case also gives negative eigenvalues for M^2. In order to remove this defect we may take the P_a as squares of some linear combinations of the W_i. It is then possible to get the mass operators M^2

with eigenvalues $n^2 m_0^2$ where n is an integer. The operators Q_a^μ and P_a do not commute with such M^2 and consequently the eigenstates of M^2 are not solutions of the wave equations. Instead, the solutions of the wave equations are infinite linear combinations of eigenstates of M^2, and each of these eigenstates is infinitely degenerate. The wave function therefore describes a state with a finite probability of finding each eigenvalue of the mass spectrum. We believe that in such a case the wave function $\psi(x)$ may be interpreted as a state of some physical field describing a collection of an arbitrary number of particles. A detailed discussion of this will be presented in a separate paper.

REFERENCES

[1] P.A.M. Dirac, *Proc. Roy. Soc.* (London), **A322**, 435 (1971).